Continuous and Contra Continuous Functions in Bi-topological Spaces

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Abstract: The concept of bi-topological spaces was first introduced by J.C. Kelly [2] in 1963. Many authors such as Levine [3] contributed as he defined the semi-open sets and semi-continuity in bi-topological spaces. Maheshwari and Prasad [5] contributed semi-open sets and semi-continuity to bi-topological spaces. The notion of β-open sets contributed by Mashhour et. al. [6] and Andrijevic [1] define Semi pre-open sets. In this paper we discuss pre-continuity and semi pre-continuity in bi-topological spaces. LellisThivager et.al. [4] introduces g*-closed sets topological spaces and initiated the concepts of ultra space by using (1,2)α-open sets in bi-topological spaces and proved that each (1,2)β-open sets is (1,2) semi-open and (1, 2) pre-open but the converse of each is not true. R-Devi and S.Sampath Kumar and M. Caldas [7] introduced and studied a class of sets and maps between bi-topological spaces Called supra α—open sets and supra α-continuous maps respectively.

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1. Introduction and Preliminaries

Throughout the paper (X, τ₁, τ₂) and (X, τ) denote bi-topological and topological space. Let A be a subject of a topological space (X, τ) then cl(A) denote closure of A and Int(A) denote interior of A. A subject A of X is said to be semi open if there exist an open set U of X such that cl(A) ⊂ U ⊂ A, τ. A subject A of X is said to be pre-open if there exist an open set U of X such that U ⊂ A ⊂ cl(U) and A is said to be semi-pre-open if there exist a pre-open set U of X such that U ⊂ A ⊂ cl(A). A function f : (X₁, τ₁) → (X₂, τ₂) is said to be semi-continues if for each U ∈ τ₂, f⁻¹(U) is semi-open in (X₁, τ₁) and A function f : (X₁, τ₁) → (X₂, τ₂) is said to be pre-continuous or β-continuous if for U ∈ τ₂, f⁻¹(U) is pre-open or β-open in (X₁, τ₁).

Definition 1.1.

(a) Pre-open Sets: A set A of a bi-topological space (X, τ₁, τ₂) said to be pre-open w. r. t. (τ₁, τ₂) if there exist U ∈ τ₁, such that A ⊂ U ⊂ cl(A) or A ⊂ int(cl(A)).

(b) A subject A of (X, τ₁, τ₂) is said to be (τ₁, τ₂) semi-open if there exist an U ∈ τ₁ such that U ⊂ A ⊂ cl(A).

(c) A subject A of (X, τ₁, τ₂) is said to be (τ₁, τ₂) semi-pre-open if there exist an (i, i) pre-open set U such that U ⊂ A ⊂ cl(A).

A subject A of x is said to be pair wise semi-open if it is (1,2) semi-open and (2,1) semi-open. The compliment of an (i, i) pre-open set is called (i, i) pre-closed set and (i, i) semi pre-open set is called (i, i) semi pre-closed set.

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Example 1.2. Let $X = [a, b, c, d]$, $\tau_1 = \{\emptyset, X, \{a, c\}\}$ and $\tau_2 = \{\emptyset, X, \{b, d\}\}$, then $A = \{b\}$ is pair wise pre-open but not $\tau_1$-open.

Definition 1.3. Let $(X, \tau_1, \tau_2)$ be a bi-topological space and $A$ is a subset of $X$ said to be

(a) Generalized closed set (g-closed set) iff $\text{cl}_{\tau_i}(A) \subseteq U$ where $A \subseteq U$ and $U$ is $\tau_i$-open set in $(X, \tau_i)$ for $i = 1$ or 2.

(b) Pair wise semi pre-continuous or pair wise is called

Example 2.2. Let $X = [a, b, c, d]$, $\tau_1 = \{\emptyset, X, \{a, c\}\}$ and $\tau_2 = \{\emptyset, X, \{b, d\}\}$, then $A = \{b\}$ is pair wise pre-open but not $\tau_1$-open.

