International Journal of Mathematics And its Applications
Volume 3, Issue 4–D (2015), 69–74.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Bi-Univalent Coefficient Estimates for Certain Subclasses of Close-to-Convex Functions

Research Article

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Abstract: In this paper,we introduce two new subclasses of the function class Σ of Bi-univalent functions defined in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Besides, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses.

MSC: 30C45.

Keywords: Bi-univalent functions, close-to-convex,quasi convex,coefficient bounds.

1. Introduction, Definitions And Preliminaries

We let $\mathcal{A}$ to denote the class of functions analytic in $\mathbb{U}$ and having the power series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Also we let $\mathcal{S}$ to denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$. The Koebe one-quarter theorem [6] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$,$(z \in \mathbb{U})$ and

$$f\left(f^{-1}(w)\right) = w, \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right)$$

where

$$h(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). Also let function $g \in \Sigma$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{3}$$

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has the inverse function of the form

$$j\left(w\right)=g^{-1}\left(w\right)=w-b_{2}w^{2}+(2b_{2}^{2}-b_{3})w^{3}-(5b_{2}^{3}-5b_{2}b_{3}+b_{4})w^{4}+\cdots. \tag{4}$$

Earlier, Brannan and Taha [4] introduced certain subclasses of bi-univalent function class Σ, namely bi-starlike functions $\mathcal{S}_{\Sigma}^{*}(\alpha)$ and bi-convex function $\mathcal{K}_{\Sigma}(\alpha)$ of order α corresponding to the function classes $\mathcal{S}^{*}(\alpha)$ and $\mathcal{K}(\alpha)$ respectively. Lewin [8] investigated the class Σ of bi-univalent functions and showed that $|a_2|<1.51$ for the functions belonging to Σ. Subsequently, Brannan and Clunie [5] conjectured that $|a_2|\leq\sqrt{2}$.

An analytic function f is subordinate to an analytic function g,written $f(z)\prec g(z)$, provided there is a schwarz function w defined on $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$ satisfying $f\left(z\right)=g\left(w(z)\right)$. Ma and Minda [9], unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1+\frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk U, $\phi(0)=1$, $\phi'(0)>0$ and $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z)=1+B_{1}z+B_{2}z^{2}+B_{3}z^{3}+\cdots,(B_{1}>0). \tag{5}$$

In this paper, we introduce the following new subclasses of Bi-univalent close-to-convex functions of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses.

Definition 1.1. *A function $f\in\Sigma$ given by (1) is said to be in the class $CC_{\Sigma}\left(\phi\right)$ if there exists a bi-convex function $g\in CV_{\Sigma}\left(\phi\right)$ given by (3) and satisfy the following conditions:*

$$\frac{zf'(z)}{g(z)}\prec\phi\left(z\right), \tag{6}$$

and

$$\frac{wh'(w)}{j(w)}\prec\phi\left(w\right) \tag{7}$$

where $h(w)$ is given by (2) and $j(w)$ is given by (4) and $z,w\in\mathbb{U}$.

Definition 1.2. *A function $f(z)$ given by (1) is said to be in the class $QC_{\Sigma}\left(\phi\right)$, if there exists a bi-convex function $g\in CV_{\Sigma}\left(\phi\right)$ such that $f\in\Sigma$*

$$\frac{\left(zf'\left(z\right)\right)'}{g'\left(z\right)}\prec\phi\left(z\right) \tag{8}$$

and

$$\frac{\left(wh'\left(w\right)\right)'}{j'\left(w\right)}\prec\phi\left(w\right) \tag{9}$$

where $h(w)$ is given by (2) and $j(w)$ is given by (4) and $z,w\in\mathbb{U}$.

Lemma 1.3. *A function $g\in\mathcal{A}$ is said to be convex function in $\mathbb{U}$ if both $g(z)$ and $g^{-1}(z)$ are convex in $\mathbb{U}$, Nehari [10], subsequently by Koepf [11] and Ian Graham and Gabriela Kohr [12], Corollary 2.2.19 gives*

$$\left|b_{3}-b_{2}^{2}\right|\leq\frac{1}{3}. \tag{10}$$

This estimate is sharp.

2. Coefficient Estimates for the Function Class $CC_\Sigma(\phi)$

Our first result provides estimates for the coefficients a_2, a_3 for functions belonging to the class $CC_\Sigma(\phi)$.

