



Tietze's Extension Theorem for Intuitionistic Fuzzy ζ -Basically Disconnected Spaces in Katetov-Tong's Sense

Research Article

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Abstract: In this paper we initiate the concept of ζ -basically disconnectedness in intuitionistic fuzzy topological spaces. We also apply these notions of ζ -basically disconnectedness to analyse and prove Tietze extension theorem.

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [14]. Fuzzy basically disconnected spaces was discussed and studied in [12]. Bruce Hutton [7] constructed an interesting L-fuzzy topological space called L-fuzzy unit interval which plays the same role in fuzzy topology. Using the concept of L-fuzzy unit interval, Tomasz Kubiaz [10, 11] extended the Urysohn lemma and Tietze extension theorem for L-fuzzy normal spaces. Atanassov [1] generalised intuitionistic fuzzy sets using the notion of fuzzy sets. On the other hand Coker [5] introduced the notion of an intuitionistic fuzzy topological spaces. In this paper we introduce and study the concept of an intuitionistic fuzzy ζ -space and intuitionistic fuzzy ζ -basically disconnected space. An approach to Tietze's extension theorem for intuitionistic fuzzy ζ -basically disconnected spaces has been established based on Kotetov and Tong [8, 9, 13].

2. Preliminaries

Definition 2.1 ([5]). An intuitionistic fuzzy set (IFS, in short) A in X is an object having the form $A = \{x, \mu_A(x), \nu_A(x) / x \in X\}$ where the functions $\mu_A : X \rightarrow I$ and $\nu_A : X \rightarrow I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A on a nonempty set X and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. Obviously every fuzzy set A on a nonempty set X is an IFSs A and B be in the form $A = \{x, \mu_A(x), 1 - \mu_A(x) / x \in X\}$.

Definition 2.2 ([5]). Let X be a nonempty set and the IFSs A and B be in the form $A = \{x, \mu_A(x), \nu_A(x) / x \in X\}$, $B = \{x, \mu_B(x), \nu_B(x) / x \in X\}$ and let $A = \{A_j : j \in J\}$ be an arbitrary family of IFSs in X . Then we define

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- (i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$.
- (ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
- (iii) $\bar{A} = \{x, \nu_A(x), \mu_A(x) / x \in X\}$. \bar{A} is the complement of A .
- (iv) $A \cap B = \{x, \mu_A(x) \cap \mu_B(x), \nu_A(x) \cup \nu_B(x) / x \in X\}$.
- (v) $A \cup B = \{x, \mu_A(x) \cup \mu_B(x), \nu_A(x) \cap \nu_B(x) / x \in X\}$.
- (vi) $1_{\sim} = \{(x, 1, 0) \mid x \in X\}$ and $0_{\sim} = \{(x, 0, 1) \mid x \in X\}$.

Definition 2.3 ([5]). An intuitionistic fuzzy topology (IFT, in short) on a nonempty set X is a family τ of an intuitionistic fuzzy set (IFS, in short) in X satisfying the following axioms:

- (i) $0_{\sim}, 1_{\sim} \in \tau$.
- (ii) $A_1 \cap A_2 \in \tau$ for any $A_1, A_2 \in \tau$.
- (iii) $\cup A_j \in \tau$ for any $A_j : j \in J \subseteq \tau$.

In this paper we denote intuitionistic fuzzy topological space (IFTS, in short) by (X, τ) , (Y, κ) or X, Y . Each IFS which belongs to τ is called an intuitionistic fuzzy open set (IFOS, in short) in X . The complement \bar{A} of an IFOS A in X is called an intuitionistic fuzzy closed set (IFCS, in short). An IFS X is called intuitionistic fuzzy clopen (IF clopen) iff it is both intuitionistic fuzzy open and intuitionistic fuzzy closed.

Definition 2.4 ([5]). Let (X, τ) be an IFTS and $A = \{x, \mu_A(x), \nu_A(x)\}$ be an IFS in X . Then the fuzzy interior and closure of A are denoted by

- (i) $cl(A) = \bigcap \{K : K \text{ is an IFCS in } X \text{ and } A \subseteq K\}$.
- (ii) $int(A) = \bigcup \{G : G \text{ is an IFOS in } X \text{ and } G \subseteq A\}$.

Note that, for any IFS A in (X, τ) , we have $cl(\bar{A}) = \overline{int(A)}$ and $int(\bar{A}) = \overline{cl(A)}$.

