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On Middle Neighborhood Graphs

Research Article

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Abstract: The middle neighborhood graph $M_{nd}(G)$ of a graph $G = (V, E)$ is the graph with the vertex set $V \cup S$ where S is the set of all open neighborhood sets of G in which two vertices u and v are adjacent if $u, v \in S$ and $u \cap v \neq \phi$ or $u \in V$ and v is an open neighborhood set of G containing u. In this paper, some properties of this new graph are established. Also characterizations are given for graphs (i) whose middle neighborhood graphs are connected, (ii) whose middle neighborhood graphs are Eulerian.

MSC: 05C.

Keywords: Open neighborhood set, neighborhood graph, middle neighborhood graph, Eulerian.

1. Introduction

The graphs considered here are finite, undirected without loops or multiple edges. We denote by p the number of vertices and q the number of edges of such a graph G. Any undefined term in this paper may be found in Kulli [1].

Let $G = (V, E)$ be a graph. For any vertex $u \in V$, the open neighborhood of u is the set $N(u) = \{v \in V : uv \in E\}$. We call $N(u)$ is the open neighborhood set of a vertex u of G. Let $V = \{u_1, u_2, \ldots, u_p\}$ and let $S = \{N(u_1), N(u_2), \ldots, N(u_p)\}$ be the set of all open neighborhood sets of G.

The neighborhood graph $N(G)$ of a graph G is the graph with the vertex set $V \cup S$ where S is the set of all open neighborhood sets of vertices of G and with two vertices u, v in $V \cup S$ adjacent if $u \in V$ and v is an open neighborhood set containing u. This concept was introduced by Kulli in [2]. Several other graph valued functions in graph theory were studied, for example, in [3–16] and also several graph valued functions in domination theory were studied, for example, in [17–26].

In Section 2, we establish some properties of middle neighborhood graph of a graph. Traversability of some graph valued functions was studied, for example, in [27–30]. In Section 3, we study traversability of middle neighborhood graphs.

2. Middle Neighborhood Graphs

We now introduce the concept of the middle neighborhood graph of a graph.

Definition 2.1. *Let $G = (V, E)$ be a graph. Let S be the set of all open neighborhood sets of vertices of G. The middle neighborhood graph $M_{nd}(G)$ of G is the graph with the vertex set $V \cup S$ in which two vertices u and v are adjacent if $u, v \in S$ and $u \cap v \neq \phi$ or $u \in V$ and v is an open neighborhood set of G containing u.*

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Example 2.2. *In Figure 1, a graph G and its middle neighborhood graph $M_{nd}(G)$ are shown. For the graph G in Figure 1, the open neighborhood sets of G are $N(1) = \{2,3,4\}$, $N(2) = \{1,3\}$, $N(3) = \{1,2\}$, $N(4) = \{1\}$.*

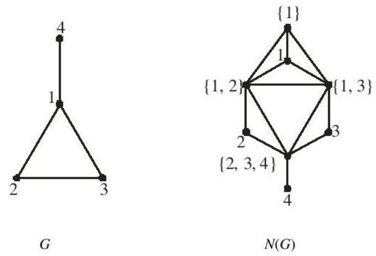

Figure 1.

Remark 2.3. *If G is a graph without isolated vertices, then G has at least two neighbourhood sets.*

Remark 2.4. *For any graph G, the neighbourhood graph N(G) of G is a spanning subgraph of $M_{nd}(G)$.*

Theorem 2.5. $M_{nd}(G) = 2pK_2$ *if and only if* $G = pK_2$, $p \geq 1$.

Proof. Suppose $G = pK_2$. Then each open neighborhood set of a vertex of G contains exactly one vertex. Thus corresponding vertex of open neighborhood set is adjacent with exactly one vertex in $M_{nd}(G)$. Since G has 2p vertices, it implies that G has 2p open neighborhood sets. Thus $M_{nd}(G)$ has 4p vertices and the degree of each vertex is one. Hence $M_{nd}(G) = 2pK_2$.

Conversely suppose $M_{nd}(G) = 2pK_2$. We now prove that $G = pK_2$. On the contrary, assume $G \neq pK_2$. Then there exists at least one open neighbourhood set containing at least two vertices of G. Then $M_{nd}(G)$ contains a subgraph P_3. Thus $M_{nd}(G) \neq 2pK_2$, which is a contradiction. Hence $G = pK_2$. □

We need the following result.

