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Continuous and Contra Continuous Functions in Bi-topological Spaces

Research Article

Parvinder Singh[1]*

1 P.G. Department of Mathematics, S.G.G.S. Khalsa College, Mahilpur (Hoshiarpur), India.

Abstract: The concept of bi-topological spaces was first introduced by J.C. Kelly [2] in 1963. Many authors such as Levine [3] contributed as he defined the semi-open sets and semi-continuity in bi-topological spaces. Maheshwari and Prasad [5] contributed semi-open sets and semi-continuity to bi-topological spaces. The notion of β-open sets contributed by Mashhour et. al. [6] and Andrijevic [1] define Semi pre-open sets. In this paper we discuss pre-continuity and semi pre-continuity in bi-topological spaces. LellisThivager et.al. [4] introduces g^*-closed sets topological spaces and initiated the concepts of ultra space by using $(1,2)\alpha$-open sets in bi-topological spaces and proved that each $(1,2)\alpha$-open sets is (1, 2) semi-open and (1, 2) pre-open but the converse of each is not true. R-Devi and S.Sampath Kumar and M. Caldas [7] introduced and studied a class of sets and maps between bi-topological spaces Called supra $\alpha-$open sets and supra α-continuous maps respectively.

Keywords: Bi-topological Spaces, Continuous Functions, Contra Continuous Functions.

1. Introduction and Preliminaries

Throughout the paper (X, τ_1, τ_2) and (X, τ) denote bi-topological and topological space. Let A be a subject of a topological space (X, τ) then cl(A) denote closure of A and Int(A) denote interior of A. A subject A of X is said to be semi open if there exist an open set U of X such that $U \subset A \subset cl(U)$ and A is said to be Semi-pre open if there exist a pre-open set U of X such that $U \subset A \subset cl(u)$. A function $f : (X_1, \tau_1) \to (X_2, \tau_2)$ is said to be semi-continues if for each $U \in \tau_2, f^{-1}(U)$ is semi-open in (X_1, τ_1) and A function $f : (X_1, \tau_1) \to (X_2, \tau_2)$ is said to be pre-continuous or β-continuous if for $U \in \tau_2, f^{-1}(U)$ is pre-open or β-open in (X_1, τ_1).

Definition 1.1.

(a) ***Pre-open Sets:*** *A set A of a bi-topological space (X, τ_1, τ_2) said to be pre-open w. r. t. (τ_1, τ_2) if there exist $U \in \tau_i$, such that $A \subset U \subset cl_i(A)$ or $A \subset int_i(cl_j(A))$.*

(b) *A subject A of (X, τ_1, τ_2) is said to be (τ_i, τ_j) semi-open if there exist an $U \in \tau_i$ such that $U \subset A \subset cl_j(U)$.*

(c) *A subject A of (X, τ_1, τ_2) is said to be (τ_i, τ_j) semi pre-open if there exist an (i, i) pre-open set U such that $U \subset A \subset cl_j(U)$.*

A subject A of x is said to be pair wise semi-open if it is $(1, 2)$ semi-open and $(2, 1)$ semi-open. The compliment of an (i, i) pre-open set is called (i, i) pre-closed set and (i, i) semi pre-open set is called (i, i) semi pre-closed set.

* *E-mail: parvinder070@gmail.com*

Example 1.2. *Let $X = [a, b, c, d]$, $\tau_1 = \{\emptyset, X, \{a\}, \{a, c\}\}$ and $\tau_2 = \{\emptyset, X, \{b\}, \{b, d\}\}$, then $A = \{b\}$ is pair wise pre-open but not τ_1-open.*

Definition 1.3. *Let (X, τ_1, τ_2) be a bi-topological space and A is a subset of X said to be*

(a) Generalized closed set (g-closed set) iff $cl_{\tau i}(A) \subseteq U$ where $A \subseteq U$ and U is τ_i-open set in (X, τ_i) for i = 1 or 2.

(b) Semi generalized closed set (sg-closed set) if $scl_{\tau i}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open set in (X, τ_i) for i= 1 or 2.

(c) Generalized semi closed (gs-closed) set Iff $scl_{\tau i}(A) \subseteq U$ where $A \subseteq U$ and U is open set in (X, τ_i) for i = 1 or 2.

(d) ψ Closed set if $scl_{\tau i}(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open set in (X, τ_i) for i=l or 2.