Definition 2.5 ([6]). Let a and b be two real numbers in $[0, 1]$ satisfying the inequality $a + b \leq 1$. Then the pair $\langle a, b \rangle$ is called an intuitionistic fuzzy pair. Let $\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle$ be two intuitionistic fuzzy pairs. Then

- (i) $\langle a_1, b_1 \rangle \leq \langle a_2, b_2 \rangle$ if and only if $a_1 \leq a_2$ and $b_1 \geq b_2$.
- (ii) $\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$ if and only if $a_1 = a_2$ and $b_1 = b_2$.
- (iii) If $\{\langle a_i, b_i \rangle : i \in J\}$ is a family of intuitionistic fuzzy pairs, then $\cup \langle a_i, b_i \rangle = \langle \cup a_i, \cap b_i \rangle$ and $\cap \langle a_i, b_i \rangle = \langle \cap a_i, \cup b_i \rangle$.
- (iv) The complement of an intuitionistic fuzzy pair $\langle a, b \rangle$ is the intuitionistic fuzzy pair defined by $\overline{\langle a, b \rangle} = \langle b, a \rangle$.
- (v) $1_{\sim} = \langle 1, 0 \rangle$ and $0_{\sim} = \langle 0, 1 \rangle$.

Definition 2.6 ([4]). Let X be a nonempty set and $A \subset X$. The characteristic function of A is denoted and defined by

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$

Definition 2.7 ([2]). Let (X, τ) be a fuzzy topological space and be a fuzzy set in X . Then λ is called fuzzy G_δ if $\lambda = \bigwedge_{i=1}^{\infty} \lambda_i$ where each $\lambda_i \in T$. The complement of fuzzy G_δ is F_σ .

Definition 2.8 ([3]). A fuzzy bitopological space (X, τ_1, τ_2) is said to be pairwise fuzzy basically disconnected if τ_1 -closure of each τ_2 -fuzzy open, τ_2 -fuzzy F_σ is τ_2 -fuzzy open and τ_2 -closure of each τ_1 -fuzzy open, τ_1 -fuzzy F_σ is τ_1 -fuzzy open.

Definition 2.9 ([5]). An IFTS X is called fuzzy compact if and only if every fuzzy open cover has a finite subcover.

3. An Intuitionistic Fuzzy ζ -basically Disconnected Spaces

Definition 3.1. Let (X, τ) be an intuitionistic fuzzy noncompact space. Let Ψ be a collection of all intuitionistic fuzzy sets which are both intuitionistic fuzzy closed set and intuitionistic fuzzy compact set in X . Let $U^+ = \{A \in \Psi : A \cap U \neq 0_\sim, U \in \tau\}$ and $V^- = \{A \in \Psi : A \cap V = 0_\sim : V \text{ is an intuitionistic fuzzy compact set in } X\}$. Then the collection $\zeta = \{A : \bar{A} \in U^+\} \cup \{B : \bar{B} \in V^-\}$ is said to be intuitionistic fuzzy ζ -structure on X and the pair (X, ζ) is said to be an intuitionistic fuzzy ζ -space.

Notation 3.2. Each member of an intuitionistic fuzzy ζ -structure is an intuitionistic fuzzy ζ -open set. The complement of an intuitionistic fuzzy ζ -open set is an intuitionistic fuzzy ζ -closed set.

Example 3.3. Let $X = \{a, b\}$ be a nonempty set. Let $G_1 = \langle x, (0.4, 0.4), (0.4, 0.4) \rangle$ and $G_2 = \langle x, (0.5, 0.5), (0.4, 0.4) \rangle$ be IFS of X . Then the family $\tau = \{0_\sim, 1, G_1, G_2\}$ be IFT on X . Thus (X, τ) be an intuitionistic fuzzy noncompact space. Now, $\Psi = \{0_\sim, 1, \bar{G}_1, \bar{G}_2\}$ is the collection of all IFS which are both intuitionistic fuzzy closed set and intuitionistic fuzzy compact set in X . Consider $U^+ = \{\bar{G}_1, \bar{G}_2, 1_\sim\}$ and $V^- = \{0_\sim, \bar{G}_1, \bar{G}_2, 1_\sim\}$. Then $\zeta = \{0_\sim, 1_\sim, \bar{G}_1, \bar{G}_2\}$ is an intuitionistic fuzzy ζ -structure. Thus (X, ζ) is an intuitionistic fuzzy ζ -space.

