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The New Proof of Euler's Inequality Using Spieker Center

Research Article

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Abstract: If R is the Circumradius and r is the Inradius of a non-degenerate triangle then due to EULER we have an inequality referred as "Euler's Inequality" which states that $R \geq 2r$, and the equality holds when the triangle is Equilateral. In this article let us prove this famous inequality using the idea of 'Spieker Center '.

Keywords: Euler's Inequality, Circumcenter, Incenter, Circumradius, Inradius, Cleaver, Spieker Center, Medial Triangle, Stewart's Theorem.

1. Historical Notes [4]

In 1767 Euler analyzed and solved the construction problem of a triangle with given orthocenter, circum center, and in center. The collinearlity of the Centroid with the orthocenter and circum center emerged from this analysis, together with the celebrated formula establishing the distance between the circum center and the in center of the triangle. The distance d between the circum center and in center of a triangle is given by $d^2 = R(R - 2r)$ where R, r are the circum radius and in radius, respectively. This formula oftenly called as Euler's triangle formula.

An immediate consequence of this theorem is $R \geq 2r$, which is often referred as Euler's triangle inequality. According to Coxeter, although this inequality had been published by Euler in 1767, it had appeared earlier in 1746 in a publication by William Chapple. This ubiquitous inequality occurs in the literature in many different equivalent forms.
For Example,

$$1 \geq 8 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2},$$

$$r_a + r_b + r_c \geq 9r,$$

$$abc \geq (a + b - c)(b + c - a)(c + a - b),$$

$$(x + y)(y + z)(z + x) \geq 8xyz,$$

Where $a, b, c, A, B, C, r_a, r_b, r_c$ denote the sides, angles and exradii of the triangle respectively, and x, y, z are arbitrary nonnegative real numbers such that $a = y + z$, $b = x + z$, $c = x + y$. A proof for the last inequality form follows immediately

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from the product of the three obvious inequalities $y + z \geq 2\sqrt{yz}$, $z + x \geq 2\sqrt{zx}$ and $x + y \geq 2\sqrt{xy}$.

Many other simple approaches are known (some of them can be found in [2], [4], [11] and [12]). Here in this article let us prove this inequality with less obvious approach, by making use of properties of Spieker Center.

2. Formal Definitions

Definition 2.1 (Medial Triangle)**.** *If $\triangle ABC$ is a given triangle and let D, E and F are the mid points of the sides BC, CA and AB, then the triangle formed by joining the points D,E and F is called as Medial triangle of the triangle ABC.*

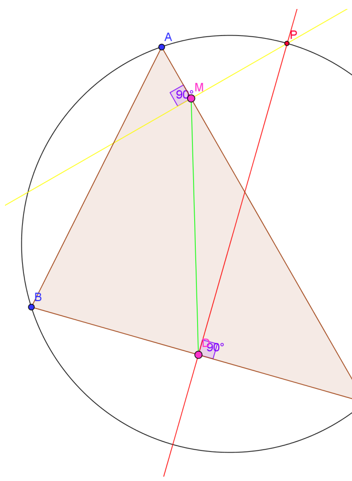
Definition 2.2 (Stewart's Theorem)**.** *In geometry, Stewart's Theorem [6] yields a relation between a lengths of the sides of the triangle and the length of a cevian of the triangle. Its name is in honor of the Scottish mathematician Matthew Stewart who published the theorem in 1746.*

Let a, b and c be the lengths of the sides of a triangle. Let be the length of a cevian to the side of length a. If the cevian divides a into two segments of lengths m and n, with m adjacent to c and n adjacent to b, then Stewart's Theorem [6] states that

$$l^2 = \frac{mb^2}{a} + \frac{nc^2}{a} - mn$$

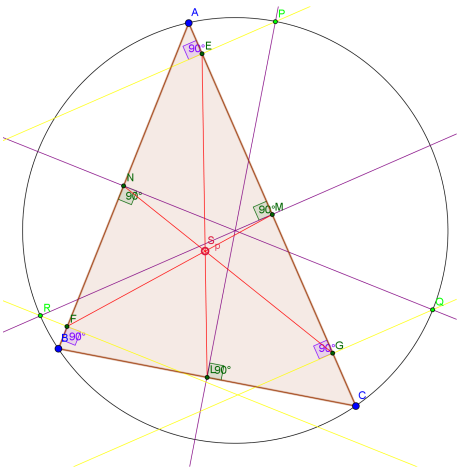
Definition 2.3 (Cleaver)**.** *Cleaver of the triangle is a line segment that bisects the perimeter of the triangle and has one endpoint at the midpoint of one of the three sides. In the triangle we have three cleavers and all the three cleavers are concurrent at Spieker center.*

