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Some Relations on Hikami's Mock Theta Functions

Research Article

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Abstract: In this paper we obtain relations connecting Hikami's mock theta functions,partial mock theta functions and infinite products analogous to the identities of Ramanujan.

Keywords: Mock theta functions, partial mock theta functions.

1. Introduction

Ramanujan's last mathematical creation was his mock theta functions which he discovered during the last years of his life. Ramanujan gave a list of seventeen mock theta functions and lebelled them as third, fifth and seven orders without giving any reason for his classification. A mock theta function is a function $f(q)$ defined by a q-series, convergent for $|q| < 1$ which satisfies the following two conditions :

(1) For every root of unity ξ there is a $\theta-$ function $\theta_\xi(q)$ such that the difference $f(q) - \theta_\xi(q)$ is bounded as $q \to \xi$ radially.

(2) There is no single $\theta-$function which works for all ξi.e. for every $\theta-$ function $\theta(q)$ there is some root of unity ξ for which $f(q) - \theta(q)$ is unbounded as $q \to \xi$ radially.

For mock theta function $\overline{\psi}_1(q)$,

$$\overline{\psi}_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_{2n}} \tag{1}$$

the partial mock theta function will be defined and denoted as

$$\overline{\psi}_{1,N}(q) = \sum_{n=0}^{N} \frac{q^{n^2}}{(-q;q)_{2n}} \tag{2}$$

Hikami [15] introduced Mock theta functions of order two as:

$$D_5(q) = \sum_{n=0}^{\infty} \frac{q^n(-q;q)_n}{(q;q^2)_{n+1}} \tag{3}$$

Hikami [14] introduced Mock theta functions of order four and eight as:

$$D_5(q) = \sum_{n=0}^{\infty} \frac{q^n(-q^2;q^2)_n}{(q^{n+1};q)_{n+1}} \tag{4}$$

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$$I_{12}(q) = \sum_{n=0}^{\infty} \frac{q^{2n}(-q;q^2)_n}{(q^{n+1};q)_{n+1}} \tag{5}$$

$$I_{13}(q) = \sum_{n=0}^{\infty} \frac{q^{n}(-q;q^2)_n}{(q^{n+1};q)_{n+1}} \tag{6}$$

$D_6(q)$ is of order four, $I_{12}(q)$ and $I_{13}(q)$ are of order eight. Hikami's partial mock theta functions are

$$D_{5,m}(q) = \sum_{n=0}^{m} \frac{q^{n}(-q;q)_n}{(q;q^2)_{n+1}} \tag{7}$$

$$D_{6,m}(q) = \sum_{n=0}^{m} \frac{q^{n}(-q^2;q^2)_n}{(q^{n+1};q)_{n+1}} \tag{8}$$

$$I_{12,m}(q) = \sum_{n=0}^{m} \frac{q^{2n}(-q;q^2)_n}{(q^{n+1};q)_{n+1}} \tag{9}$$

$$I_{13,m}(q) = \sum_{n=0}^{m} \frac{q^{n}(-q;q^2)_n}{(q^{n+1};q)_{n+1}} \tag{10}$$

The following q-notations have been used for $|q^k| < 1$

$$(a;q^k)_n = \prod_{j=0}^{n-1}(1-aq^{kj}); n \geq 1$$

$$(a;q^k)_0 = 1$$

$$(a;q^k)_\infty = \prod_{j=0}^{\infty}(1-aq^{kj})$$

$$(a)_n = (a;q)_n$$

$$(a_1, a_2 \ldots, a_m; q^k)_n = (a_1, q^k)_n (a_2;q^k)_n \ldots (a_m;q^k)_n.$$

Ramanujan, in chapter 16 of his second notebook defined theta functions as follows [6, 20]:

$$\chi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2,q^2)_\infty}{(q;q^2)_\infty} \tag{11}$$

An identity due to Euler is [9]

$$\sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q;q^2)_{n+1}} = (-x;q)_\infty \tag{12}$$

The special cases of the above identity are

$$L(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n} = \frac{(q^2,q^2,q^4;q^4)_\infty}{(q;q)_\infty} \tag{13}$$

$$T(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} = \frac{(q,q^3,q^4;q^4)_\infty}{(q;q)_\infty} \tag{14}$$

Jackson [17] discovered the following identity

$$U(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}} = \frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty} \tag{15}$$