Definition 1.4. Let $(X, \tau_1, \tau_2)$ and $(X', \tau_1', \tau_2')$ be two bi-topological spaces. Then a function $f : (X_1, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ is called

(a) Pair wise continuous if $f^{-1}(U)$ is $\tau_i$ open-set in $(X, \tau_1, \tau_2)$ for each $\tau'_i$ open set $U$ of $(X', \tau_1', \tau_2')$ for $i = 1, 2$.

(b) Pair wise semi continuous if $f^{-1}(U)$ is $(\tau_i, \tau_i)$ semi-open in $(X, \tau_1, \tau_2)$ for each $\tau'_i$ open set $U$ of $(X', \tau_1', \tau_2')$ for $i \neq i$ and $I, i = 1, 2$.

(c) Generalized semi closed (gs-closed) set Iff $\text{scl}_{\tau_i}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open set in $(X, \tau_i)$ for $i = 1$ or 2.

(d) $\psi$ Closed set if $\text{scl}_{\tau_i}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\psi$-open set in $(X, \tau_i)$ for $i = 1$ or 2.

2. Main Results

Theorem 2.1. Let $f : (X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ be a pair wise continuous function. If $A$ is an $(i, j)$ pre-open set of $X$ then $f(A)$ is $(i, j)$ pre-open in $X'$.

Proof. Let $A$ be $(i, j)$ pre-open in $X$, there exist $\in \tau_i$ such that $A \subseteq U \subseteq \text{cl}_{\tau_j}(A)$. As $f$ is pair wise open, $f(U) \in \tau_i$. Also $f$ is pair wise continuous we have $f(A) \subseteq f(U) \subseteq f(\text{cl}_{\tau_j}(A)) \subseteq \text{cl}_{\tau_j}(f(A))$. Shows that $f(A)$ is $(i, j)$ pre-open in $X'$. Let $A$ be $(i, j)$ semi pre-open in $X$. Then by definition there exist an $(i, j)$ pre-open set $U$ such that $U \subseteq A \subseteq \text{cl}_{\tau_j}(A)$. As $f$ is pair wise continuous we have $f(U) \subseteq f(A) \subseteq f(\text{cl}_{\tau_j}(A)) \subseteq \text{cl}_{\tau_j}(f(U))$. By first part we can say that $f(U)$ is $(i, j)$ pre-open in $X$. Thus $f(A)$is $(i, j)$ semi-open in $X'$.

Example 2.2. Let $X = [a, b, c, d]$, $\tau_1 = \{\emptyset, X, \{a, c\}\}$ and $\tau_2 = \{\emptyset, X, \{b, d\}\}$ and $\tau'_1 = \{\emptyset, X, \{b, c\}\}$ and $\tau'_2 = \{\emptyset, X, \{c, d\}\}$. Let $f(X, \tau_1, \tau_2) \to (X', \tau'_1, \tau'_2)$ be the identity map. Then induced function $f : (X, \tau_1) \to (X, \tau'_1)$ and $f : (X, \tau_2) \to (X, \tau'_2)$ are both pre-continuous. Here $f$ is not pair wise pre-continuous, because there is $\{b\} \in \tau'_1$ such that $f^{-1}b$ is not $(i, j)$ pre-open.
Theorem 2.3. Let \( f(X, \tau_1, \tau_2) \to (X', \tau'_1, \tau'_2) \) be a pair wise continuous open function, \( A \) is an \((i, j)\) pre-open set of \( X \) then \( f(A) \) is \((i, j)\) pre-open set in \( X' \).