Theorem 2.6 ([2])**.** *Let G be a connected graph. The neighborhood graph N(G) of G is connected if and only if G contains an odd cycle.*

Theorem 2.7. *Let G be a connected graph. The middle neighborhood graph $M_{nd}(G)$ of G is connected if and only if G contains an odd cycle.*

Proof. Let G be a connected graph. Suppose G contains an odd cycle. By Theorem 2.6, N(G) is connected. Since by Remark **??**, N(G) is a spanning subgraph of $M_{nd}(G)$, it implies that $M_{nd}(G)$ is connected.

Conversely suppose $M_{nd}(G)$ is connected. By Remark **??**, N(G) is a spanning subgraph of $M_{nd}(G)$. Therefore N(G) is connected. Hence by Theorem 2.6, a connected graph G contains an odd cycle. □

Corollary 2.8. *For a nontrivial bipartite graph, $M_{nd}(G)$ is not connected.*

Theorem 2.9. $M_{nd}(G) = G \cup K_p$ *if and only if* $G = K_{1,p-1}$, $p \geq 2$.

Proof. Let $G = K_{1,p-1}$, $p \geq 2$. Let $V(G) = \{v_1, v_2, \ldots, v_{p-1}\}$. Let $deg\ v = p-1$ and $deg\ v_i = 1$, $1 \leq i \leq p-1$. Then $N(v) = \{v_1, v_2, \ldots, v_{p-1}\}$, $N(v_i) = \{v\}$, $1 \leq i \leq p-1$. Therefore $V(M_{nd}(G)) = \{v, v_1, v_2, \ldots, v_{p-1}, N(v), N(v_1), N(v_2), \ldots, N(v_{p-1})\}$. By Theorem 2.7, $M_{nd}(G)$ is disconnected. The vertex N(v) is adjacent with $v_1, v_2, \ldots, v_{p-1}$ and no two vertices of $v_1, v_2, \ldots, v_{p-1}$ are adjacent in $M_{nd}(G)$. This produces $K_{1,p-1}$ in $M_{nd}(G)$. Also v lies in $N(v_1), N(v_2), \ldots, N(v_{p-1})$ and $N(v_i) \cap N(v_j) \neq \phi$, $1 \leq i \leq p-1$, $1 \leq j \leq p-1$, $i \neq j$. Then the vertex v is adjacent with $N(v_1), N(v_2), \ldots, N(v_{p-1})$ and every pair of vertices of $N(v_1), N(v_2), \ldots, N(v_{p-1})$ are adjacent in $M_{nd}(G)$. This produces K_p in $M_{nd}(G)$. Thus the resulting graph is $K_{1,p-1} \cup K_p$. Hence $M_{nd}(G) = G \cup K_p$.

Conversely suppose $M_{nd}(G) = G \cup K_p$. Since $M_{nd}(G)$ is disconnected, G has no odd cycles. Suppose G has even cycles. Then any component of $M_{nd}(G)$ is not K_p, a contradiction. Thus G has no even cycles. Hence G must be a tree. We now prove that $G = K_{1,p-1}$. On the contrary, G is not a star. Then $\Delta(G) < p-1$. Therefore open neighborhood set of vertex of G contains at most $p-2$ vertices. Then in any component of $M_{nd}(G)$, the degree of any vertex is at most $p2$. Thus $M_{nd}(G)$ does not contain K_p as a component, which is a contradiction. Thus $G = K_{1,p-1}$. □

Proposition 2.10. *If v is an end vertex of G, then the corresponding vertex of v in $M_{nd}(G)$ is an end vertex.*

Proof. Let v be an end vertex of G. Then v is adjacent with exactly one vertex of G, say u. Then $N(v) = \{u\}$. Thus the corresponding vertex of v in $M_{nd}(G)$ is adjacent with exactly one vertex $N(v)$. Hence the corresponding vertex of v is an end vertex in $M_{nd}(G)$. □

Theorem 2.11. *For any graph G without isolated vertices, $N(G) \subseteq M_{nd}(G)$. Furthermore, equality holds if and only if every pair of open neighborhood sets of vertices of G are disjoint.*

Proof. By Remark 2.4,

$$N(G) \subseteq M_{nd}(G). \tag{1}$$

We now prove the second part. Suppose the equality in (1) is attained. Let $N_1, N_2, \ldots, N_p$ be the open neighborhood sets of vertices of G. By Remark 2.3, we see that $p \geq 2$. Since $N(G) = M_{nd}(G)$, it implies that no two open neighborhood sets of vertices of G have a vertex in common. Thus every pair of open neighborhood sets of vertices of G are disjoint.