This identity was independently discovered by slater [21] who also discovered its companion identity

$$V(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q)_{2n+1}}=\frac{(-q,-q^7,q^8;q^8)_\infty}{(q^2;q^2)_\infty} \tag{16}$$

The famous Roger's Ramanujan identities are

$$M(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q,q^4;q^5)_\infty} \tag{17}$$

$$N(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}=\frac{1}{(q^2,q^3;q^5)_\infty} \tag{18}$$

The identity analogous to the Rogers-Ramanujan Identity is the so- called Gollnitz-Gordon identity given by [10, 11]

$$E(q)=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n}=\frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty} \tag{19}$$

$$\eta(q)=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}=\frac{1}{(q^3,q^4,q^5;q^8)_\infty} \tag{20}$$

Hahn defined the septic analogues of the Roger's-Ramanujan functions as [13, 14]

$$X(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}}=\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty} \tag{21}$$

$$Y(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n+1}}=\frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty} \tag{22}$$

$$Z(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n+1}}=\frac{(q,q^6,q^7;q^7)_\infty}{(q^2;q^2)_\infty} \tag{23}$$

The nonic analogous of Rogers-Ramanujan functions are [5].

$$O(q)=\sum_{n=0}^{\infty}\frac{(q;q)_{3n}q^{3n^2}}{(q^3;q^3)_n(q^3;q^3)_{2n}}=\frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty} \tag{24}$$

$$Q(q)=\sum_{n=0}^{\infty}\frac{(q;q)_{3n}(1-q^{3n+2})q^{3n(n+1)}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}}=\frac{(q^7,q^7,q^9;q^9)_\infty}{(q^3;q^3)_\infty} \tag{25}$$

$$W(q)=\sum_{n=0}^{\infty}\frac{(q;q)_{3n+2}q^{3n(n+1)}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}}=\frac{(q,q^8,q^9;q^9)_\infty}{(q^3;q^3)_\infty} \tag{26}$$

2. Methodology

We shall make use of the following known identity of Srivastava [22]:

$$\sum_{m=0}^{\infty}\delta_m\sum_{r=0}^{m}\alpha_r=\left(\sum_{r=0}^{\infty}\alpha_r\right)\left(\sum_{m=0}^{\infty}\delta_m\right)-\sum_{r=0}^{\infty}\alpha_{r+1}\sum_{m=0}^{r}\delta_m \tag{27}$$

3. Result

(A) Taking $\delta_m = q^{\frac{m(m+1)}{2}}$ in (27) and by (11) we have

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty} q^{\frac{m(m+1)}{2}}\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}\chi_m(q) \tag{28}$$

(I) Taking $\alpha_r = \dfrac{q^r(-q;q)_r}{(q;q^2)_{r+1}}$ in (28) and making use of (3) and (7) we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}D_5(q) = \sum_{m=0}^{\infty} q^{\frac{m(m+1)}{2}} D_{5,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q;q)_{r+1}}{(q;q^2)_{r+2}}\chi_m(q)$$

(II) Taking $\alpha_r = \dfrac{q^r(-q^2;q^2)_r}{(q^{r+1;q})_{r+1}}$ in (28) and making use of (4) and (8) we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}D_6(q) = \sum_{m=0}^{\infty} q^{\frac{m(m+1)}{2}} D_{6,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q^2;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}\chi_m(q) \tag{29}$$

(III) Taking $\alpha_r = \dfrac{q^{2r}(-q;q^2)_r}{(q^{r+1;q})_{r+1}}$ in (28) and making use of (5) and (9) we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}I_{12}(q) = \sum_{m=0}^{\infty} q^{\frac{m(m+1)}{2}} I_{12,m}(q) + \sum_{r=0}^{\infty}\frac{q^{2(r+1)}(-q;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}\chi_m(q) \tag{30}$$

(IV) Taking $\alpha_r = \dfrac{q^r(-q;q^2)_r}{(q^{r+1;q})_{r+1}}$ in (28) and making use of (6) and (10) we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}I_{13}(q) = \sum_{m=0}^{\infty} q^{\frac{m(m+1)}{2}} I_{13,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}\chi_m(q) \tag{31}$$

(B) Taking $\delta_m = \dfrac{q^{m^2}}{(q^2;q^2)_m}$ in (27) and by (13) we get

$$\frac{(q^2,q^2,q^4;q^4)_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^2;q^2)_m}\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}L_m(q) \tag{32}$$