Definition 1.4. *Let (X, τ_1, τ_2) and (X', τ_1', τ_2') be two bi-topological spaces. Then a function $f : (X_1, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ is called*

(a) Pair wise continuous if $f^{-1}(U)$ is τ_i open-set in (X, τ_1, τ_2) for each τ_i' open set U of (X', τ_1', τ_2') for $i = 1, 2$.

(b) Pair wise semi continuous if $f^{-1}(U)$ is (τ_i, τ_i) semi-open in (X, τ_1, τ_2) for each τ_i' open set U of (X', τ_1', τ_2') for $i \neq i$ and I, $i = 1, 2$.

(c) Pair wise pre continuous if $f^{-1}(U)$ is (τ_i, τ_i) pre open in (X, τ_1, τ_2) for each τ_i' open set U of (X', τ_1', τ_2'), $i \neq I$ and I, $i = 1, 2$.

(d) Pair wise semi pre-continuous or pair wise β-continuous if $f^{-1}(U)$ is (τ_i, τ_i) semi pre-open in (X, τ_1, τ_2) for each τ_i' open-set U of (X', τ_1', τ_2') for $i \neq j$ and $i = 1, 2$. That is

τ_i open(closed)	$\longrightarrow$	*(τ_i, τ_i) semi-open (Semi-closed)*
$\downarrow$		$\downarrow$
τ_i, τ_i Pre-open(pre-closed)	$\longrightarrow$	*(τ_i, τ_i) semi pre-open (Semi pre-closed)*

2. Main Results

Theorem 2.1. *Let $f : (X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ be a pair wise continuous function. If A is an (i, j) pre -open set of X then f(A) is (i, j) pre open in X'.*

Proof. Let A be (i, j) pre-open in X, there exist $\in \tau_i$ such that $A \subset U \subset cl_j(A)$. As f is pair wise open, $f(U) \in \tau_i$. Also f is pair wise continuous we have $f(A) \subset f(U) \subseteq f(cl_j(A)) \subset cl_j(f(A))$. Shows that $f(A)$ is (i, j) pre-open in X'. Let A be (i, j) semi pre-open in X. Then by definition there exist an (i, j) pre-open set U such that $U \subset A \subset cl_j(A)$. As f is pair wise continuous we have $f(U) \subset f(A) \subseteq f(cl_j(U)) \subset cl_j(f(U))$. By first part we can say that $f(U)$ is (i, j) pre-open in X. Thus f(A)is (i,j) semi-open in X'. □

Example 2.2. *Let $X = [a, b, c, d]$, $\tau_1 = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\tau_2 = \{\emptyset, X, \{a\}\{b, d\}, \{a, c\}, \{a, b, d\}\}$ and $\tau_1' = \{\emptyset, X, \{b\}, \{b, c\}\}$ and $\tau_2' = \{\emptyset, X, \{d\}, \{a, c\}, \{a, c, d\}\}$.*

Let $f(X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ be the identity map. Then induced function $f : (X, \tau_1) \to (X, \tau_1')$ and $f : (X, \tau_2) \to (X, \tau_2')$ are both pre-continuous. Here f is not pair wise pre-continuous, because there is $\{b\} \in \tau_1'$ such that $f^{-1}b$ is not (i, j) pre-open.

Theorem 2.3. *Let $f(X, \tau_1, \tau_2) \rightarrow (X', \tau_1', \tau_2')$ be a pair wise continuous open function, A is an (i, j) pre-open set of X then $f(A)$ is (i, j) pre-open set in X'.*

Proof. Let A be (i, j) pre-open in X. then by definition there exist $U \in \tau_i$ such that $A \subset U \subset cl_j(A)$. As f is pair wise open then $f(U) \in \tau_1'$. Also f is pair wise continuous we have $f(A) \subset f(U) \subset f(cl_i(A)) \subset cl_i(f(A))$. Shows that $f(A)$ is (i, j) pre-open in X'. □

Remark 2.4. *If A is (i, j) semi pre-open set of X then $f(A)$ is (i, j) semi pre-open in X'.*

Theorem 2.5. *Let $f(X, \tau_1, \tau_2) \rightarrow (X', \tau_1', \tau_2')$ be pair wise pre-continuous open function. A be an (i, j) pre-open set of X', then $f^{-1}(A)$ is (i, j) pre-open in X.*

Proof. Let A is (i, j) pre-open in X', then there exist $V\imath\tau_1'$ such that $A \subset V \subseteq cl_j(A)$. Since f is pair wise open then $f^{-1}(A) \subset f^{-1}(V) \subset f^{-1}(cl_j(A)) \subset cl_j(f^{-1}(A))$. As f is pair wise pre-continuous $f^{-1}(V)$ is (i, j) pre open in X. Thus by last theorem $f^{-1}(A)$ is (i, j) pre-open in X. □

Remark 2.6. *If A is an (i, j) semi pre-open set of X' then $f^{-1}(A)$ is (i, j) semi pre-open in X.*

Theorem 2.7. *Let f be a function such that $f(X, \tau_1, \tau_2) \rightarrow (X', \tau_1', \tau_2')$ then the following statements are equivalent:*

(a) f is pair wise pre-continuous.

