International Journal of Mathematics And its Applications
Volume 3, Issue 4–E (2015), 33–40.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Approximating Fixed Point in CAT(0) Space by s-iteration Process for a Pair of Single Valued and Multivalued Mappings

Research Article

Manjula Tripathi[1] and Anil Kumar Dubey[2]*

1 Sr. Lecturer, Government Polytechnic College, Durg, India.

2 Assistant Professor, Bhilai Institute of Technology, Durg, India.

Abstract: Suppose K is a closed convex subset of a complete CAT(0) space X. T is mapping from K to X. $F(T)$ is set of fixed point of T which is nonempty. Sequence $\{x_n\}$ is defined by an element $x_1 \in k$ such that

$$\begin{aligned} x_{n+1} &= P((1-\alpha_n)Tx_n \oplus \alpha_n y_n) \\ y_n &= P((1-\beta_n)x_n \oplus \beta_n Tx_n) \quad \forall \geq 1 \end{aligned}$$

where P is the nearest point projection from X onto k. $\{\alpha_n\}, \{\beta_n\}$ are real sequences in (0,1) with the condition

$$\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n) = \infty$$

Then $\{x_n\}$ converges to some point x^* in $F(T)$. This result is extension of the result of Abdul Rehman Razani and saeed Shabhani. [Approximating fixed points for nonself mappings in CAT(0) spaces Springer 2011:65]

Keywords: S-iteration, CAT(0) spaces, fixed point condition E, nonself mapping, condition C.

1. Introduction

In 2009 Agrawal etc introduced the S-iteration process [1] as follows. They showed that S-iteration process is faster than the picard km iteration. E be a convex subset of linear space X. T be nonself mapping from E to X. F(T) be nonempty set of fixed points of T. $\{x_n\}$ be the iterative sequence generated by $x_1 \in E$ and is defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n Ty_n \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n \quad \forall \geq 1 \end{aligned}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in (0,1) satisfying the condition

$$\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n) = \infty$$

* *E-mail: anilkumardby70@gmail.com*

In 2010 haowang and Panyanak [2] studied the iterative scheme defined by the following process. Let K be a nonempty closed convex subset of a complete CAT(0) space with the nearest point projection P from X onto K and $T : K \to X$ be a nonexpansive nonself mapping with the nonempty fixed point set $\{x_n\}$ is a sequence generated by $x_1 \in k$ such that.

$$x_{n+1} = P((1-\alpha_n)x_n \oplus \alpha_n TP[(1-\beta_n)x_n \oplus \beta_n Tx_n)]$$

where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(\in, 1-\in) \in\in (0,1)$ then $\{x_n\}$ is Δ-convergent to a fixed point of T.

In this paper this result is extended for the S-iteration process.

Fixed point for CAT(0) space:- Let K be a nonempty subset of a CAT(0) space X and let $T : K \to X$ be a nonself mapping. A point $x \in K$ is called a fixed point of T if $Tx = x$. Set of fixed points is denoted by $F(T)$. Mapping T is nonexpansive if for each $x, y \in k$

$$d(Tx, Ty) \leq d(x, y)$$

Condition C : Suzuki [3] introduced condition C in 2008 which states that mapping T is said to satisfy condition C if

$$\begin{aligned} d(x, Tx) &\leq d(x, y) \quad \textit{implies} \\ d(Tx, Ty) &\leq d(x, y) \quad \forall x, y \in K \end{aligned}$$

Condition E : In 2011, Falset etc [4] introduced condition E which states that

Let K be a bounded closed convex subset of a complete CAT(0) space X. A non self mapping $T : K \to X$ is satisfying condition E if there exist $\mu \leq 1$ such that

$$d(x, Ty) \leq \mu(Tx, x) + d(x, y) \quad \forall x, y \in K, \quad \mu \geq 1.$$

Proposition 1.1. *Every non expansive mapping satisfies condition (c) but the converge is not true.*

Proposition 1.2. *Every non expansive mapping satisfies condition (E) but the converge is not true.*

CAT(0) Spaces - Let (x, d) be a metric space. A geodesic path joining two points $x, y \in k$ or more briefly a geodesic from x to y is a mapping $c : [0, l] \to X$ such that $[0, l]CR, C(0) = xC(l) = y$ and

$$d(c(t), c(t^{'})) = |t - t^{'}| \quad \forall t, t^{'} \in [0, l]$$

this implies that $d(x, y) = l$.