(i) Taking $\alpha_r = \frac{q^r(-q;q)_r}{(q;q^2)_{r+1}}$ in (32) and using (3) and (7) we get

$$\frac{(q^2,q^2,q^4;q^4)_\infty}{(q;q)_\infty}D_5(q) = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^2;q^2)_m}D_{5,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q;q)_{r+1}}{(q;q^2)_{r+2}}L_m(q) \tag{33}$$

(ii) Taking $\alpha_r = \frac{q^r(-q^2;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (32) and using (4) and (8) we get

$$\frac{(q^2,q^2,q^4;q^4)_\infty}{(q;q)_\infty}D_6(q) = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^2;q^2)_m}D_{6,m}(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)}(-q^2;q^2)_{r+1}}{(q^{(r+2)};q)_{r+2}}L_m(q) \tag{34}$$

(iii) Taking $\alpha_r = \frac{q^r(-q;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (32) and using (5) and (9) we get

$$\frac{(q^2,q^2,q^4;q^4)_\infty}{(q;q)_\infty}I_{12}(q) = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^2;q^2)_m}I_{12,m}(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)}(-q;q^2)_{r+1}}{(q^{(r+2)};q)_{r+2}}L_m(q) \tag{35}$$

(iv) Taking $\alpha_r = \frac{q^r(-q;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (6) and using (10) we get

$$\frac{(q^2,q^2,q^4;q^4)_\infty}{(q;q)_\infty}I_{13}(q) = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^2;q^2)_m}I_{13,m}(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)}(-q;q^2)_{r+1}}{(q^{(r+2)};q)_{r+2}}L_m(q) \tag{36}$$

(C) Taking $\delta_m = \dfrac{q^{m(m+1)}}{(q^2;q^2)_m}$ in (27) and by (14) we get

$$\frac{(q,q^3,q^4;q^4)_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{m(m+1)}}{(q^2;q^2)_m}\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}\,T_m(q) \tag{37}$$

(I) Taking $\alpha_r = \frac{q^r(-q;q)_r}{(q;q^2)_{r+1}}$ in (37) and making use of (3) and (7) we get

$$\frac{(q,q^3,q^4;q^4)_\infty}{(q;q)_\infty}D_5(q) = \sum_{m=0}^{\infty}\frac{q^{m(m+1)}}{(q^2;q^2)_m}D_{5,m}(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)}(-q;q)_{r+1}}{(q;q^2)_{r+2}}\,T_m(q) \tag{38}$$

(II) Taking $\alpha_r = \frac{q^r(-q^2;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (37) and making use (4) and (8) we get

$$\frac{(q,q^3,q^4;q^4)_\infty}{(q;q)_\infty}D_6(q) = \sum_{m=0}^{\infty}\frac{q^{m(m+1)}}{(q^2;q^2)_m}D_{6,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q^2;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}T_m(q) \tag{39}$$

(III) Taking $\alpha_r = \frac{q^{2r}(-q;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (37) and making use of (5) and (9) we get

$$\frac{(q,q^3,q^4;q^4)_\infty}{(q;q)_\infty}I_{12}(q) = \sum_{m=0}^{\infty}\frac{q^{m(m+1)}}{(q^2;q^2)_m}I_{12,m}(q) + \sum_{r=0}^{\infty}\frac{q^{2(r+1)}(-q;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}T_m(q) \tag{40}$$

(IV) Taking $\alpha_r = \frac{q^r(-q;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (37) and making me of (6) and (10) we get

$$\frac{(q,q^3,q^4;q^4)_\infty}{(q;q)_\infty}I_{13}(q) = \sum_{m=0}^{\infty}\frac{q^{m(m+1)}}{(q^2;q^2)_m}I_{13,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}T_m(q) \tag{41}$$

(D) Taking $\delta_m = \frac{q^{m^2}}{(q;q)_m}$ in (27) and by (17) we get

$$\frac{1}{(q,q^4;q^5)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}M_m(q) \tag{42}$$

(i) Taking $\alpha_r = \frac{q^r(-q;q)_r}{(q;q^2)_{r+1}}$ in (42) and using (3) and (7) we get

$$\frac{1}{(q,q^4;q^5)_\infty}D_5(q) = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}D_{5,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q;q)_{r+1}}{(q;q^2)_{r+2}}M_m(q) \tag{43}$$