Definition 3.4. Let (X, ζ) be an intuitionistic fuzzy ζ -fuzzy space. For an IFS A on X , the intuitionistic fuzzy ζ -closure of A and the intuitionistic fuzzy ζ -interior of A are defined by $IF\zeta cl(A) = \cap\{B : B = \langle x, \mu_B, \nu_B \rangle \text{ is an } IF\zeta CS \text{ in } X \text{ and } A \subseteq B\}$, $IF\zeta int(A) = \cup\{B : B = \langle x, \mu_B, \nu_B \rangle \text{ is an } IF\zeta OS \text{ in } X \text{ and } B \subseteq A\}$.

Remark 3.5. Let (X, ζ) be an intuitionistic fuzzy ζ -space. Then for any IFS A in X ,

$$(i) IF\zeta int(A) \subseteq A \subseteq IF\zeta cl(A)$$

$$(ii) IF\zeta cl(\bar{A}) = \overline{IF\zeta int(A)}, IFint(\bar{A}) = \overline{IF\zeta cl(A)}.$$

Definition 3.6. Let (X, ζ) be an intuitionistic fuzzy ζ -space. An IFS A is said to be an intuitionistic fuzzy ζG_δ set (in short, $IF\zeta G_\delta S$) if $A = \cap_{i=1}^{\infty} A_i$, where each $A_i \in \zeta$. The complement of intuitionistic fuzzy ζG_δ set is said to be an intuitionistic fuzzy ζF_δ set (in short, $IF\zeta F_\delta S$) set.

Notation 3.7. Let (X, ζ) be an intuitionistic fuzzy ζ -space.

(i) An intuitionistic fuzzy ζ open F_δ (in short, $IF\zeta OF_\delta$) set is an IFS which is both $IF\zeta OS$ and $IF\zeta F_\delta$.

(ii) An intuitionistic fuzzy ζ closed G_δ (in short, $IF\zeta CG_\delta$) set is an IFS which is both $IF\zeta CS$ and $IF\zeta G_\delta$.

(iii) An intuitionistic fuzzy ζ closed open $G_\delta F_\sigma$ (in short, $IF\zeta COGF$) set is an intuitionistic fuzzy set which is both $IF\zeta OF_\sigma$ and $IF\zeta CG_\sigma$.

Definition 3.8. Let (X, ζ) be an intuitionistic fuzzy ζ -space. Then (X, τ) is said to be an intuitionistic fuzzy ζ -basically disconnected space, if the intuitionistic fuzzy ζ closure of every $IF\zeta OF_\sigma$ set is an $IF\zeta OS$.

Example 3.9. In the above example 3.3, (X, ζ) is intuitionistic fuzzy ζ -basically disconnected space.

Theorem 3.10. Let (X, ζ) be an intuitionistic fuzzy ζ -space. Then the following statements are equivalent.

- (i) (X, ζ) is an intuitionistic fuzzy ζ -basically disconnected space.
- (ii) For each $IF\zeta CG_\delta$ set A , we have $IF\zeta int(A)$ is an $IF\zeta CS$.
- (iii) For each $IF\zeta OF_\sigma$ set A , we have $IF\zeta cl(IF\zeta int(\bar{A})) = \overline{IF\zeta cl(A)}$.
- (iv) For an $IF\zeta OF_\sigma$ set A and for any IFS B with $IF\zeta cl(A) = \bar{B}$, we have $IF\zeta cl(B) = \overline{IF\zeta cl(A)}$.

Proof. (i) \Rightarrow (ii). Let A be an $IF\zeta CG_\delta$ set. Then \bar{A} is an $IF\zeta OF_\sigma$ set. By assumption (i) $IF\zeta cl(\bar{A})$ is an $IF\zeta OS$. Now, $IF\zeta cl(\bar{A}) = \overline{IF\zeta int(A)}$. Hence $IF\zeta int(A)$ is an $IF\zeta CS$.

(ii) \Rightarrow (iii). Let A be an $IF\zeta OF_\sigma$ set. Then \bar{A} is an $IF\zeta CG_\delta$ set. By assumption (ii) $IF\zeta int(\bar{A})$ is an $IF\zeta CS$. Consider $IF\zeta cl(IF\zeta int(\bar{A})) = IF\zeta int(\bar{A}) = \overline{IF\zeta cl(A)}$.