The image α of mapping c is called a geodesic segment joining x and y. The space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic c . X is uniquely geodesic if there is exactly one geodesic for $x, y \in k$.

A subset $y \subset X$ is said to be convex if every geodesic segment (ie image α of c), joining any two points of y is contained in y itself.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ is a geodesic metric space (X,d) consists of three points in X which are vertices of Δ and geodesic segment between each pair of vertices as the edges of Δ.

A comparison triangle $\overline{\Delta}$ for the triangle $\overline{\Delta}(x_1, x_2, x_3)$ in (X,d) is a triangle $\Delta(x_1, x_2, x_3)$ such that $\overline{\Delta}(x_1, x_2, x_3) = \Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane E^2 such that

$$d_{E^2}(\overline{x_i}, \ \ \overline{x_j}) = d(\overline{x_i}, \ \ \overline{x_j}) \quad \forall i, j \in (1, 2, 3).$$

A geodesic metric space is said to be a CAT(0) space if all geodesic triangle of approximate size satisfy the following condition.

Let Δ be a geodesic triangle in X and let $\overline{\Delta}$ be comparison triangle for Δ then Δ is said to satisfy the CAT(0) inequality if $\forall x, y \in \Delta$ and $x, y \in \overline{\Delta}$

$$d(x,y) \leq d_{E^2}(\overline{x}, \overline{y})$$

If (x_1, y_1, y_2) are points in CAT(0) space and if y_0 is the middle point of the geodesic regment $[y_1, y_2]$ then the CAT(0) inequality implies that

$$d(x,y_0)^2 \leq \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2^2) - \frac{1}{4}d(y_1,y_2)^2$$

A geodesic space is a CAT(0) space if and only if it satisfies CN inequality.

Proposition 1.3. *Let K be a bounded closed convex subset of a complete CAT(0) space X and $T : K \to X$. If T is satisfying condition (c) then*

$$d(x,Ty) \leq 3d(Tx,x) + d(x,y)$$

Lemma 1.4 ([5])**.** *Let (X,d) be a CAT(0) space.*

(1) Let K be a convex subset of X which is complete in the induced metric. Then for every $x \in X$, there exists a unique point $p(x) \in k$ such that $d(x,p(x)) = \inf\{d(x,y) : y \in k\}$ moreover map $x \to p(x)$ is a nonexpansive retract from X onto K.

(2) For $x, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that $d(x,z) = td(x,y), \quad d(y,z) = (1-t)d(x,y) \quad z = (1-t)x \oplus ty$

(3) For every $x, y, z \in X$ and $t \in [0,1]$

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z)$$

(4) For $x, y, z \in X$ and $t \in [0,1]$

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2$$

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \to X$ we set

$$r(x, \{x_n\}) = \lim_{n\to\infty} \sup d(x, x_n)$$

The asymptotic radius $r(\{x_n\}$ is given by

$$r(\{x_n\}) = \inf(r(x, \{x_n\}) \quad x \in X)$$

The asymptotic centre $A(\{x_n\}) of \{x_n\}$ is the set

$$A(x_n) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})$$

or in other words

$$A(x_n) = \{x \in X : \lim_{n\to\infty} \sup d(x, x_n) = \inf(r(x, x_n))$$

In a CAT(0) space X, $A(\{x_n\})$ consists of exactly one point.

Definition 1.5 ([5]). *A sequence $\{x_n\}$ in a CAT(0) space X is said Δ converges to $x \in X$, if x is the unique asymptotic centre $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$.*

Lemma 1.6. *Let (X,d) be a CAT(0) space*

(1) [7] Every bounded sequence in X has a Δ- convergent subsequence.

(2) [8] If K is a closed convex subset of X and if $\{x_n\}$ is bounded sequence in K, then the asymptotic centre of $\{x_n\}$ is in K.

(3) [] If $\{x_\}$ is a bounded sequence in X with $A(\{x_n\}) = (x)$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = (u)$ and the sequence $\{d(x_m, u\}$ converges then $x = u$.