(ii) Taking $\alpha_r = \frac{q^r(-q^2;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (42) and using (4) and (8) we get

$$\frac{1}{(q,q^4;q^5)_\infty}D_6(q) = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}D_{6,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q;q)_{r+1}}{(q^{r+2};q)_{r+2}}M_m(q) \tag{44}$$

(iii) Taking $\alpha_r = \dfrac{q^{2r}(-q;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (42) and using (5) and (9) we get

$$\frac{1}{(q,q^4;q^5)_\infty}I_{12}(q) = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}I_{12,m}(q) + \sum_{r=0}^{\infty}\frac{q^{2(r+1)}(-q;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}M_m(q) \tag{45}$$

(iv) Taking $\alpha_r = \frac{q^r(-q;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (42) and using (6) and (10) we get

$$\frac{1}{(q,q^4;q^5)_\infty}I_{13}(q) = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}I_{13,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}M_m(q) \tag{46}$$

(E) Taking $\delta_m = \frac{q^{m(m+1)}}{(q;q)_m}$ in (27) and by (18) we have

$$\frac{1}{(q^2,q^3;q^5)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{m(m+1)}}{(q;q)_m}\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}N_m(q) \tag{47}$$

(I) Taking $\alpha_r = \frac{q^r(-q;q)_r}{(q;q^2)_{r+1}}$ in (47) and making use of (3) and (7) we have

$$\frac{1}{(q^2,q^3;q^5)_\infty}D_5(q) = \sum_{m=0}^{\infty}\frac{q^{m(m+1)}}{(q;q)_m}D_{5,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q;q)_{r+1}}{(q;q^2)_{r+2}}N_m(q) \tag{48}$$

(II) Taking $\alpha_r = \frac{q^r(-q^2;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (47) and making use of (4) and (8) we have

$$\frac{1}{(q^2,q^3;q^5)_\infty}D_6(q) = \sum_{m=0}^{\infty}\frac{q^{m(m+1)}}{(q;q)_m}D_{6,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q^2;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}N_m(q) \tag{49}$$

(III) Taking $\alpha_r = \frac{q^{2r}(-q;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (47) and making use of (5) and (9) we get

$$\frac{1}{(q^2,q^3;q^5)_\infty}I_{12}(q) = \sum_{m=0}^{\infty}\frac{q^{m(m+1)}}{(q;q)_m}I_{12,m}(q) + \sum_{r=0}^{\infty}\frac{q^{2(r+1)}(-q;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}N_m(q) \tag{50}$$

(IV) Taking $\alpha_r = \frac{q^r(-q;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (47) and making use of (6) and (10) we get

$$\frac{1}{(q^2,q^3;q^5)_\infty}I_{13}(q) = \sum_{m=0}^{\infty}\frac{q^{m(m+1)}}{(q;q)_m}I_{13,m}(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)}(-q;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}N_m(q) \tag{51}$$

(F) Taking $\delta_m = \dfrac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}$ in (27) and by (21) we get

$$\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}\chi_m(q) \tag{52}$$

(i) Taking $\alpha_r = \frac{q^r(-q;q)_r}{(q;q^2)_{r+1}}$ in (52) and using (5) and (7) we get

$$\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}D_5(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}D_{5,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q;q)_{r+1}}{(q;q^2)_{r+2}}X_m(q) \tag{53}$$

(ii) Taking $\alpha_r = \frac{q^r(-q^2;q^2)_r}{(q^{r+1}q)_{r+1}}$ in (52) and using (4) and (8) we get

$$\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}D_6(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}D_{6,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q^2;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}X_m(q) \tag{54}$$

(iii) Taking $\alpha_r = \frac{q^{2r}(-q;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (52) and using (5) and (9) we get

$$\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}I_{12}(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}I_{12,m}(q) + \sum_{r=0}^{\infty}\frac{q^{2(r+1)}(-q;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}X_m(q) \tag{55}$$

(iv) Taking $\alpha_r = \frac{q^r(-q;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (52) and using (6) and (10) we get

$$\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}I_{13}(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}I_{13,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}X_m(q) \tag{56}$$

(G) Taking $\delta_m = \dfrac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m}}$ in (27) and by (22) and (??) we get

$$\frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m}}\sum_{r=0}^{\infty}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}Y_m(q) \tag{57}$$