(iii) \Rightarrow (iv). Let A be an $IF\zeta OF_\sigma$ set and for any IFS B such that $IF\zeta cl(A) = \bar{B}$. By (iii), $IF\zeta cl(IF\zeta int(\bar{A})) = \overline{IF\zeta cl(A)} = IF\zeta int(\bar{A})$. That is, $IF\zeta cl(B) = IF\zeta int(\bar{A})$.

(iv) \Rightarrow (i). Let A be any $IF\zeta OF_\sigma$ set. Let $\overline{IF\zeta cl(A)} = B$. By (iv), it follows that $IF\zeta cl(B) = \overline{IF\zeta cl(A)}$. That is, $\overline{IF\zeta cl(A)}$ is an $IF\zeta CS$. This implies that $IF\zeta cl(A)$ is an $IF\zeta OS$. Hence (X, ζ) is intuitionistic fuzzy ζ -basically disconnected space. \square

Theorem 3.11. Let (X, ζ) be an intuitionistic fuzzy ζ space. Then (X, ζ) is intuitionistic fuzzy ζ -basically disconnected space if and only if for each $IF\zeta OF_\sigma$ set A and $IF\zeta CG_\delta$ set B such that $A \subseteq B$, $IF\zeta cl(A) \subseteq IF\zeta int(B)$.

Proof. Let A be an $IF\zeta OF_\sigma$ set and B be $IF\zeta CG_\delta$ set such that $A \subseteq B$. Then by (ii) of Theorem 3.10, $IF\zeta int(B)$ is an $IF\zeta CS$. Also, since A is an $IF\zeta OF_\sigma$ set, $IF\zeta cl(A) \subseteq IF\zeta int(B)$.

Conversely, Let B be any $IF\zeta CG_\delta$ set. Then $IF\zeta int(B)$ is an $IF\zeta OS$ and $IF\zeta int(B) \subseteq B$. By assumption, $IF\zeta cl(IF\zeta int(B)) \subseteq IF\zeta int(B)$. Also we know that $IF\zeta cl(IF\zeta int(B)) \subseteq IF\zeta int(B)$. This implies that $IF\zeta cl(IF\zeta int(B)) = IF\zeta int(B)$. Therefore, $IF\zeta int(B)$ is an $IF\zeta CS$. Hence by (ii) of Theorem 3.10, it follows that (X, ζ) is intuitionistic fuzzy ζ -basically disconnected space. \square

Remark 3.12. Let (X, ζ) be an intuitionistic fuzzy ζ -basically disconnected space. Let $\{A_i, \bar{B}_i / i \in N\}$ be collection such that A_i 's are $IF\zeta OF_\sigma$ and B_i 's are $IF\zeta CG_\sigma$ sets. If $A_i \subseteq A \subseteq B_j$ and $A_i \subseteq A \subseteq B_j$ for all $i, j \in N$, then there exists an $IF\zeta COGF$ set C such that $IF\zeta cl(A_i) \subseteq C \subseteq IF\zeta int(B_j)$ for all $i, j \in N$.

Proof. By Theorem 3.11, $IF\zeta cl(A_i) \subseteq IF\zeta cl(A) \cap IF\zeta int(B) \subseteq IF\zeta int(B_j)$ for all $i, j \in N$. Letting $C = IF\zeta cl(A) \cap IF\zeta int(B)$ in the above, we have C is an $IF\zeta COGF$ set satisfying the required conditions. \square

Theorem 3.13. Let (X, ζ) be an intuitionistic fuzzy ζ -basically disconnected space. Let $\{A_q\}_{q \in Q}$ and $\{B_q\}_{q \in Q}$ be monotone increasing collections of $IF\zeta OF_\sigma$ an sets and $IF\zeta CG_\delta$ of (X, τ) . Suppose that $A_{q_1} \subseteq B_{q_2}$ whenever $q_1 < q_2$ (Q is the set of all rational numbers). Then there exists a monotone increasing collection $\{C_q\}_{q \in Q}$ of an $IF\zeta COGF$ sets of (X, τ) such that $IF\zeta cl(A_{q_1}) \subseteq C_{q_2}$ and $C_{q_1} \subseteq IF\zeta int(B_{q_2})$ whenever $q_1 < q_2$.