Proof. Let \( A \) be \((i, j)\) pre-open in \( X \). Then by definition there exist \( U \in \tau_i \) such that \( A \subset U \subset cl_j(A) \). As \( f \) is pair wise open then \( f(U) \subset \tau'_i \). Also \( f \) is pair wise continuous we have \( f(A) \subset f(U) \subset f(cl_j(A)) \subset cl_j(f(A)) \). Shows that \( f(A) \) is \((i, j)\) pre-open in \( X' \).

Remark 2.4. If \( A \) is \((i, j)\) semi pre-open set of \( X \) then \( f(A) \) is \((i, j)\) semi pre-open in \( X' \).

Theorem 2.5. Let \( f(X, \tau_1, \tau_2) \to (X', \tau'_1, \tau'_2) \) be pair wise continuous open function. \( A \) be an \((i, j)\) pre-open set of \( X' \), then \( f^{-1}(A) \) is \((i, j)\) pre-open in \( X \).

Proof. Let \( A \) be \((i, j)\) pre-open in \( X' \), then there exist \( V \in \tau'_i \) such that \( A \subset V \subset cl_j(A) \). Since \( f \) is pair wise open then \( f^{-1}(A) \subset f^{-1}(V) \subset f^{-1}(cl_j(A)) \subset cl_j(f^{-1}(A)) \). As \( f \) is pair wise continuous \( f^{-1}(V) \) is \((i, j)\) pre open in \( X \). Thus by last theorem \( f^{-1}(A) \) is \((i, j)\) pre-open in \( X \).

Remark 2.6. If \( A \) is an \((i, j)\) semi pre-open set of \( X' \) then \( f^{-1}(A) \) is \((i, j)\) semi pre-open in \( X \).

Theorem 2.7. Let \( f \) be a function such that \( f(X, \tau_1, \tau_2) \to (X', \tau'_1, \tau'_2) \) then the following statements are equivalent:

(a) \( f \) is pair wise continuous.

(b) Inverse image of each \( \tau'_i \) closed set of \( X' \) is \((i, j)\) pre-closed in \( X \).

(c) For each \( x \in X \) and each \( A \in \tau'_i \) containing \( f(x) \), there exist an \((i, j)\) pre-open set \( B \) of \( X \) containing \( x \) such that \( f(B) \subset A \).

(d) \((i, j)\) – pcl \( f^{-1}(B) \) \( \subseteq f^{-1}(cl_i(B)) \) for every subset \( B \) of \( X' \).

(e) \( f(i, j) \) – pcl \( f(A) \) \( \subseteq cl_i(f(A)) \) for every subset \( A \) of \( X \) for \( i \neq j \) and \( i, j = 1, 2 \).

Theorem 2.8. Let \( f(X, \tau_1, \tau_2) \to (X', \tau'_1, \tau'_2) \) be a function and \( X = U{U_\lambda \in \tau_1 \cap \tau_2/\lambda \in \Delta} \). Then \( f \) is pair wise continuous if the restriction \( f/U_\lambda : (U_\lambda, \tau_1/U_\lambda, \tau_2/U_\lambda) \to (X', \tau'_1, \tau'_2) \) is pair wise continuous.

Remark 2.9. Let \( f(X, \tau_1, \tau_2) \to (X', \tau'_1, \tau'_2) \) be a function and \( X = U{U_\lambda \in \tau_1 \cap \tau_2/\lambda \in \Delta} \), then \( f \) is pair wise continuous iff \( f/U_\lambda : (U_\lambda, \tau_1/U_\lambda, \tau_2/U_\lambda) \to (X', \tau'_1, \tau'_2) \) is pair wise continuous for \( \lambda \in \Delta \).

Definition 2.10 (Product Topology). Let \( \{X_\lambda, \tau_\lambda(\gamma), \tau_\lambda(\gamma)/\lambda \in \Delta\} \) be a family of bi-topological spaces. Then \( (X, \tau_1, \tau_2) \) is called product space where \( X = \prod X_\lambda \) and \( \tau_1, \tau_2 \) denote product topology for \( \tau_\lambda(\gamma)/\gamma \in \Delta \) for \( i = 1, 2 \).