Conversely suppose every pair of open neighborhood sets of vertices of G are disjoint. Then any two vertices corresponding to open neighborhood sets cannot be adjacent in $M_{nd}(G)$. Thus $M_{nd}(G) \subseteq N(G)$ and since $N(G) \subseteq M_{nd}(G)$, we see that $N(G) = M_{nd}(G)$. □

3. Traversability

Observation 3.1. *If v is a vertex of a graph G, then the degree of the corresponding vertex of v in $M_{nd}(G)$ is the same as the degree of v in G.*

Observation 3.2. *If N(v) is an open neighborhood set of v containing the vertices $u_1, u_2, \ldots, u_n$, $n \geq 1$, then the degree of the corresponding vertex of N(v) in $M_{nd}(G)$ is equal to $deg_G(u_1) + deg_G(u_2) + \cdots + deg_G(u_n)$.*

We need the following result.

Theorem 3.3. *A connected graph G is eulerian if and only if every vertex of G has even degree.*

Remark 3.4. *If G is eulerian, then $M_{nd}(G)$ need not be eulerian. For example, for the eulerian graph C_6, the middle neighborhood graph $M_{nd}(C_6)$ is disconnected, by Corollary 2.8. Thus $M_{nd}(C_6)$ is not eularian.*

We obtain a characterization of graphs whose middle neighborhood graphs are eulerian.

Theorem 3.5. *Let G be a nontrivial connected graph. The middle neighborhood graph $M_{nd}(G)$ of G is eulerian if and only if the following conditions hold:*

(i) G has an odd cycle, and

(ii) G is eulerian.

Proof. Suppose $M_{nd}(G)$ is eulerian. On the contrary, suppose condition (i) is not satisfied. Then G has only even cycles or no cycles. By Theorem 2.7, $M_{nd}(G)$ is not connected. Thus $M_{nd}(G)$ is not eulerian, which is a contradiction. This proves (i). Now suppose (ii) is not satisfied. Then G has a vertex v of odd degree. By Observation 3.1, the corresponding vertex of v in $M_{nd}(G)$ is odd. Thus $M_{nd}(G)$ is not eulerian, a contradiction. This proves (ii).

Conversely suppose the given conditions are satisfied. Suppose (i) holds. Then by Theorem 2.7, $M_{nd}(G)$ is connected. Suppose (ii) holds. By Theorem 3.3, the degree of each vertex of G is even. If v is a vertex of G, then the degree of v in G is even. By Observation 3.1, the degree of the corresponding vertex of v in $M_{nd}(G)$ is the degree of v in G, which is even. Also by Observation 3.2, if N(v) is an open neighborhood set of v in G containing vertices $u_1, u_2, \ldots, u_n$; $n \geq 1$, then the degree of the corresponding vertex of N(v) in $M_{nd}(G) = deg_G u_1 + deg_G u_2 + \cdots + deg_G u_n$.

Since $deg_G u_1, deg_G u_2, \ldots, deg_G u_n$ are even and also n is even, it implies that the degree of the corresponding vertex of N(v) is even in $M_{nd}(G)$. Since v is arbitrary, it implies that the degree of every vertex of $M_{nd}(G)$ is even. By Theorem 3.3, $M_{nd}(G)$ is eulerian. □

References

[1] V.R.Kulli, *College Graph Theory*, Vishwa International Publications, Gulbarga, India, (2012).

[2] V.R.Kulli, *The neighborhood graph of a graph*, International Journal of Fuzzy Mathematical Archive, 8(2)(2015), 93-99..

[3] V.R.Kulli, *On common edge graphs*, J. Karnatak University Sci., 18(1973), 321-324.

[4] V.R.Kulli, *The block point tree of a graph*, Indian J. Pure Appl. Math., 7(1976), 620-624.