Proof. Let us arrange all rational numbers into a sequence $\{q_n\}$ (without repetitions). For every $n \geq 2$, we shall define inductively a collection $\{C_{q_i} / 1 \leq i < n\} \subset \Omega^X$ such that $IF\zeta cl(A_{q_i}) \subseteq C_{q_i}$, $C_{q_i} \subseteq IF\zeta int(B_{q_i})$ if $q_i < q$, for all $i < n$. By Theorem 3.11 the countable collections $\{IF\zeta cl(A_{q_i})\}$ and $\{IF\zeta int(B_{q_i})\}$ satisfy $IF\zeta cl(A_{q_1}) \subseteq IF\zeta int(B_{q_2})$ if $q_1 < q_2$. By Remark 3.12, there exists an $IF\zeta COGF$ set D_1 such that $IF\zeta cl(A_{q_1}) \subseteq D_1 \subseteq IF\zeta int(B_{q_2})$.

Letting $C_{q_1} = D_1$, we get (S_2) . Assume that IFSs C_{q_i} , are already defined for $i < n$ and satisfy (S_n) . Define $E = \cup\{C_{q_i}/i < n, q_i < q_n\} \cup A_{q_n}$ and $F = \cap\{C_{q_j}/j < n, q_j < q_n\} \cup B_{q_n}$. Then $IF\zeta cl(C_{q_i}) \subseteq IF\zeta cl(E) \subseteq IF\zeta int(C_{q_j})$ and $IF\zeta cl(C_{q_i}) \subseteq IF\zeta int(F) \subseteq IF\zeta int(C_{q_j})$ whenever $q_i < q_n < q_j (i, j < n)$, as well as $A_q \subseteq IF\zeta cl(E) \subseteq B'_q$ and $A_q \subseteq IF\zeta int(F) \subseteq B'_q$ whenever $q < q_n < q'$. This shows that the countable collections $\{C_{q_i}/i < n, q_i < q_n\} \cup A_q/q < q_n\}$ and $\{C_{q_j}/j < n, q_j < q_n\} \cup \{B_q/q > q_n\}$ together with E and F fulfil the conditions of Remark 3.12. Hence, there exists an $IF\zeta COGF$ set D_n such that if $IF\zeta cl(D_n) \subseteq B_q, q_n < q, A_q \subseteq IF\zeta int(D_n), q < q_n, IF\zeta cl(C_{q_i}) \subseteq IF\zeta int(D_n)$ if $q_i < q_n, IF\zeta cl(D_n) \subseteq IF\zeta int(C_{q_i})$ if $q_n < q_j$ where $1 \leq i, j \leq n - 1$. Letting $C_{q_n} = D_n$ we obtain IFSs $C_{q_1}, C_{q_2}, \dots, C_{q_n}$ that satisfy (S_{n+1}) . Therefore, the collection $\{C_{q_i}/i = 1, 2, \dots\}$ has the required property. \square

4. Tietzes Extension Theorem for an Intuitionistic Fuzzy Basically Disconnected Space

Notation 4.1. The family of all IFSs in \mathfrak{R} is denoted by $\zeta_{\mathfrak{R}}$.

Definition 4.2. An intuitionistic fuzzy real line is the set of all monotone decreasing IFS $A \in \zeta_{\mathfrak{R}}$ satisfying $\cup\{A(t) : t \in \mathfrak{R}\} = 1_{\sim}$ and $\cap\{A(t) : t \in \mathfrak{R}\} = 0_{\sim}$ after the identification of an IFSs $A, B \in \mathfrak{R}_I(I)$ if and only if $A(t-) = B(t-)$ and $A(t+) = B(t+)$ for all $t \in \mathfrak{R}$ where $A(t-) = \cap\{A(s) : s < t\}$ and $A(t+) = \cup\{A(s) : s > t\}$. The intuitionistic fuzzy unit interval $I(I)$ is a subset of $\mathfrak{R}(I)$ such that $[A] \in I(I)$ if the membership and nonmembership of an IFS line $\mathfrak{R}(I)A \in \zeta_{\mathfrak{R}}$ are

$$\text{defined by } \mu_A(t) = \begin{cases} 1, & t < 0; \\ 0, & t > 1. \end{cases} \text{ and } \nu_A(t) = \begin{cases} 0, & t < 0; \\ 1, & t > 1. \end{cases} \text{ respectively.}$$

The natural intuitionistic fuzzy topology on $\mathfrak{R}(I)$ is generated from the subbasis $\{L_t, R_t : s < t\}$ where $L_t, R_t : \mathfrak{R}(I) \rightarrow I(I)$ are given by $L_t[A] = \overline{A(t-)}$ and $R_t[A] = A(t+)$ respectively.