2. Main Result

Lemma 2.1 ([5]). *Let K be a nonempty closed convex subset of a complete CAT(0) space X and $T : K \to X$ be a nonself mapping, satisfying condition (E). Suppose $\{x_n\}$ is a bounded sequence in K such that $\lim_n d(x_n, Tx_n) = 0$ and $\{d(x_n, v)\}$ converges for all $v \in F(T)$ then*

$$w_w(x_n) \subset F(T)$$

where $w_w(x_n) = \cup A(\{u_n\})$ and the union is taken over all subsequence $\{u_n\} of \{x_n\}$. Moreover $w_w(x_n)$ consists of exactly one point.

Theorem 2.2. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and $T : K \to X$ be a nonself mapping satisfying condition E with $x^* \in F(T) = \{x \in K : Tx = x\} F(T)$ is nonempty set. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\in, 1-\in]$ for some $\in \in (0, 1)$ starting from arbitrary $x_1 \in K$. Define the sequence*

$$\begin{aligned} x_{n+1} &= P((1-\alpha_n)Tx_n \oplus \alpha_n Ty_n) \\ y_n &= P((1-\beta)x_n \oplus \beta_n Tx_n) \quad \forall n \quad \geq 1 \end{aligned}$$

Then $\lim_{n\to\infty} d(x_n, x^*)$ exists.

Proof.

$$\begin{aligned} d(x_{n+1}, x^*) &= d(P((1-\alpha_n)Tx_n \oplus \alpha_n Ty_n, x^*)) \\ &\leq d((1-\alpha_n)Tx_n \oplus \alpha Ty_n x^*) \\ &\leq (1-\alpha_n)d(Tx_n, x^*) \oplus \alpha_n d(Ty_n, x^*) \\ &\leq (1-\alpha_n)d(x_n, x^*) + \alpha_n d(y_n, x^*) \\ &\leq (1-\alpha_n)d(x_n, x^*) + \alpha_n d(P(1-\beta_n)x_n \oplus \beta_n Tx_n, x^*) \\ &\leq (1-\alpha_n)d(x_n, x^*) + \alpha_n d((1-\beta_n)x_n \oplus \beta_n Tx_n, x^*) \\ &\leq (1-\alpha_n)d(x_n, x^*) + \alpha_n(1-\beta_n)d(x_n, x^*) + \alpha_n\beta_n(Tx_n, x^*) \\ &\leq (1-\alpha_n)d(x_n, x^*) + \alpha_n(1-\beta_n)d(x_n, x^*) + \alpha_n\beta_n d(x_n, x^*) \\ &= (1-\alpha_n + \alpha_n - \alpha_n\beta_n + \alpha_n\beta_n)d(x_n, x^*) \\ &= d(x_n, x^*) \end{aligned}$$

Hence $d(x_{n+1}, x^*) \leq d(x_n, x^*) \quad \forall n \geq 1$ □

So the sequence $\{d(x_n, x^*)\}_{=1}^{\infty}$ is bounded and decreasing. Hence $\lim_{n\to\infty} d(x_n, x^*)$ exists.

Theorem 2.3. *Let K be a nonempty closed convex subset of complete CAT(0) space X and $T : K \to X$ be a nonself mapping satisfying condition E with $F(T) \neq \phi$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\in, 1-\in]$ for some $\in\in (0, 1)$. Starting from arbitrary $x_1 \in K$. Define the sequence*

$$\begin{aligned} x_{n+1} &= P((1-\alpha_n)Tx_n \oplus \alpha_n Ty_n) \\ y_n &= P((1-\beta)x_n \oplus \beta_n Tx_n) \quad \forall n \quad \geq 1 \end{aligned}$$

Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0.$

Proof. By theorem sequence $\{d(x_n, x^*)\}_{n=}^{\infty}$ bounded and decreasing so $\lim_{n\to\infty} d(x_n, x^*$ exists where $x^* \in F(T)$. Let

$$\lim_{n\to\infty} d(x_n, x^*) = r$$

Now $d(x_n, Tx_n) \leq d(x^*, x_n) + d(x^* Tx_n)$

(a) If $r = 0$

$$d(x_n, Tx_n) \leq d(x^*, Tx_n)$$

By conditions E for some $\mu \geq 1$

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x^*, Tx_n) \\ &\leq \mu(dx^*, Tx^*) + d(x^*, x_n) \end{aligned}$$

Here $r = 0$ is $d(x_n, x^*) = 0$

So $Lt_{n\to\infty} d(x_n, Tx_n) = 0$.

