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Closed Form Solution for the Time-Fractional Schrödinger Equation via Laplace Transform

Research Article

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Abstract: In this work, we apply an efficient technique based on coupling of iterative method and Laplace transform method to obtain the exact solution of the linear and nonlinear time- fractional Schrödinger equations. This method is termed as iterative Laplace transform method. The time -fractional derivative is considered in the Caputo sense and the solutions are found in closed form, in terms of Mittag-Leffler functions. Several numerical examples have been considered to show the effectiveness of the proposed method and the results are compared with the exact solution depicting high accuracy and efficiency.

MSC: 26A33, 33E12, 35R11, 44A10.

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1. Introduction

Fractional differential equations are generalizations of classical differential equations of integer order. In last two decades, fractional differential equations have been the focus of many studies due to their frequent appearance in various applications in mechanics, modeling, identification, control theory, signal Processing, economics, physics, mathematical biology, visco-elasticity and other areas of science [5, 11, 14] etc. In recent years, many researchers have paid attention to study the solutions of fractional linear and nonlinear partial differential equations using various methods combined with the Laplace transform such as Laplace variational iteration method (LVIM) [1, 26], homotopy analysis transform method (HATM)[3, 6], Laplace decomposition method (LDM) [7, 15] and homotopy perturbation transform method (HPTM) [24, 25] etc. The above mentioned methods provide immediate and visible symbolic terms of numerical approximate solutions as well as of analytical solutions to both linear and nonlinear fractional differential equations.

An iterative method was introduced in 2006 by Daftardar-Gejji and Jafari to solve numerically the nonlinear functional equations [7, 22]. By now, the iterative method has been used to solve many non-linear differential equations of integer and fractional order [19] and fractional boundary value problem [23].

In recent, Jafari et al. [9] developed the iterative Laplace transform method for searching numerical solutions of a system of fractional partial differential equations. The iterative Laplace transform method (ILTM) was successfully applied to solve fractional Fokker-Planck equations [13] and fractional Heat and Wave- Like equations [20].

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In the present paper, we consider the linear time-fractional Schrdinger equations of the form:

$$D_t^\alpha u(x,t) + i u_{xx}(x,t) = 0, \quad u(x,t) = g(x), \quad i = \sqrt{-1}, \tag{1}$$

and the nonlinear time- fractional Schrdinger partial differential equations are

$$iD_t^\alpha u(x,t) + u_{xx}(x,t) \pm \lambda \left|u(x,t)\right|^2 u(x,t) = 0, \quad u(x,t) = g(x), \quad i = \sqrt{-1} \tag{2}$$

and

$$iD_t^\alpha u(x,t) + u_{xx}(x,t) \pm \lambda \left|u(x,t)\right|^{2r} u(x,t) = 0, \;\; u(x,t) = g(x), \;\; r \geq 1, \;\; i = \sqrt{-1}\,, \;\; 0 \leq \lambda \in R \tag{3}$$

(where $0 < \alpha \leq 1$) with a cubic and power law nonlinearities respectively.

2. Preliminaries

In this section, we give some basic definitions and properties of fractional calculus and Laplace transform theory, which shall be used in this paper:

Definition 2.1. *The Caputo fractional derivative [10, 12] of function $u(x,t)$ is defined as*

$$\begin{aligned} D_t^\alpha u(x,t) &= \frac{1}{\Gamma(m-\alpha)} \int\limits_0^t (t-\eta)^{m-\alpha-1} u^{(m)}(x,\eta) d\eta, \;\; m-1 < \alpha \leq m, \;\; m \in N, \\ &= J_t^{m-\alpha} D^m u\left(x,t\right). \end{aligned} \tag{4}$$

Here $D^m \equiv \frac{d^m}{dt^m}$ and J_t^α stands for the Riemann-Liouville fractional integral operator of order $\alpha > 0$ [12] defined as

$$J_t^\alpha u\left(x,t\right) = \frac{1}{\Gamma\left(\alpha\right)} \int\limits_0^t \left(t-\eta\right)^{\alpha-1} u\left(x,\eta\right) d\eta, \;\; \eta > 0, \left(m-1 < \alpha \leq m\right), m \in N. \tag{5}$$