Let $\triangle ABC$ is a given triangle whose Circumcenter is S. let P be the point of intersection of Perpendicular bisector SD of the side BC with the Circumcircle of $\triangle ABC$. If M, N are the foot of perpendiculars drawn from P to the sides AB and AC respectively then the points M, N and D are collinear (by the property of simson line with respect to the point P) and The line through the points M, N and D is called as Cleaver [5].



In the above figure $\triangle ABC$ is a given triangle, S is its Circumcenter, SD is the perpendicular bisector of the side BC, P is the point of intersection of Circumcircle of $\triangle ABC$ with the perpendicular bisector SD of side BC, M is the feet of perpendicular drawn from P to the side AC and MD is the required Cleaver.

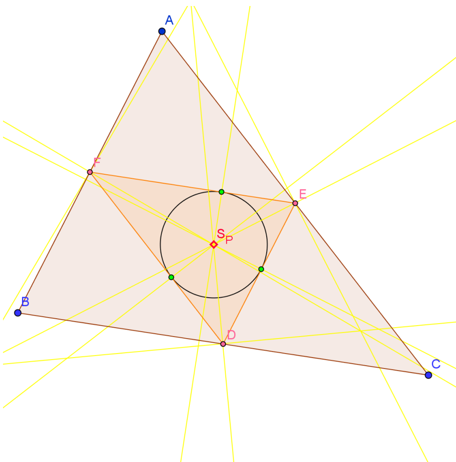
Definition 2.4 (Spieker Center (S_P)). *In geometry, the Spieker center is a special point associated with a plane triangle. It is defined as the center of mass of the perimeter of the triangle. The Spieker center of a triangle is the concurrent point of three cleavers of the triangle. The point is named in honor of the* 19^{th} *century German geometer Theodor Spieker. The Spieker center is a triangle center and it is listed as the point X(10) in Clark Kimberling's Encyclopedia of Triangle Centers.*



In the above figure $\triangle ABC$ is given triangle, the lines PL, QN, RM are perpendicular bisectors of the sides BC,CA,AB with P,Q, R lies on the circumcircle of $\triangle ABC$, the points E,G,F are the foot of the perpendiculars drawn from the points P,Q,R to the sides AC, AC, AB respectively, now the lines EL,GN, FM are Cleavers and their point of intersection S_P is our Spieker center. The most interesting property of this triangle center is as follows

"The Spieker Center of the given triangle ABC is the Incenter of the medial triangle of triangle ABC"[4].

That is, the Spieker center of triangle ABC is the center of the circle inscribed in the medial triangle of triangle ABC. This circle is known as the Spieker circle.



In the above figure the points D,E,F are the mid points of the sides BC,CA, AB of ABC and clearly Spieker center (S_P) is the incenter of medial triangle $\triangle DEF$.

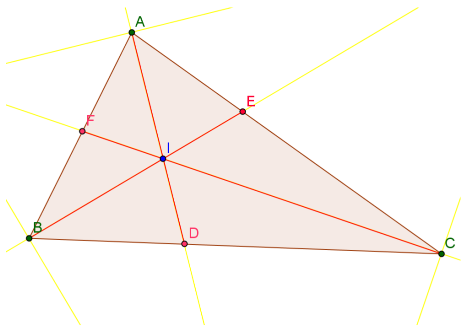
The Spieker center (S_P) is collinear with the incenter(I), the Centroid(G) and the Nagel Point (N_G) of the triangle ABC. Moreover, $N_GS_P : S_PG : GI = 3 : 1 : 2$. Before going to prove our main theorem related to Spieker center, let us prove some lemma's.

Lemma 2.5. *If I is the Incenter of the triangle ABC whose sides are a, b and c and M be any point in the plane of the triangle then*

$$IM^2 = \frac{aAM^2 + bBM^2 + cCM^2 - abc}{a+b+c}$$

Proof. Let $\triangle ABC$ is the given triangle whose lengths of sides are $BC = a$, $CA = b$ and $AB = c$ respectively. Let AD, BE are the angular bisectors of the angles A, B and I (Incenter) is their point of intersection.