Theorem 2.11. Necessary and sufficient condition that a non empty subject \( A_\lambda \) of \( X_\lambda \) for \( \lambda = \lambda_1, \lambda_2 \ldots \lambda_n \). Then \( A = \bigcap_{i=1}^n A_\lambda \).

Proof. Let \( A \) be \((\tau_1, \tau_2)\) pre-open subset of \( X \) as natural projection is open and continuous and surjective then \( \tau_i(\lambda_k) \), pre-open for \( K = 1, 2, 3 \ldots n \). Proves that condition is sufficient. Now let \( A_\lambda \) be \( \tau_i(\lambda_k) \) pre-open for each \( K = 1, 2, 3 \ldots n \) then by definition of pre-open set there exist \( \tau_\lambda(\lambda_k) \) an open set \( B_\lambda \) such that \( A_\lambda \subset B_\lambda \subset \tau_\lambda(\lambda_k) - cl((A_\lambda)) \). Then we have \( A \subset \bigcap_{k=1}^n B_\lambda \times \bigcap_{k=1}^n X_\lambda \subset \bigcap_{k=1}^n \tau_\lambda(\lambda_k) - cl((A_\lambda)) \times \bigcap_{k=1}^n X_\lambda = \tau_i - cl(A) \). As \( \bigcap_{k=1}^n B_\lambda \times \bigcap_{k=1}^n X_\lambda \) is \( \tau_i \) open where \( A \) is \((\tau_1, \tau_2)\) pre-open in \( X \). Now \( \{X_\lambda, \tau_\lambda(\gamma), \tau_\lambda(\gamma)/\lambda \in \Delta\} \) and \( \{Y_\lambda, \tau'_\lambda(\lambda), \tau'_\lambda(\lambda)/\lambda \in \Delta\} \) be two bi-topological spaces with same set of indices. Then \( f_\lambda : (X_\lambda, \tau_\lambda(\lambda), \tau_\lambda(\lambda)) \to (Y_\lambda, \tau'_\lambda(\lambda), \tau'_\lambda(\lambda)) \) is a function for each \( \lambda \in \Delta \).

Then \( f(X, \tau_1, \tau_2) \to (X'_1, \tau'_1, \tau'_2) \) be the product function defined by \( f(X_\lambda) = \{f_\lambda \} \) for each \( \{X_\lambda \} \in X = \prod X_\lambda \) where \( \tau_i \) and \( \tau'_i \) are the product topologies for \( i=1,2 \). Proves that condition is sufficient.
Definition 2.12 (Pair Wise Contra-Continuous Functions). A function \( f : (X_1, \tau_1, \tau_2) \to (X', \tau'_1, \tau'_2) \) called pair wise \( c \)-continuous function if \( f^{-1}(U) \) is \( \tau_i, \tau'_j \)-locally closed in \( (X, \tau_1, \tau_2) \) for each \( \tau'_1 \). Open set \( U \) of \( (X', \tau'_1, \tau'_2) \) and \( i \neq i \) and pair wise Re-continuous functions if \( f^{-1}(U) \) is \( \tau_i, \tau'_j \)-regular closed in \( (X, \tau_1, \tau_2) \) for each. \( \tau'_1 \)-open set \( U \) of \( (X', \tau'_1, \tau'_2) \) for \( i \neq i \).

Definition 2.13. A function \( f : (X_1, \tau_1, \tau_2) \to (X', \tau'_1, \tau'_2) \) called \( ij \)-contra continuous functions if \( f^{-1}(U) \) is \( \tau_i, \tau'_j \)-closed in \( (X, \tau_1, \tau_2) \) for each \( \tau'_j \). Open set \( U \) of \( (X', \tau'_1, \tau'_2) \) for \( i = 1, 2 \) and \( i \neq j \). \( f \) is said to be pair wise contra-continuous if it is both \( ij \)-contra continuous and same as \( ji \)-contra continuous function from \( (X, \tau_1, \tau_2) \) to \( (X', \tau'_1, \tau'_2) \).