Definition 2.2. *The Laplace transform of a function $f(t)$, $t > 0$ is defined as [10, 12]*

$$L[f(t)] = F(t) = \int\limits_0^\infty e^{-st} f(t) dt. \tag{6}$$

Definition 2.3. *The Laplace transform of $D_t^\alpha u(x,t)$ is given as [10, 12]*

$$L[D_t^\alpha u(x,t)] = L[u(x,t)] - \sum_{k=o}^{m-1} u^k(x,0)\, s^{\alpha-k-1}, \quad m-1 < \alpha \leq m, m \in N, \tag{7}$$

Where $u^k(x,0)$ is the k-order derivative of $u(x,t)$ at $t = 0$.

Definition 2.4. *The Mittag-Leffler function which is a generalization of exponential function is defined as [12];*

$$E_\alpha\left(z\right) = \sum_{n=0}^\infty \frac{z^n}{\Gamma\left(\alpha n + 1\right)} \;\;, \quad \left(\alpha \in C, \;\; \mathrm{Re}(\alpha) > 0\right). \tag{8}$$

a further generalization of (8) is given in the form [4]:

$$E_{\alpha,\beta}\left(z\right) = \sum_{n=0}^\infty \frac{z^n}{\Gamma\left(\alpha n + \beta\right)}; \quad \left(\alpha, \beta \in C, R\left(\alpha\right) > 0, R\left(\beta\right) > 0\right). \tag{9}$$

3. Basic Idea of Iterative Laplace Transform Method

To illustrate the basic idea of this method, we consider the following fractional nonlinear nonhomogeneous partial differential equation with the prescribed initial conditions written in an operator form as:

$$D_t^{\alpha}\, u(x,t) + R\, u(x,t) + N\, u(x,t) = g(x,t), \qquad m-1 < \alpha \leq m, \quad m \in N, \tag{10}$$

$$u^k(x,0) = h_k(x), \quad k = 0,1,2,\ldots,m-1 \tag{11}$$

Where $D_t^{\alpha} u(x,t)$ is Caputo fractional derivative of order α; $m-1 < \alpha \leq m$, defined by Equation (4), R is a linear operator which might include other fractional derivatives order less than α, N is a non-linear operator which might include other fractional derivatives of order less than α and $g(x,t)$ is the source term. Applying the Laplace transform (denoted by throughout the present paper) on both sides of Equation (10), we get

$$L\left[D_t^{\alpha}\, u(x,t)\right] + L\left[R\, u(x,t) + N u(x,t)\right] = L\left[g(x,t)\right]. \tag{12}$$

Using the differentiation property of the Laplace transform, we have

$$L\left[u(x,t)\right] = \frac{1}{s^{\alpha}} \sum_{k=0}^{m-1} s^{\alpha-1-k} u^k(x,0) + \frac{1}{s^{\alpha}} L\left[g(x,t)\right] - \frac{1}{s^{\alpha}} L\left[R\, u(x,t) + N\, u(x,t)\right]. \tag{13}$$

Taking inverse Laplace transform of Equation (13) implies

$$u(x,t) = L^{-1}\left[\frac{1}{s^{\alpha}}\left(\sum_{k=0}^{m-1} s^{\alpha-1-k} u^k(x,0) + L\left[g(x,t)\right]\right)\right] - L^{-1}\left[\frac{1}{s^{\alpha}} L\left[R u(x,t) + N u(x,t)\right]\right], \tag{14}$$

Now we apply the Iterative method,

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) \tag{15}$$

Since R is a linear operator,

$$R\left(\sum_{i=0}^{\infty} u_i(x,t)\right) = \sum_{i=0}^{\infty} R(u_i(x,t) \tag{16}$$

and the nonlinear operator N is decomposed as

$$N\left(\sum_{i=0}^{\infty} u_i(x,t)\right) = N(u_0(x,t)) + \sum_{i=1}^{\infty}\left\{N(\sum_{k=0}^{i} u_k(x,t)) - N(\sum_{k=0}^{i-1} u_k(x,t))\right\} \tag{17}$$