(b) Inverse image of each τ_i' closed set of X' is (i, j) pre-closed in X.

(c) For each $x \in X$ and each $A \in \tau_i'$ containing $f(x)$, there exist an (i, j) pre-open set B of X containing x such that $f(B) \subset A$.

(d) $(i, j) - pcl(f^{-1}(B)) \subseteq f^{-1}(cl_i(B))$ for every sub set B of X'.

(e) $f(i, j) - pcl(A) \subseteq cl_i(f(A))$ for every sub set A of X for $i \neq j$ and i, j=1, 2.

Theorem 2.8. *Let $f(X, \tau_1, \tau_2) \rightarrow (X', \tau_1', \tau_2')$ be a function and $X = U\{U_\lambda \in \tau_1 \cap \tau_2/\lambda \in \Delta\}$. Then f is pair wise pre-continuous if the restriction $f/U_\lambda : (U_\lambda, \tau_1/U_\lambda, \tau_2/U_\lambda) \rightarrow (X', \tau_1', \tau_2')$ is pair wise pre-continuous.*

Remark 2.9. *Let $f(X, \tau_1, \tau_2) \rightarrow (X', \tau_1', \tau_2')$ be a function and $X = U\{U_\lambda \in \tau_1 \cap \tau_2/\lambda \in \Delta\}$, then f is pairwise pre-continuous iff $f/U_\lambda : (U_\lambda, \tau_1/U_\lambda, \tau_2/U_\lambda) \rightarrow (X', \tau_1', \tau_2')$ is pair wise pre continuous for $\lambda \in \Delta$.*

Definition 2.10 (Product Topology). *Let $\{X_\lambda, \tau_1(\gamma), \tau_2(\gamma)/\lambda \in \Delta\}$ be a family of bi-topological spaces. Then (X, τ_1, τ_2) is called product space where $X = \prod_\lambda X_\lambda$ and τ_1, τ_2 denote product topology for $\{\tau_i(\gamma)/\gamma \in \Delta\}$ for $i = 1, 2$.*

Theorem 2.11. *Necessary and sufficient condition that a non empty subject A_λ of X_λ for $\lambda = \lambda_1, \lambda_2 \ldots \lambda_n$, Then $A = \prod_{i=n}^{n} A_{\lambda_i} x \prod_{\lambda \neq \lambda_i} X_\lambda$.*

Proof. Let A be (τ_i, τ_j) pre-open subset of X as natural projection is open and continuous and surjective then $\tau_i(\lambda_k)$, pre-open for $K = 1, 2, 3 \ldots n$. Proves that condition is sufficient. Now let A_{λ_k} be $\tau_i(\lambda_k)$ pre-open for each $K = 1, 2, 3 \ldots n$ then by definition of pre-open set there exist $\tau_i(\lambda_k)$ an open set B_{λ_k} such that $A_{\lambda_k} \subset B_{\lambda_k} \subset \tau_i(\lambda_k) - cl((A_{\lambda_k})$. Then we have $A \subset \prod_{k=1}^{n} B_{\lambda_k} x \prod_{\lambda \neq \lambda_k} X_\lambda \subset \prod_{k=1}^{n} \tau_i(\lambda_k) - cl((A_{\lambda_k}) x \prod_{\lambda \neq \lambda_k} X_\lambda = \tau_i - cl(A)$. As $\prod_{k=1}^{n} B_{\lambda_k} x \prod_{\lambda \neq \lambda_k} X_\lambda$ is τ_i open where A is (τ_i, τ_j) is pre-open in X. Now $\{X_\lambda, \tau_1(\gamma), \tau_2(\gamma)/\lambda \in \Delta\}$ and $\{Y_\lambda, \tau_1'(\lambda), \tau_2'(\lambda)/\lambda \in \Delta\}$ be two bi-topological spaces with same set of indices. Then $f_\lambda : (X_\lambda, \tau_1(\lambda), \tau_2(\lambda)) \rightarrow (Y_\lambda, \tau_1'(\lambda), \tau_2'(\lambda))$ is a function for each $\lambda \in \Delta$.
Then $f(X, \tau_1, \tau_2) \rightarrow (X_1', \tau_1', \tau_2')$ be the product function defined by $f(X_\lambda) = \{f_\lambda X_\lambda\}$ for each $\{X_\lambda\} \in X = \prod X_\lambda$ where τ_i and τ_i' are the product topologies for i=1,2. Proves that condition is sufficient. □