Definition 4.3. Let (X, ζ) be intuitionistic fuzzy ζ -space. A function $f : X \rightarrow \mathfrak{R}(I)$ is said to be lower (resp. upper) intuitionistic fuzzy ζ -continuous function if $f^{-1}(\mathfrak{R}_t)(f^{-1}(L_t))$ is an $IF\zeta OF_{\sigma}$ set, for each $t \in \mathfrak{R}$.

Notation 4.4. Let X be any nonempty set and $A \in \zeta^X$. Then for $x \in X, < \mu_A(x), \nu_A(x) >$ is denoted by A^{\sim} .

Definition 4.5. Let X be any nonempty set. An intuitionistic fuzzy characteristic function of an IFS $A \in \zeta^X$ is a map $\psi_A : X \rightarrow I(I)$ is defined by $\psi_A(x) = A^{\sim}$, for each $x \in X, t \in \mathfrak{R}$.

Theorem 4.6. Let (X, τ) be intuitionistic fuzzy ζ space and let A be an IFS in X . Let $f : X \rightarrow \mathfrak{R}(I)$ be such that

$$f(x)(t) = \begin{cases} 1^{\sim}, & t < 0 \\ A^{\sim}, & 0 \leq t \leq 1 \\ 0^{\sim}, & t > 1 \end{cases} \text{ for all } x \in X \text{ and } t \in \mathfrak{R}. \text{ Then } f \text{ is an lower (resp. upper) intuitionistic fuzzy } \zeta\text{-continuous function if and only if } A \text{ is } IF\zeta OF_{\sigma} \text{ set.}$$

Proof. $f^{-1}(\mathfrak{R}_t) = \begin{cases} 1^{\sim}, & t < 0 \\ A^{\sim}, & 0 \leq t \leq 1 \\ 0^{\sim}, & t > 1 \end{cases}$ implies that f is an lower intuitionistic fuzzy ζ -continuous function if and only if A

is an $IF\zeta OF_{\sigma}$ set.

$f^{-1}(\overline{L_t}) = \begin{cases} 1^{\sim}, & t < 0 \\ A^{\sim}, & 0 \leq t \leq 1 \\ 0^{\sim}, & t > 1 \end{cases}$ implies that f is an upper intuitionistic fuzzy ζ -continuous function if and only if A is an

$IF\zeta CG_{\delta}$ set. Hence the proof is complete. \square

Remark 4.7. Let (X, ζ) be intuitionistic fuzzy ζ -space. Let ψ_A be an intuitionistic fuzzy characteristic function of an IFS A in X . Then ψ_A is a lower (resp. upper) intuitionistic fuzzy ζ continuous function if and only if A is an $IF\zeta OF_{\sigma}$ set.

Definition 4.8. Let (X, ζ) be intuitionistic fuzzy ζ -space. A function $f : X \rightarrow \mathfrak{R}(I)$ is said to be a strongly intuitionistic fuzzy ζ -continuous function if $f^{-1}(\mathfrak{R}_t)$ is an $IF\zeta OF_\sigma$ and $f^{-1}(\overline{L}_t)$ is both $IF\zeta OF_\sigma$ and $IF\zeta CG_\delta$ for each $t \in \mathfrak{R}$.

Notation 4.9. The collection of all strongly intuitionistic fuzzy ζ -continuous functions in an intuitionistic fuzzy ζ space (X, ζ) with values in $I(I)\zeta_s$.

Theorem 4.10 (Intuitionistic Fuzzy ζ -insertion Theorem). Let (X, ζ) be an intuitionistic fuzzy ζ -space. Then the following statements are equivalent.

- (i) (X, ζ) is an intuitionistic fuzzy ζ -basically disconnected space.
- (ii) If $g, h : X \rightarrow R(I)$, g is lower intuitionistic fuzzy ζ -continuous function, h is upper intuitionistic fuzzy ζ -continuous function and $g \subseteq h$, then there exists an $f \in \zeta_s$ such that $g \subseteq f \subseteq h$.
- (iii) If A and B are $IF\zeta OF_\sigma$ sets such that $B \subseteq A$, then there exists strongly intuitionistic fuzzy ζ -continuous function $f : X \rightarrow \mathfrak{R}(I)$ such that $B \subseteq f^{-1}(\overline{L}_1) \subseteq f^{-1}(R_0) \subseteq A$.