Theorem 2.1. *If $f \in CC_\Sigma(\phi)$, then*

$$|a_2| \le B_1 + \sqrt{B_1(1+B_1) + |B_1 - B_2|} \quad \textit{and} \tag{11}$$

$$|a_3| \le \frac{2B_1(B_1+6)+1}{9} + |B_1 - B_2| + 2B_1\sqrt{B_1(1+B_1) + |B_1 - B_2|}. \tag{12}$$

Proof. Since $f \in CC_\Sigma(\phi)$, there exists two analytic functions $r, s : \mathbb{U} \to \mathbb{U}$, with $r(0) = 0 = s(0)$, such that

$$\frac{zf'(z)}{g(z)} = \phi(r(z)) \quad \text{and} \tag{13}$$

$$\frac{wh'(w)}{j(w)} = \phi(s(w)). \tag{14}$$

Define the functions p and q by

$$p(z) = \frac{1+r(z)}{1-r(z)} = 1 + p_1 z + p_2 z^2 + \cdots \quad \text{and} \quad q(z) = \frac{1+s(z)}{1-s(z)} = 1 + q_1 z + q_2 z^2 + \cdots. \tag{15}$$

Or equivalently,

$$r(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \left(p_3 + \frac{p_1}{2}\left(\frac{p_1^2}{2} - p_2\right) - \frac{p_1 p_2}{2}\right)z^3 + \cdots\right] \tag{16}$$

and

$$s(z) = \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right)z^2 + \left(q_3 + \frac{q_1}{2}\left(\frac{q_1^2}{2} - q_2\right) - \frac{q_1 q_2}{2}\right)z^3 + \cdots\right]. \tag{17}$$

It is clear that p and q are analytic in $\mathbb{U}$ and $p(0) = 1 = q(0)$. Also p and q have positive real part in $\mathbb{U}$ and hence $|p_i| \le 2$ and $|q_i| \le 2$. In the view of (13), (14) and (15), clearly,

$$\frac{zf'(z)}{g(z)} = \phi\left(\frac{p(z)-1}{p(z)+1}\right) \tag{18}$$

and

$$\frac{wh'(w)}{j(w)} = \phi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{19}$$

Using (16) and (17) together with (3), one can easily verify that

$$\phi\left(\frac{p(z)-1}{p(z)+1}\right) = 1 + \frac{B_1 p_1}{2} z + \left[\frac{B_1}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2\right]z^2 + \cdots \tag{20}$$

and

$$\phi\left(\frac{q(w)-1}{q(w)+1}\right) = 1 + \frac{B_1 q_1}{2} w + \left[\frac{B_1}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{B_2 q_1^2}{4}\right]w^2 + \cdots. \tag{21}$$

Since $f \in \Sigma$ has the Maclaurin series given by (1), computation shows that its inverse $h = f^{-1}$ and $j = g^{-1}$ has the expansion given by (2). It follows from (18), (19), (20) and (21) that

$$2a_2 - b_2 = \frac{1}{2}B_1 p_1, \tag{22}$$

$$b_2^2 - b_3 - 2a_2b_2 + 3a_3 = \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2p_1^2 \tag{23}$$

and

$$b_2 - 2a_2 = \frac{1}{2}B_1q_1, \tag{24}$$

$$b_3 - b_2^2 - 2a_2b_2 + 6a_2^2 - 3a_3 = \frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2q_1^2, \tag{25}$$

From (22) and (24), it follows that

$$p_1 = -q_1, \tag{26}$$

Now (23), (25) and using (22), (26) gives

$$\left(a_2 - \frac{B_1p_1}{2}\right)^2 = \frac{B_1^2p_1^2}{4} - \frac{B_1(p_2+q_2)}{4} + \frac{(B_1-B_2)p_1^2}{4}. \tag{27}$$

Using the fact that $|p_2| \le 2$ and $|q_2| \le 2$ gives the desired estimate on $|a_2|$,

$$|a_2| \le \left(B_1 + \sqrt{B_1(1+B_1) + |B_1 - B_2|}\right).$$

Subtracting (25) from (23), gives

$$a_3 = a_2^2 + \frac{1}{3}\left(b_3 - b_2^2\right) + \frac{B_1}{12}(p_2 - q_2) \tag{28}$$

Using the fact that $|p_2| \le 2$ and $|q_2| \le 2$ and Lemma 1.3, then which yields the estimate (12). □

3. Coefficient Estimates for the Function Class $QCV_\Sigma(\phi)$

Theorem 3.1. *If $f \in QCV_\Sigma(\phi)$, then*

$$|a_2| \le \frac{B_1}{3} + \frac{1}{6}\sqrt{7B_1^2 + 12(B_1 + |B_1 - B_2|)} \quad \textit{and} \tag{29}$$

$$|a_3| \le \frac{B_1(16 + 11B_1) + 4}{36} + \frac{|B_1 - B_2|}{3} + \frac{B_1}{9}\sqrt{7B_1^2 + 12(B_1 + |B_1 - B_2|)} \quad . \tag{30}$$

Proof. Since $f \in QCV_\Sigma(\phi)$, there exists two analytic functions $r, s : \mathbb{U} \to \mathbb{U}$, with $r(0) = 0 = s(0)$, satisfying

$$\frac{(zf'(z))'}{g'(z)} = \phi(r(z)) \quad \text{and} \tag{31}$$

$$\frac{(wh'(w))'}{j'(w)} = \phi(s(w)) \tag{32}$$

From (20) and (21), (31) and (32) it follows that

$$4a_2 - 2b_2 = \frac{1}{2}B_1p_1, \tag{33}$$

$$9a_3 - 3b_3 + 4b_2^2 - 8a_2b_2 = \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2p_1^2, \tag{34}$$

$$2b_2 - 4a_2 = \frac{1}{2}B_1q_1 \quad \text{and} \tag{35}$$

$$18a_2^2 - 9a_3 - 8a_2b_2 - 2b_2^2 + 3b_3 = \frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2q_1^2. \tag{36}$$