[5] V.R.Kulli, *The block-line forest of a graph*, Journal of Computer and Mathematical Sciences, 6(4)(2015), 200-205.

[6] V.R.Kulli, *On line block graphs*, International Research Journal of Pure Algebra, 5(4)(2015), 40-44.

[7] V.R.Kulli, *On block line graphs, middle line graphs and middle block graphs*, International Journal of Mathematical Archive, 6(5) (2015), 80-86.

[8] V.R.Kulli, *The semifull graph of a graph*, Annals of Pure and Applied Mathematics, 10(1)(2015), 99-104.

[9] V.R.Kulli, *On full graphs*, Journal of Computer and Mathematical Sciences, 6(5)(2015), 261-267.

[10] V.R.Kulli, *On qlick transformation graphs*, International Journal of Fuzzy Mathematical Archive, 8(1)(2015), 29-35.

[11] V.R.Kulli, *On semifull line graphs and semifull block graphs*, Journal of Computer and Mathemetical Sciences, 6(7)(2015), 388-394.

[12] V.R.Kulli, *On full line graph and the full block graph of a graph*, International Journal of Mathemetical Archive, 6(8)(2015), 91-95.

[13] V.R.Kulli and B.Basavanagoud, *On the quasivertex total graph of a graph*, J. Karnatak University Sci., 42(1998), 1-7.

[14] V.R.Kulli and M.S.Biradar, *The middle blict graph of a graph*, International Research Journal of Pure Algebra, 5(7)(2015), 111-117.

[15] V.R.Kulli and M.H.Muddebihal, *Lict and litact graph of a graph*, J. Analysis and Computation, 2(2006), 33-43.

[16] V.R.Kulli and K.M.Niranjan, *The semisplitting block graph of a graph*, Journal of Scientific Research, 2(2)(2010), 485-488.

[17] V.R.Kulli, *The edge dominating graph of a graph*, In Advances in Domination Theory I, V.R.Kulli, ed., Vishwa International Publications, Gulbarga, India, (2012), 127-131.

[18] V.R.Kulli, *The semientire total dominating graph*, In Advances in Domination Theory II, V.R.Kulli, ed., Vishwa International Publications, Gulbarga, India, (2013) ,75-80.

[19] V.R.Kulli, *The total dominating graph*, Annals of Pure and Applied Mathematics, 10(1)(2015) 123-128.

[20] V.R.Kulli, *On entire dominating transformation graphs and fuzzy transformation graphs*, International Journal of Fuzzy Mathematical Archive, 8(1)(2015), 43-49.

[21] V.R.Kulli, *Entire edge dominating transformation graphs*, International Journal of Advanced Research in Computer Science and Technology, 3(2)(2015), 104-106.

[22] V.R.Kulli, *Entire total dominating transformation graphs*, International Research Journal of Pure Algebra, 5(5)(2015), 50-53.

[23] V.R.Kulli and B.Janakiram, *The common minimal dominating graph*, Indian J.Pure Appl. Math, 27(2)(1996), 193-196.

[24] V.R.Kulli and B.Janakiram, *The minimal dominating graph*, Graph Theory Notes of New York, New York Academy of Sciences, 28(1995), 12-15.

[25] V.R.Kulli, B.Janakiram and K.M.Niranjan, *The vertex minimal dominating graph*, Acta Ciencia Indica, 28(2002), 435-440.

[26] V.R.Kulli, B.Janakiram and K.M.Niranjan, *The dominating graph*, Graph Theory Notes of New York, New York Academy of Sciences, 46 (2004), 5-8.

[27] V.R.Kulli and D.G.Akka, *Traversability and planarity of total block graphs*, J Mathematical and Physical Sciences, 11(1977), 365-375.

[28] V.R.Kulli and D.G.Akka, *Traversability and planarity of semitotal block graphs*, J Math. and Phy. Sci., 12(1978), 177-178.

[29] B.Basavanagoud and V.R.Kulli, *Traversability and planarity of quasi-total graphs*, Bull. Cal. Math. Soc., 94(1)(2002), 1-6.

[30] B.Basavanagoud and V.R.Kulli, *Hamiltonian and eulerian properties of plick graphs*, The Mathematics Student, 74(1-4)(2004), 175-181.