Proof. (i) \Rightarrow (ii) Define $A_r = h^{-1}(L_r)$ and $B_r = g^{-1}(\overline{R}_r)$, for all $r \in Q$ (Q is the set of all rationals). Clearly, $\{A_r\}_{r \in Q}$ and $\{B_r\}_{r \in Q}$ are monotone increasing families of an B sets and $IF\zeta CG_\delta$ sets of (X, ζ) . Moreover $A_r \subseteq B_s$ if $r < s$. By Theorem 3.13, there exists a monotone increasing family $\{C_r\}_{r \in Q}$ of an $IF\zeta COGF$ sets of such (X, ζ) that $IF\zeta cl(A_r) \subseteq C_s$ and $C_r \subseteq IF\zeta int(B_s)$ whenever $r < s$ ($r, s \in Q$). Letting $V_t = \bigcap_{r < t} \overline{C}_r$ for $t \in \mathfrak{R}$, we define a monotone decreasing family $\{V_t/t \in \mathfrak{R}\} \subseteq \zeta_x$. Moreover we have $IF\zeta cl(V_t) \subseteq IF\zeta int(V_s)$ whenever $s < t$. We have,

$$\begin{aligned} \bigcup_{t \in \mathfrak{R}} V_t &= \bigcup_{t \in \mathfrak{R}} \bigcap_{r < t} \overline{C}_r \supseteq \bigcup_{t \in \mathfrak{R}} \bigcap_{r < t} \overline{B}_r = \bigcup_{t \in \mathfrak{R}} \bigcap_{r < t} g^{-1}(R_r) \\ &= \bigcup_{t \in \mathfrak{R}} g^{-1}(\overline{L}_t) = g^{-1}\left(\bigcup_{t \in \mathfrak{R}} \overline{L}_t\right) = 1_{\sim} \end{aligned}$$

Similarly, $\bigcap_{t \in \mathfrak{R}} V_t = 0_{\sim}$. Now define a function $f : X \rightarrow \mathfrak{R}(I)$ possessing required conditions. Let $f(x)(t) = V_t(x)$, for all $x \in X$ and $t \in \mathfrak{R}$. By the above discussion, it follows that f is well defined. To prove f is a strongly intuitionistic fuzzy ζ -continuous function. Observe that $\bigcup_{s > t} V_s = \bigcup_{s > t} IF\zeta int(V_s)$ and $\bigcap_{s < t} V_s = \bigcap_{s < t} IF\zeta cl(V_s)$. Then $f^{-1}(R_t) = \bigcup_{s > t} V_s = \bigcup_{s > t} IF\zeta int(V_s)$ is an $IF\zeta COGF$ and $f^{-1}(\overline{L}_t) = \bigcap_{s < t} V_s = \bigcap_{s < t} IF\zeta cl(V_s)$ is an $IF\zeta COGF$ set. Therefore, f is strongly intuitionistic fuzzy ζ -continuous function. To conclude the proof it remains to show that $g \subseteq f \subseteq h$. That is $g^{-1}(\overline{L}_t) \subseteq f^{-1}(\overline{L}_t) \subseteq h^{-1}(\overline{L}_t)$ and $g^{-1}(R_t) \subseteq f^{-1}(R_t) \subseteq h^{-1}(R_t)$ for each $t \in \mathfrak{R}$. We have,

$$\begin{aligned} g^{-1}(\overline{L}_t) &= \bigcap_{s < t} g^{-1}(\overline{L}_s) = \bigcap_{s < t} \bigcap_{r < s} g^{-1}(R_r) \\ &= \bigcap_{s < t} \bigcap_{r < s} \overline{B}_r \subseteq \bigcap_{s < t} \bigcap_{r < s} \overline{C}_r = \bigcap_{s < t} V_s = f^{-1}(\overline{L}_t). \end{aligned}$$