The equations (33) and (35) yield

$$p_1 = -q_1 \tag{37}$$

The equations (34), (36) and using (33), (37) we get

$$\left(a_2 - \frac{B_1 p_1}{6}\right)^2 = \frac{7B_1^2 p_1^2}{144} + \frac{p_1^2 (B_1 - B_2)}{12} - \frac{B_1 (p_2 + p_2)}{12} \tag{38}$$

Using the familiar inequalities $|p_i| \leq 2, |q_i| \leq 2$ and (37) gives

$$|a_2| \leq \frac{B_1}{3} + \frac{1}{6}\sqrt{7B_1^2 + 12 (B_1 + |B_1 - B_2|)}$$

Subtracting (36) from (34) using (37)

$$a_3 = \frac{B_1 (p_2 - q_2)}{36} + a_2^2 + \frac{b_3 - b_2^2}{3}. \tag{39}$$

Using the fact that $|p_i| \leq 2$, $|q_i| \leq 2$ and Lemma 1.3, in (39). which yields the estimate (30). □

4. Coefficient Bounds for the Function Class $M_\Sigma(\alpha, \phi)$

Theorem 4.1. *Let f given by (1) be in the class $M_\Sigma(\alpha, \phi)$, then*

$$|a_2| \leq \frac{B_1}{1+2\alpha} + \sqrt{\frac{(3\alpha^2 + 3\alpha + 1) B_1^2}{(1+2\alpha)^2 (1+\alpha)^2} + \frac{B_1 + |B_1 - B_2|}{(1+2\alpha)}} \quad \text{and} \tag{40}$$

$$\begin{aligned} |a_3| \leq & \frac{(4\alpha^2 + 5\alpha + 2) B_1^2}{(1+2\alpha)^2 (1+\alpha)^2} + \frac{4B_1 - 3|B_1 - B_2|}{3(1+2\alpha)} + \frac{1}{9} \\ & + \frac{2B_1}{(1+2\alpha)}\sqrt{\frac{(3\alpha^2 + 3\alpha + 1) B_1^2}{(1+2\alpha)^2 (1+\alpha)^2} + \frac{B_1 + |B_1 - B_2|}{(1+2\alpha)}}. \end{aligned} \tag{41}$$

Proof. Since $f \in M_\Sigma(\alpha, \phi)$, there exists two analytic functions $r, s : \mathbb{U} \to \mathbb{U}$, with $r(0) = 0 = s(0)$, such that

$$(1-\alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)} = \phi(r(z)) \quad \text{and} \tag{42}$$

$$(1-\alpha)\frac{wh'(w)}{j(z)} + \alpha\frac{(wh'(w))'}{j'(w)} = \phi(s(w)). \tag{43}$$

From (20) and (21), (42) and (43) it follows that

$$(1+\alpha)(2a_2 - b_2) = \frac{1}{2}B_1 p_1, \tag{44}$$

$$(1+2\alpha)(3a_3 - b_3) + (1+3\alpha)\left(b_2^2 - 2a_2 b_2\right) = \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2 p_1^2, \tag{45}$$

$$(1+\alpha)(b_2 - 2a_2) = \frac{1}{2}B_1 q_1, \tag{46}$$

and

$$\begin{aligned} &(1+2\alpha)\left(6a_2^2 - 3a_3 + b_3\right) - (1+3\alpha)\,2a_2 b_2 - (1+\alpha)\,b_2^2 \\ &= \frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2 q_1^2. \end{aligned} \tag{47}$$

The equations (44) and (46) yield

$$p_1 = -q_1 \tag{48}$$

The equations (45), (47) and using (44) (48) we get

$$\left(a_2 - \frac{B_1 p_1}{2(1+\alpha)}\right)^2 = \frac{\left(3\alpha^2+3\alpha+1\right) B_1^2 p_1^2}{4(1+2\alpha)^2 (1+\alpha)^2} - \frac{B_1 (p_2+q_2)}{4(1+2\alpha)} + \frac{p_1^2 (B_1 - B_2)}{4(1+2\alpha)} \tag{49}$$

Which yields the desired estimation of $|a_2|$ in (40). Subtracting (47) from (45) using (48)

$$a_3 = \frac{B_1 (p_2 - q_2)}{12(1+2\alpha)} + a_2^2 + \frac{b_3 - b_2^2}{3}. \tag{50}$$

Using the fact that $|p_i| \leq 2$, $|q_i| \leq 2$ and using Lemma 1.3 in (50), which yields the desired estimation of $|a_3|$.

□

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