(b) If $r > 0$

$$\begin{aligned} d(y_n, x^*)^2 &= d(P(1-\beta_n)x_n \oplus \beta_n Tx_n), x^*)^2 \\ &\leq d((1-\beta_n)x_n \oplus \beta_n Tx_n), x^*)^2 \quad as \quad \|Tx - Ty\| \leq \|x - y\| \\ &\leq (1-\beta_n)d(x_n, x^*)^2 + \beta_n d(Tx_n, x^*)^2 - \beta_n(1-\beta_n)d(x_n, Tx_n)^2 \\ &\leq (1-\beta_n)d(x_n, x^*)^2 + \beta_n d(Tx_n, x^*)^2 \\ &\leq (1-\beta_n)d(x_n, x^*)^2 + \beta_n(\mu d(Tx, x^*) + d(x_n, x*))^2 \end{aligned}$$

by condition E for some $\mu \geq 1$

$$\leq d(x_n, x^*)^2 \tag{1}$$

So we have the result

$$d(y_n, x^*) \leq d(x_n, x^*)$$

Now consider

$$\begin{aligned}
d(x_{n+1},x^*)^2 &= d(P(1-\alpha_n)Tx_n \oplus \alpha_n Ty_n, x^*)^2\\
&\leq d((1-\alpha_n)Tx_n + \alpha_n Ty_n, x^*)^2\\
&\leq (1-\alpha_n)d(Tx_n,x^*)^2 + \alpha_n d(Ty_n,x^*)^2 - \alpha_n(1-\alpha_n)d(Tx_n,Ty_n)^2\\
&\leq (1-\beta_n)d(x_n,x^*)^2 + \alpha_n(\mu d(Tx^*,x^*) + d(y_n,x^*))^2 - \alpha_n(1-\alpha_n)d(Tx_n,Ty_n)^2\\
&\leq (1-\alpha_n)d(x_n,x^*)^2 + \alpha_n d(y_n,x^*)^2 - \alpha_n(1-\alpha_n)d(Tx_n,Ty_n)^2\\
&\leq d(x_n,x^*)^2 - \alpha_n(1-\alpha_n)d(Tx_n,Ty_n)^2\\
&\leq d(x_n,x^*)^2 - \alpha_n(1-\alpha_n)d(x_n,y_n)^2\\
Now \quad d(x_n,y_n) &= d(x_n,(1-\beta_n)x \oplus \beta_n Tx_n)\\
&= (1-\beta_n)d(x_n,x_n) + \beta_n d(x_n,Tx_n)\\
&= \beta_n d(x_n,Tx_n).\\
So \quad d(x_{n+1},x^*)^2 &\leq d(x_n,x^*)^2 - \alpha_n(1-\alpha_n\beta_n d(x_n,Tx_n)^2\\
&\leq d(x_n,x^*)^2 - \alpha_n(1-\alpha_n)\beta d(x_n,Tx_n)^2
\end{aligned}$$

taking limit when $n \to \infty$

$$\lim_{n\to\infty} d(x_n,Tx_n) = 0$$

□

Theorem 2.4. *Let K be nonempty closed convex subset of a complete CAT(0) space X and $T : K \to X$ be a nonself mapping, satisfying condition (E) with $F(T) \neq \phi$. Assume $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[\in, 1-\in]$ for some $\in\in (0,1)$. Starting from arbitrary $x_1 \in K$. Define the sequence $\{x_n\}$ by*

$$\begin{aligned}
x_{n+1} &= P((1-\alpha_n)Tx_n \oplus \alpha_n Ty_n)\\
y_n &= P((1-\beta)x_n \oplus \beta_n Tx_n) \quad \forall \quad n \geq 1
\end{aligned}$$

Then $\{x_n\}$ is $\Delta-$ convergent to some point x^* in $F(T)$

Proof. By theorem (2.3) $\lim_{n\to\infty} d(x_n,Tx_n) = 0$. Theorem (2.2) shows that $\{d(x_n,v)\}$ is bounded and decreasing sequence for each $v \in F(T)$ and so it is convergent.
By Lemma (2.1), if K be a nonempty closed convex subset of a complete CAT(0) space X and $T : K \to X$ be a nonself mapping satisfying condition E, suppose $\{x_n\}$ is a bounded sequence in K such that $\lim d(x_n,Tx_n) = 0$ and $\{d(x_n,v)\}$ converges for all $v \in F(T)$ then $w_w(x_n) \subset F(T)$ where $w_w(x_n) = \cup A\{u_n\}$ and the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$, then $w_w(x_n)$ consists exactly one point so the sequence $\{x_n\}$ is $\Delta-$ convergent to some point x^* in $F(T)$. □

Definition 2.5. *A mapping $T : K \to X$ is said to satisfy condition if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0 \forall r > 0$. Such that*

$$d(x,Tx) \leq f(d(x,F(T)))$$

where $x \in K$.