Definition 2.12 (Pair Wise Contra-Continuous Functions). *A function $f(X,\tau_1,\tau_2)\to(X',\tau_1',\tau_2')$ called pair wise c continuous function if $f^{-1}(U)$ is τ_i,τ_j-locally closed in (X,τ_1,τ_2) for each τ_1' Open set U of (X',τ_1',τ_2') and $i\neq i$ and pair wise Rc-continuous functions if $f^{-1}(U)$ is τ_i,τ_j-regular closed in (X,τ_1,τ_2) for each. τ_1'-open set U of (X',τ_1',τ_2') for $i\neq i$.*

Definition 2.13. *A function $f(X,\tau_1,\tau_2)\to(X',\tau_1',\tau_2')$ called ij-contra continuous functions if $f^{-1}(U)$ is τ_i-closed set in (X,τ_1,τ_2) for each τ_j' Open set U of (X',τ_1',τ_2') for $i=1,2$ and $i\neq j$. f is said to be pair wise contra-continuous if it is both ij-contra continuous and same as ji contra-continuous function from (X,τ_1,τ_2) to (X',τ_1',τ_2').*

Example 2.14. *Let $X=[a,b,c]$, $\tau_1=\{\emptyset,X,\{b\},\{a,b\}\}$ and $\tau_2=\{\emptyset,X,\{b,c\}\}$, $X'=\{1,2,3\}$, $\tau_1'=\{\emptyset,X',\{1\}\}$ and $\tau_2'=\{\emptyset,X',\{2\}\}$. Then let $f(X,\tau_1,\tau_2)\to(X',\tau_1',\tau_2')$ defined by $f(a)=1$, $f(b)=3$, $f(c)=2$ then f is pair wise contra-continuous.*

Example 2.15. *Let $X=[a,b,c]$, $\tau_1=\{\emptyset,X,\{a\},\{a,c\}\}$ and $\tau_2=\{\emptyset,X,\{c\}\}$, then $X'=\{1,2,3\}$, $\tau_1'=\{\emptyset,X,\{1\}\}$ and $\tau_2'=\{\emptyset,X,\{2\}\}$ define $f(X,\tau_1,\tau_2)\to(X',\tau_1',\tau_2')$ such that $f(a)=1$, $f(b)=3$, $f(c)=2$. Then f is 1-continuous but not 2,1 contra-continuous, because $\{1\}$ is τ_1' open set of X' but $f^{-1}(1)=a$ is not τ_2-closed in X and 2-continuous but not 12 contra-continuous because $\{2\}$ is τ_2' open set of X' but $f^{-1}\{(2)\}=\{c\}$ is not τ_1 closed set of X.*

Theorem 2.16. *A function $f(X,\tau_1,\tau_2)\to(X',\tau_1',\tau_2')$, the following statements are equivalent.*

(a) f is ij-contra continuous.

(b) For $x\in X$ and each τ_j' closed set V in X' with $f(x)\in V$, then there exist a τ_i-open set U of X such that $x\in U$ and $f(U)\subset V$.

(c) The inverse image of each V a τ_j' closed set in X' is U an τ_i- open set of X.

Proof. Let $x\in X$ and V is τ_j' closed set X' such that $f(x)\in V$ then $\bar{V}$ is τ_j' open set of X'. As f is ij- contra continuous then $\bar{U}=f^{-1}(\bar{V})=(f^{-1}(V))^c$ is τ_i closed set of X there for $U=f^{-1}(V)$ is τ_i open set in X such that $f(U)\subset V$ and $x\in U$ show that $(a)=(b)$. Further let V be a τ_j' closed set of X' then by definition ij-contra continuous function f there is a τ_i-open set U of X sit $f^{-1}(V)=U$ is τ_i-open set of X.
Shows that (b)=(c). Now let V be a τ_j'-open set of Y then $\bar{V}$ is τ_j'-closed set in Y. Then $f^{-1}(\bar{V})=(f^{-\bar{1}}(V))$ is τ_i-open set in X. Shows that $f^{-1}(V)$ is τ_i-closed set in X, as f is ij-contra continuous function shows that $(c)=(a)$. This completes the proof. ☐

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