Example 2.14. Let \( X = [a, b, c], \tau_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \) and \( \tau_2 = \{\emptyset, X, \{b, c\}\} \), \( X' = \{1, 2, 3\}, \tau'_1 = \{\emptyset, X', \{1\}\} \) and \( \tau'_2 = \{\emptyset, X', \{2\}\} \). Let \( f(X, \tau_1, \tau_2) \to (X', \tau'_1, \tau'_2) \) defined by \( f(a) = 1, f(b) = 3, f(c) = 2 \) then \( f \) is pair wise contra-continuous.

Example 2.15. Let \( X = [a, b, c], \tau_1 = \{\emptyset, X, \{a\}, \{a, c\}\} \) and \( \tau_2 = \{\emptyset, X, \{c\}\} \) then \( X' = \{1, 2, 3\}, \tau'_1 = \{\emptyset, X, \{1\}\} \) and \( \tau'_2 = \{\emptyset, X, \{2\}\} \) define \( f(X, \tau_1, \tau_2) \to (X', \tau'_1, \tau'_2) \) such that \( f(a) = 1, f(b) = 3, f(c) = 2 \). Then \( f \) is \( 1 \)-continuous but not \( 2,1 \) contra-continuous, because \( \{1\} \) is \( \tau'_1 \)-open set of \( X' \) but \( f^{-1}(1) = a \) is not \( \tau_2 \)-closed in \( X \) and \( 2 \)-continuous but not \( 12 \) contra-continuous because \( \{2\} \) is \( \tau'_2 \)-open set of \( X' \) but \( f^{-1}(\{2\}) = \{c\} \) is not \( \tau_1 \)-closed set of \( X \).

Theorem 2.16. A function \( f(X, \tau_1, \tau_2) \to (X', \tau'_1, \tau'_2) \), the following statements are equivalent.

(a) \( f \) is \( ij \)-contra continuous.

(b) For \( x \in X \) and each \( \tau'_j \)-closed set \( V \) in \( X' \) with \( f(x) \in V \), then there exist a \( \tau_i \)-open set \( U \) of \( X \) such that \( x \in U \) and \( f(U) \subset V \).

(c) The inverse image of each \( V \) a \( \tau'_j \)-closed set in \( X' \) is \( U \) an \( \tau_i \)-open set of \( X \).

Proof. Let \( x \in X \) and \( V \) is \( \tau'_j \)-closed set \( X' \) such that \( f(x) \in V \) then \( V \) is \( \tau'_j \)-open set of \( X' \). As \( f \) is \( ij \)-contra continuous then \( U = f^{-1}(V) = (f^{-1}(V))^c \) is \( \tau_i \)-closed set of \( X \) there for \( U = f^{-1}(V) \) is \( \tau_i \)-open set in \( X \) such that \( f(U) \subset V \) and \( x \in U \) show that \( (a) = (b) \). Further let \( V \) be a \( \tau'_j \)-closed set of \( X' \) then by definition \( ij \)-contra continuous function \( f \) there is a \( \tau_i \)-open set \( U \) of \( X \) s.t. \( f^{-1}(V) = U \) is \( \tau_i \)-open set of \( X \).

Shows that \( (b) = (c) \). Now let \( V \) be a \( \tau'_j \)-open set of \( Y \) then \( V \) is \( \tau'_j \)-closed set in \( Y \). Then \( f^{-1}(V) = (f^{-1}(V))^c \) is \( \tau'_i \)-open set in \( X \). Shows that \( f^{-1}(V) \) is \( \tau'_i \)-closed set in \( X \), as \( f \) is \( ij \)-contra continuous function shows that \( (c) = (a) \). This completes the proof. \( \square \)

References