Theorem 2.6. *Let K be nonempty closed convex subset of a complete CAT(0) space X and $T : K \to X$ be a nonself mapping satisfying condition E with $F(T) \neq \phi$, Assume $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[\in, 1- \in]$ for some $\in \in (0,1)$. Starting from arbitrary $x_1 \in K$. Define the sequence $\{x_n\}$ by*

$$\begin{aligned} x_{n+1} &= P((1-\alpha_n)Tx_n \oplus \alpha_n T y_n) \\ y_n &= P((1-\beta)x_n \oplus \beta_n T x_n) \quad \forall \leq 1 \end{aligned}$$

If T satisfies condition E then $\{x_n\}$ is converges strongly to a fixed point of T.

Theorem 2.7. *Let K be nonempty compact convex subset of a complete CAT(0) space X and $T : K \to X$ be a nonself mapping satisfying condition E with $F(T) \neq \phi$, Assume $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[\in, 1- \in]$ for some $\in \in (0,1)$. Starting from arbitrary $x_1 \in K$. Define the sequence $\{x_n\}$ by*

$$\begin{aligned} x_{n+1} &= P((1-\alpha_n)Tx_n \oplus \alpha_n T y_n) \\ y_n &= P((1-\beta)x_n \oplus \beta_n T x_n) \quad \forall \quad n \geq 1 \end{aligned}$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 2.8. *Let K be nonempty closed convex subset of a complete CAT(0) space X and $S, T : K \to X$ be two nonself mappings satisfying condition E with $F(T) \neq \phi$, Assume $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[\in, 1- \in]$ for some $\in \in (0,1)$. Starting from arbitrary $x_1 \in K$. Define the sequence $\{x_n\}$ by*

$$\begin{aligned} x_{n+1} &= P((1-\alpha_n)Tx_n \oplus \alpha_n T y_n) \\ y_n &= P((1-\beta)x_n \oplus \beta_n T x_n) \quad \forall \leq 1 \end{aligned}$$

Then $\{x_n\}$ is convergent to a common fixed point of S and T.

‘Above theorems defines the $\Delta-$ convergence of a defined sequence to a common fixed point of two nonself nonexpansive mappings.

References

[1] Ravi P. Agrawal, D.O.Regan and D.R.Sahu, *Fixed Point theory for Lipschitzian type mapping with applications*, springer, London/ Newyork, (2009).

[2] W.Haowong and B.Panyanak, *Approximating fixed points of nonexpansive nonself mappig in CAT(0) space*, Fixed Point Th. Appl. ID 367274(2010).

[3] T.Suzuki, *Fixed point theorem and convergence theorems for some generalized nonexpansive mappings* Math. Anal. Appl., 340(2008), 1088-1095.

[4] G.Falset, J.C.Fuster and T.Suzuki, *Fixed point theory for a class of generalized nonexpansive mapping*, J. Math. Anal. Appl., 375(2011), 185-195.

[5] Razani and Shabani, *Approximating fixed points for nonself mappings in CAT(0) space 2011-65*, Fixed point theory and appl, springer open journal.

[6] Shabani and S.J.Hoseini Ghoricheh, *Approximating fixed point of generalized nonexpansive nonself mappings in CAT(0) spaces*, Math. Sci. Journal, 7(1)(2011), 89-95.

[7] Kirk and W.Panyanak, *A concept of convergence in geodesic space*, Nonlinear Anal., 68-36893696 (2008) doc (10.1016/na 2007.04.011).

[8] Dhompongsa, S.Kirk and W.A.Panyanak, *Nonexpansive set valued mapping in metric and Banach space*, J. Nonlinear convex anal., 8(2007), 35-45.