Substituting (15), (16) and (17) in (14), we get

$$\begin{aligned}\sum_{i=0}^{\infty} u_i(x,t) = {} & L^{-1}\left[\tfrac{1}{s^{\alpha}}\left(\sum_{k=0}^{m-1} s^{\alpha-1-k} u^k(x,0) + L\left[g(x,t)\right]\right)\right] - \\ & L^{-1}\left[\tfrac{1}{s^{\alpha}} L\left[\sum_{i=0}^{\infty} R\left(u_i(x,t)\right) + N\left(u_0(x,t)\right) + \sum_{i=1}^{\infty}\left\{N(\sum_{k=0}^{i} u_k(x,t)) - N(\sum_{k=0}^{i-1} u_k(x,t))\right\}\right]\right],\end{aligned} \tag{18}$$

We define the recurrence relations as

$$\begin{aligned} u_0(x,t) &= L^{-1}\left[\frac{1}{s^{\alpha}}\left(\sum_{k=0}^{m-1} s^{\alpha-1-k} u^k(x,0) + L\left[g(x,t)\right]\right)\right] \\ u_1(x,t) &= -L^{-1}\left[\frac{1}{s^{\alpha}} L\ \left[R\ \left(u_0(x,t)\right) + N\left(u_0(x,t)\right)\right]\right] \\ u_{m+1}(x,t) &= -L^{-1}\left[\frac{1}{s^{\alpha}} L\ \left[R\ \left(u_m(x,t)\right) - \left\{N(\sum_{k=0}^{m} u_k(x,t)) - N(\sum_{k=0}^{m-1} u_k(x,t))\right\}\right]\right], \quad m \geq 1 \end{aligned} \tag{19}$$

Therefore the m-term approximate solution of (10)-(11) in series form is given by

$$u(x,t) \cong u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots + u_m(x,t), \qquad m = 1,2,\ldots \tag{20}$$

4. Numerical Examples

In this section, the time-fractional Schrödinger equations are solved by iterative Laplace transform method (ILTM).

Example 4.1. *We consider the following linear time- fractional schrdinger equation:*

$$D_t^{\alpha} u + i u_{xx} = 0, \qquad 0 < \alpha \le 1, \tag{21}$$

with the initial condition

$$u(x,0) = \mathrm{e}^{3ix}. \tag{22}$$

Applying the Laplace transform in Equation (21) and making use of (22), we get

$$L\left[u(x,t)\right] = \frac{\mathrm{e}^{3ix}}{s} - \frac{1}{s^{\alpha}} L\left[i u_{xx}\right], \tag{23}$$

Taking inverse Laplace transform of Equation (23) implies

$$u(x,t) = \mathrm{e}^{3ix} - L^{-1}\left[\frac{1}{s^{\alpha}} L\left[i u_{xx}\right]\right] \tag{24}$$

Now, applying the Iterative method. Substituting (15)-(17) into (23) and applying (19), we obtain the components of the solution as follows:

$$u_0(x,t) = \mathrm{e}^{3ix} \tag{25}$$

$$\begin{aligned} u_1(x,t) &= -L^{-1}\left[\frac{1}{s^{\alpha}} L\left[i \frac{\partial^2 u_0}{\partial x^2}\right]\right] \\ &= \frac{(9i)\, e^{3ix} t^{\alpha}}{\Gamma(\alpha+1)} \end{aligned} \tag{26}$$

$$\begin{aligned} u_2(x,t) &= -L^{-1}\left[\frac{1}{s^{\alpha}} L\left[i \frac{\partial^2 (u_0+u_1)}{\partial x^2}\right]\right] + L^{-1}\left[\frac{1}{s^{\alpha}} L\left[i \frac{\partial^2 u_0}{\partial x^2}\right]\right] \\ &= \left[\frac{(9i)\, e^{3ix} t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(9i)^2 e^{3ix} t^{2\alpha}}{\Gamma(2\alpha+1)}\right] - \left[\frac{(9i)\, e^{3ix} t^{\alpha}}{\Gamma(\alpha+1)}\right] \\ &= \frac{(9i)^2 e^{3ix} t^{2\alpha}}{\Gamma(2\alpha+1)} \end{aligned} \tag{27}$$