(I) Taking $\alpha_r = \frac{q^r(-q;q)_r}{(q;q^2)_{r+1}}$ in (57) and using (3) and (7) we get

$$\frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty}D_5(q) = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m}}D_{5,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q;q)_{r+1}}{(q;q^2)_{r+2}}Y_m(q) \tag{58}$$

(II) Taking $\alpha_r = \dfrac{q^r(-q^2;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (57) and by (4) and (8) we get

$$\frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty}D_6(q) = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m}}D_{6,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q^2;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}Y_m(q) \tag{59}$$

(III) Taking $\alpha_r = \frac{q^{2r}(-q;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (57) and using (5) and (9) we get

$$\frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty}I_{12}(q) = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m}}I_{12,m}(q) + \sum_{r=0}^{\infty}\frac{q^{2(r+1)}(-q;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}Y_m(q) \tag{60}$$

(IV) Taking $\alpha_r = \frac{q^r(-q;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (57) and using (6) and (10) we get

$$\frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty}I_{13}(q) = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m}}I_{13,m}(q) + \sum_{r=0}^{\infty}\frac{q^{2(r+1)}(-q;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}Y_m(q) \tag{61}$$

(H) Taking $\delta_m = \dfrac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m+1}}$ in (27) and using (23) we get

$$\frac{(q,q^6,q^7;q^7)_\infty}{(q^2;q^2)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m+1}}\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}Z_m(q) \tag{62}$$

(I) Taking $\alpha_r = \dfrac{q^r(-q;q)_r}{(q;q^2)_{r+1}}$ in (62) and using (3), (7) we get

$$\frac{(q,q^6,q^7;q^7)_\infty}{(q^2;q^2)_\infty}D_5(q) = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m+1}}D_{5,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q;q)_{r+1}}{(q;q^2)_{r+2}}Z_m(q) \tag{63}$$

(II) Taking $\alpha_r = \dfrac{q^r(-q^2;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (62) and using (4), (8) we get

$$\frac{(q,q^6,q^7;q^7)_\infty}{(q^2;q^2)_\infty}D_6(q) = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m+1}}D_{6,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q^2;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}Z_m(q) \tag{64}$$

(III) Taking $\alpha_r = \dfrac{q^{2r}(-q;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (62) and using (5), (9) we get

$$\frac{(q,q^6,q^7;q^7)_\infty}{(q^2;q^2)_\infty}I_{12}(q) = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m+1}}I_{12,m}(q) + \sum_{r=0}^{\infty}\frac{q^{2(r+1)}(-q;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}Z_m(q) \tag{65}$$

(IV) Taking $\alpha_r = \dfrac{q^r(-q;q^2)_r}{(q^{r+1};q)_{r+1}}$ in (62) and using (6), (10) we get

$$\frac{(q,q^6,q^7;q^7)_\infty}{(q^2;q^2)_\infty}I_{13}(q) = \sum_{m=0}^{\infty}\frac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m+1}}I_{13,m}(q) + \sum_{r=0}^{\infty}\frac{q^{r+1}(-q;q^2)_{r+1}}{(q^{r+2};q)_{r+2}}Z_m(q) \tag{66}$$

In the same way by assuming
$\delta_m = \dfrac{(-q;q^2)_m q^{m^2}}{(q^2;q^2)_m}, \dfrac{q^{m^2+2m}(-q;q^2)_m}{(q^2;q^2)_m}, \dfrac{q^{2m^2}}{(q;q)_{2m}}, \dfrac{q^{2m(m+1)}}{(q;q)_{2m+1}}, \dfrac{q^{3m^2}(q;q)_{3m}}{(q^3;q^3)_m(q^3;q^3)_{2m}}, \dfrac{(q;q)_{3m}(1-q^{3m+2})q^{3m(m+1)}}{(q^3;q^3)_m(q^3;q^3)_{2m+1}}$ and $\dfrac{q^{3m(m+1)}(q;q)_{3m+1}}{(q^3;q^3)_m(q^3;q^3)_{2m+1}}$

we can obtain relation connecting $D_5(q)$,$D_6(q)$,$I_{12}(q)$and $I_{13}(q)$ and the infinite products $E(q)$, $\eta(q)$,$U(q)$,$V(q)$,$O(q)$,$Q(q)$,and $W(q)$ respectively.

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