And

$$\begin{aligned} f^{-1}(\overline{L}_t) &= \bigcap_{s < t} V_s = \bigcap_{s < t} \bigcap_{r < s} \overline{C}_r \subseteq \bigcap_{s < t} \bigcap_{r < s} \overline{A}_r \\ &= \bigcap_{s < t} \bigcap_{r < s} h^{-1}(\overline{L}_r) = \bigcap_{s < t} h^{-1}(\overline{L}_s) = h^{-1}(\overline{L}_t) \end{aligned}$$

Similarly,

$$\begin{aligned} g^{-1}(R_t) &= \bigcup_{s > t} g^{-1}(R_s) = \bigcup_{s > t} \bigcup_{r < s} g^{-1}(R_r) \\ &= \bigcup_{s > t} \bigcup_{r < s} \overline{B}_r \subseteq \bigcup_{s > t} \bigcap_{r < s} \overline{C}_r = \bigcap_{s > t} V_s = f^{-1}(R_t) \end{aligned}$$

And

$$\begin{aligned} f^{-1}(R_t) &= \bigcup_{s>t} V_s = \bigcup_{s>t} \bigcap_{r<s} \overline{C_r} \subseteq \bigcup_{s>t} \bigcup_{r>s} \overline{A_r} \\ &= \bigcup_{s>t} \bigcup_{r>s} h^{-1}(\overline{L_r}) = \bigcup_{s,t} h^{-1}(R_s) = h^{-1}(R_t) \end{aligned}$$

Hence the condition (ii) is proved.

(ii) \Rightarrow (iii) Let \bar{A} be an $IF\zeta OF_\sigma$ set and B be an $IF\zeta CG_\delta$ set such that $B \subseteq A$. Then $\psi_B \subseteq \psi_A$, where ψ_A, ψ_B are lower and upper intuitionistic fuzzy ζ -continuous functions respectively. By (ii), there exists a strongly intuitionistic fuzzy ζ -continuous function $f : X \rightarrow I(I)$ such that $\psi_B \subseteq f \subseteq \psi_A$. Clearly $f(x) \in I(I)$ for all $x \in X$ and $B = \psi_B^{-1}(\overline{L_1}) \subseteq f^{-1}(\overline{L_1}) \subseteq f^{-1}(R_0) \subseteq \psi_A^{-1}(R_0) = A$. Therefore, $B \subseteq f^{-1}(\overline{L_1}) \subseteq f^{-1}(R_0) \subseteq \psi_A^{-1}(R_0) = A$.

(iii) \Rightarrow (i). Since $f^{-1}(\overline{L_1})$ and $f^{-1}(R_0)$ are $IF\zeta COGF$ sets and by Theorem 3.11, (X, τ) is an intuitionistic fuzzy ζ basically disconnected space. □

Notation 4.11. Let X be any nonempty set. Let $A \subset X$. Then an IFS ψ_A^* is of the form $\langle x, \psi_A(x), 1 - \psi_A(x) \rangle$.

Theorem 4.12 (Tietzes Extension Theorem). Let (X, ζ) is an intuitionistic fuzzy ζ -basically disconnected space. Let $A \subset X$ such that ψ_A^* is an $IF\zeta OF_\sigma$ set in X . Let $f : (A, \zeta/A) \rightarrow I(I)$ be a strongly intuitionistic fuzzy ζ -continuous function. Then f has strongly intuitionistic fuzzy ζ -continuous extension over (X, τ) .

Proof. Let $g, h : X \rightarrow I(I)$ be such that $g = f = h$ on A and $g(x) = 0_\sim, h(x) = 1_\sim$ if $x \notin A$. For every $t \in \mathfrak{R}$, we have,

$$g^{-1}(R_t) = \begin{cases} B_t \cap \psi_A^*, & t \geq 0 \\ 1_\sim, & t < 0 \end{cases}$$

where B_t is an $IF\zeta COGF$ set such that $B_t/A = f^{-1}(R_t)$ and

$$h^{-1}(L_t) = \begin{cases} C_t \cap \psi_A^*, & t \leq 1 \\ 1_\sim, & t > 1 \end{cases}.$$

Where C_t is an $IF\zeta COGF$ set such that $C_t/A = f^{-1}(L_t)$. Thus g is lower intuitionistic fuzzy ζ -continuous function and h is upper intuitionistic fuzzy ζ -continuous function with $g \subseteq h$. By Theorem 4.10, there is a strongly fuzzy ζ -continuous function $F : X \rightarrow I(I)$ such that $g \subseteq F \subseteq h$. Hence $F \equiv f$ on A . □

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