and so on for other components. Therefore, the series form solution is given by

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \dots \\ &= e^{3ix}\left[1 + \frac{(9i)\, t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(9i)^2 t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots\right] = \mathrm{e}^{3ix} \sum_{n=0}^{\infty} \frac{(9it^{\alpha})^n}{\Gamma(n\alpha+1)} \end{aligned}$$

thus, the exact solution can be given as

$$= e^{3ix} E_{\alpha}[(9i)\, t^{\alpha}] \tag{28}$$

The same result was obtained by S.T.Mohyud-Din et al. [21] using MVIM, F.Saba et.al. [6] Using HTAM and A. Kamran et al. [2] using HPM. If we put $\alpha = 1$ in Equation (28), we have

$$u(x,y) = e^{3i(x+3t)} \tag{29}$$

Which is the exactly the same solution obtained by M.M.Mousa et al. [16] using HPM.

Example 4.2. *Consider the following linear time- fractional schrödinger equation:*

$$D_t^{\alpha} u + i u_{xx} = 0, \qquad 0 < \alpha \leq 1, \tag{30}$$

with the initial condition

$$u(x,0) = 1 + 2\cosh(2x). \tag{31}$$

Applying the Laplace transform in Equation (30) and making use of (31), we get

$$L\left[D_x^{\alpha} u\,(x,t)\right] = -L[i u_{xx}], \tag{32}$$

Taking inverse Laplace transform of Equation (32) implies

$$u(x,t) = 1 + 2\cosh(2x) - L^{-1}\left[\frac{1}{s^{\alpha}} L\left[i\frac{\partial^2 u}{\partial x^2}\right]\right] \tag{33}$$

Now, applying the Iterative method. Substituting (15)-(17) into (33) and applying (19), we obtain the components of the solution as follows:

$$u_0(x,t) = 1 + 2\cosh(2x). \tag{34}$$

$$\begin{aligned} u_1(x,t) &= -L^{-1}\left[\frac{1}{s^{\alpha}} L\left[i\frac{\partial^2 u_0}{\partial x^2}\right]\right] \\ &= \frac{(-4i)\,2\cosh(2x)t^{\alpha}}{\Gamma(\alpha+1)} \end{aligned} \tag{35}$$

$$\begin{aligned} u_2(x,t) &= -L^{-1}\left[\frac{1}{s^{\alpha}} L\left[i\frac{\partial^2 (u_0+u_1)}{\partial x^2}\right]\right] + L^{-1}\left[\frac{1}{s^{\alpha}} L\left[i\frac{\partial^2 u_0}{\partial x^2}\right]\right] \\ &= \left[\frac{(-4i)\,2\cosh(2x)t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(4i)^2\,2\cosh(2x)t^{2\alpha}}{\Gamma(2\alpha+1)}\right] - \left[\frac{(-4i)\,2\cosh(2x)t^{\alpha}}{\Gamma(\alpha+1)}\right] \\ &= \frac{(4i)^2\,2\cosh(2x)t^{2\alpha}}{\Gamma(2\alpha+1)} \end{aligned} \tag{36}$$

and so on for other components. Therefore, the series form solution is given by

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \dots \\ &= 1 + 2\cosh(2x)\ \left[1 + \frac{(-4i)\,t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(4i)^2\,t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots\right]\ = 1 + 2\cosh(2x)\sum_{n=0}^{\infty}\frac{(-4it^{\alpha})^n}{\Gamma(n\alpha+1)} \end{aligned}$$

thus, the exact solution can be given as

$$= 1 + 2\cosh(2x) E_{\alpha}[(-4i)\,t^{\alpha}] \tag{37}$$

The same result was obtained by S.T.Mohyud-Din et al. [21] using MVIM, F.Saba et al. [6] using HTAM and A. Kamran et al. [2] using HPM. If we put $\alpha = 1$ in Equation (37), we have

$$u(x,y) = 1 + 2\cosh(2x)e^{-4t} \tag{38}$$

Which is the exactly the same solution obtained by M.M.Mousa et al. [16] using HPM.

Example 4.3. *Consider the following nonlinear time-fractional schrödinger equation:*

$$iD_t^\alpha u + u_{xx} + 2\left|u\right|^2 u = 0, \qquad 0<\alpha\le 1, \tag{39}$$

with the initial condition

$$u(x,0) = e^{ix}. \tag{40}$$

Applying the Laplace transform in Equation (39) and making use of (40), we get

$$L\left[u(x,t)\right] = \frac{e^{ix}}{s} + \frac{i}{s^\alpha} L\left[u_{xx} + 2\left|u\right|^2 u\right], \tag{41}$$

Taking inverse Laplace transform of Equation (41) implies

$$u(x,t) = \mathrm{e}^{ix} + L^{-1}\left[\frac{i}{s^\alpha} L\left[u_{xx} + 2\left|u\right|^2 u\right]\right], \tag{42}$$

Now, applying the Iterative method. Substituting (15)-(17) into (42) and applying (19), we obtain the components of the solution as follows:

$$u_0(x,t) = e^{ix}. \tag{43}$$

$$\begin{aligned} u_1(x,t) &= L^{-1}\left[\frac{i}{s^\alpha} L\left[\frac{\partial^2 u_0}{\partial x^2} + 2\left|u_0\right|^2 u_0\right]\right], \\ &= \frac{it^\alpha e^{ix}}{\Gamma(\alpha+1)} \end{aligned} \tag{44}$$

$$\begin{aligned} u_2(x,t) &= L^{-1}\left[\frac{i}{s^\alpha} L\left[\frac{\partial^2 (u_0+u_1)}{\partial x^2} + 2\left|(u_0+u_1)\right|^2 (u_0+u_1)\right]\right] - L^{-1}\left[\frac{i}{s^\alpha} L\left[\frac{\partial^2 u_0}{\partial x^2} + 2\left|u_0\right|^2 u_0\right]\right], \\ &= \left[\frac{i^2 t^{2\alpha} e^{ix}}{\Gamma(2\alpha+1)} + \frac{it^\alpha e^{ix}}{\Gamma(\alpha+1)}\right] - \left[\frac{it^\alpha e^{ix}}{\Gamma(\alpha+1)}\right], \\ &= \frac{(it^\alpha)^2 e^{ix}}{\Gamma(2\alpha+1)} \end{aligned} \tag{45}$$

and so on for other components. Therefore, the series form solution is given by

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \dots \\ &= e^{ix}\left[1 + \frac{(it^\alpha)}{\Gamma(\alpha+1)} + \frac{(it^\alpha)^2}{\Gamma(2\alpha+1)} + \dots\right] = e^{ix}\sum_{n=0}^{\infty}\frac{(it^\alpha)^n}{\Gamma(n\alpha+1)} \end{aligned}$$

thus, the exact solution can be given as

$$= e^{ix} E_\alpha(it^\alpha) \tag{46}$$

The same result was obtained by Z.Odibat et al. [27] using GDTM, F.Saba et al. [6] using HTAM and A.Kamran et al. [2] using HPM. If we put $\alpha = 1$ in Equation (46), we have

$$u(x,y) = e^{i(x+t)} \tag{47}$$

Which is the exactly the same solution obtained by M.M.Mousa et al. [16] using HPM.

Example 4.4. *Consider the following nonlinear time- fractional schrödinger equation:*

$$iD_t^{\alpha}u + u_{xx} - 2\left|u\right|^2 u = 0, \qquad 0 < \alpha \leq 1, \tag{48}$$

with the initial condition

$$u(x,0) = e^{ix}. \tag{49}$$

Applying the Laplace transform in Equation (48) and making use of (49), we get

$$L\left[u(x,t)\right] = \frac{e^{ix}}{s} + \frac{i}{s^{\alpha}}L\left[u_{xx} - 2\left|u\right|^2 u\right], \tag{50}$$

Taking inverse Laplace transform of Equation (50) implies

$$u(x,t) = \mathrm{e}^{ix} + L^{-1}\left[\frac{i}{s^{\alpha}}L\left[u_{xx} - 2\left|u\right|^2 u\right]\right], \tag{51}$$

Now, applying the Iterative method. Substituting (15)-(17) into (51) and applying (19), we obtain the components of the solution as follows:

$$u_0(x,t) = e^{ix}. \tag{52}$$

$$\begin{aligned} u_1(x,t) &= L^{-1}\left[\frac{i}{s^{\alpha}}L\left[\frac{\partial^2 u_0}{\partial x^2} - 2\left|u_0\right|^2 u_0\right]\right], \\ &= \frac{(-3i)\,t^{\alpha}e^{ix}}{\Gamma(\alpha+1)} \end{aligned} \tag{53}$$

$$\begin{aligned} u_2(x,t) &= L^{-1}\left[\frac{i}{s^{\alpha}}L\left[\frac{\partial^2 (u_0+u_1)}{\partial x^2} - 2\left|(u_0+u_1)\right|^2 (u_0+u_1)\right]\right] - L^{-1}\left[\frac{i}{s^{\alpha}}L\left[\frac{\partial^2 u_0}{\partial x^2} - 2\left|u_0\right|^2 u_0\right]\right], \\ &= \left[\frac{(3i)^2\,t^{2\alpha}e^{ix}}{\Gamma(2\alpha+1)} + \frac{(-3i)\,t^{\alpha}e^{ix}}{\Gamma(\alpha+1)}\right] - \left[\frac{(-3i)\,t^{\alpha}e^{ix}}{\Gamma(\alpha+1)}\right], \\ &= \frac{(3it^{\alpha})^2\,e^{ix}}{\Gamma(2\alpha+1)} \end{aligned} \tag{54}$$

and so on for other components. Therefore, the series form solution is given by

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots \\ &= e^{ix}\left[1 + \frac{(-3it^{\alpha})}{\Gamma(\alpha+1)} + \frac{(3it^{\alpha})^2}{\Gamma(2\alpha+1)} + \ldots\right] = e^{ix}\sum_{n=0}^{\infty}\frac{(-3it^{\alpha})^n}{\Gamma(n\alpha+1)} \end{aligned}$$

thus, the exact solution can be given as

$$= e^{ix}E_{\alpha}\left(-3it^{\alpha}\right). \tag{55}$$

The same result was obtained by S.T.Mohyud-Din et.al. [21] using MVIM, F. Saba et.al. [6] using HTAM and A.Kamran et al. [2] using HPM. If we put $\alpha = 1$ in Equation (55) we have

$$u(x,y) = e^{i(x-3t)}, \tag{56}$$

Which is the exactly the same solution obtained by M.M.Mousa et al. [16] using HPM.

Example 4.5. *Consider the following nonlinear time- fractional schrdinger equation:*

$$iD_t^{\alpha}u + u_{xx} + 2\left|u\right|^{2r}u = 0, \qquad\qquad 0 < \alpha \leq 1, \tag{57}$$

with the initial condition

$$u(x,0) = \left(2\left(r+1\right)\sec h^2(2rx)\right)^{\frac{1}{2r}} \quad , \qquad\qquad r \geq 1. \tag{58}$$

Applying the Laplace transform in Equation (57) and making use of (58), we get

$$L\left[u(x,t)\right] = \frac{\left(2\left(r+1\right)\sec h^2(2rx)\right)^{\frac{1}{2r}}}{s} + \frac{i}{s^{\alpha}}L\left[u_{xx} + 2\left|u\right|^{2r}u\right], \tag{59}$$

Taking inverse Laplace transform of Equation (59) implies

$$u(x,t) = \left(2\left(r+1\right)\sec h^2(2rx)\right)^{\frac{1}{2r}} + L^{-1}\left[\frac{i}{s^{\alpha}}L\left[u_{xx} + 2\left|u\right|^{2r}u\right]\right], \tag{60}$$

Now, applying the Iterative method. Substituting (15)-(17) into (60) and applying (19), we obtain the components of the solution as follows:

$$u_0(x,t) = \left(2\left(r+1\right)\sec h^2(2rx)\right)^{\frac{1}{2r}}. \tag{61}$$

$$\begin{aligned} u_1(x,t) &= L^{-1}\left[\frac{i}{s^{\alpha}}L\left[\frac{\partial^2 u_0}{\partial x^2} + 2\left|u_0\right|^{2r}u_0\right]\right], \\ &= \frac{(4i)\,t^{\alpha}\left(2\left(r+1\right)\sec h^2(2rx)\right)^{\frac{1}{2r}}}{\Gamma\left(\alpha+1\right)}. \end{aligned} \tag{62}$$

$$\begin{aligned} u_2(x,t) &= L^{-1}\left[\frac{i}{s^{\alpha}}L\left[\frac{\partial^2\left(u_0+u_1\right)}{\partial x^2} + 2\left|\left(u_0+u_1\right)\right|^{2r}\left(u_0+u_1\right)\right]\right] - L^{-1}\left[\frac{i}{s^{\alpha}}L\left[\frac{\partial^2 u_0}{\partial x^2} + 2\left|u_0\right|^{2r}u_0\right]\right], \\ &= \left[\frac{16i^2t^{2\alpha}\left(2\left(r+1\right)\sec h^2(2rx)\right)^{\frac{1}{2r}}}{\Gamma\left(2\alpha+1\right)} + \frac{(4i)\,t^{\alpha}\left(2\left(r+1\right)\sec h^2(2rx)\right)^{\frac{1}{2r}}}{\Gamma\left(\alpha+1\right)}\right] - \left[\frac{(4i)\,t^{\alpha}\left(2\left(r+1\right)\sec h^2(2rx)\right)^{\frac{1}{2r}}}{\Gamma\left(\alpha+1\right)}\right], \\ &= \frac{\left(4it^{\alpha}\right)^2\left(2\left(r+1\right)\sec h^2(2rx)\right)^{\frac{1}{2r}}}{\Gamma\left(2\alpha+1\right)} \end{aligned} \tag{63}$$

and so on for other components. Therefore, the series form solution is given by

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots \\ &= \left(2\left(r+1\right)\sec h^2(2rx)\right)^{\frac{1}{2r}}\left[1 + \frac{\left(4it^{\alpha}\right)}{\Gamma\left(\alpha+1\right)} + \frac{\left(4it^{\alpha}\right)^2}{\Gamma\left(2\alpha+1\right)} + \ldots\right] = \left(2\left(r+1\right)\sec h^2(2rx)\right)^{\frac{1}{2r}}\sum_{n=0}^{\infty}\frac{\left(4it^{\alpha}\right)^n}{\Gamma(n\alpha+1)} \end{aligned}$$

thus, the exact solution can be given as

$$= \left(2\left(r+1\right)\sec h^2(2rx)\right)^{\frac{1}{2r}}E_{\alpha}\left(4it^{\alpha}\right). \tag{64}$$

The same result was obtained by F.Saba et.al. [6] Using HTAM. If we put $\alpha = 1$ in Equation (64) we have

$$u(x,y) = 2\sec h(2x)e^{4it}, \tag{65}$$

Which is the exactly the same solution obtained by M.M.Mousa et al. [16] using HPM.

5. Conclusion

The solutions of the linear and nonlinear time- fractional Schrödinger equations in terms of Mittag-Leffler functions by the use of iterative Laplace transform method were derived. The solutions are obtained in series form that rapidly converges in a closed exact formula with simply computable terms. The calculations are simple and straightforward. The method was tested on five examples on different situations.

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