Step : 1 We know by vertical angular bisector theorem, BD : DC = AB : AC = c : b and CE : EA = a : c. Thus $BD = \frac{ca}{c+b}$,

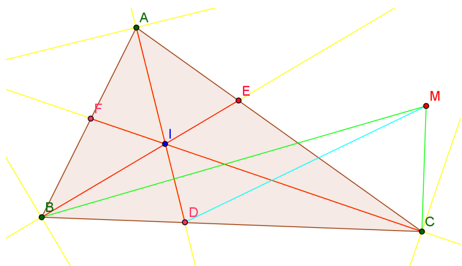


$CD = \frac{ba}{c+b}$ and $CE = \frac{ab}{c+a}$, $EA = \frac{cb}{c+b}$. And AD is the cevian for the $\triangle ABC$, so by Stewart's theorem, we have

$$AD^2 = \frac{BD.AC^2}{BC} + \frac{CD.AB^2}{BC} - BD.DC = \frac{bc(b+c+a)(b+c-a)}{(b+c)^2}$$

Now by Menelaus Theorem, since the line BIE is a transversal for $\triangle ADC$, $\frac{AE}{EC}\frac{CB}{BD}\frac{DI}{IA} = 1 \Rightarrow AI : ID = (b+c) : a$. Thus $AI = \frac{b+c}{c+b+a}AD$ and $ID = \frac{a}{c+b+a}AD$.

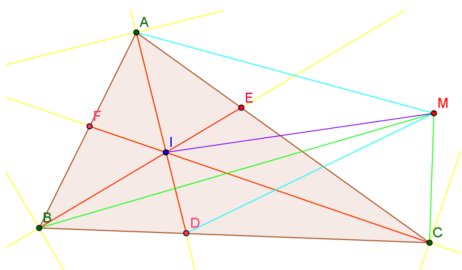
Step : 2



Let M be any point of in the plane of $\triangle ABC$, since DM is cevian for $\triangle BMC$. Hence by applying Stewart's theorem for $\triangle BMC$, we get

$$DM^2 = \frac{BD.CM^2}{BC} + \frac{CD.BM^2}{BC} - BD.DC = \frac{cCM^2}{b+c} + \frac{bBM^2}{b+c} - \frac{a^2bc}{(b+c)^2}.$$

Step : 3



Now for $\triangle AMD$, IM is a cevian, so again by Stewart's theorem, we have

$$\begin{aligned} IM^2 &= \frac{AI.DM^2}{AD} + \frac{ID.AM^2}{AD} - AI.ID \\ &= \left(\frac{b+c}{a+b+c}\right)\left(\frac{cCM^2}{b+c} + \frac{bBM^2}{b+c} - \frac{a^2bc}{(b+c)^2}\right) + \frac{aAM^2}{a+b+c} - \frac{a(a+b)AD^2}{(a+b+c)^2} \qquad \text{(Using Step 1 and 2)} \\ &= \left[\sum_{a,b,c} \frac{aAM^2}{a+b+c}\right] - \frac{abc(a+b+c-a)}{(a+b+c)(b+c)} \end{aligned}$$

Therefore,

$$IM^2 = \frac{aAM^2 + bBM^2 + cCM^2 - abc}{a+b+c}$$

□

Lemma 2.6. *If a, b, c are the sides of the triangle ABC, and if s,R,r and $\triangle$ are semi perimeter, Circumradius, Inradius and area of the triangle ABC respectively then*

1. $ab + bc + ca = r^2 + s^2 + 4Rr$
2. $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$
3. $a^3 + b^3 + c^3 + abc = 2s(s^2 - 3r^2 - 4Rr)$
4. $s^2 \geq 3(r^2 + 4Rr)$ *and the equality holds when* $a = b = c$.

Proof. We know that $2s = a + b + c$, $\triangle = rs$ and $abc = 4R\triangle = 4Rrs$. And by Heron's formula for the area of the triangle, we have $\triangle^2 = s(s-a)(s-b)(s-c) = s[s^3 - s^2(a+b+c) + s(ab+bc+ca) - abc]$. This can be rewritten as $r^2s^2 = s^2\left[-s^2 + (ab+bc+ca) - 4Rr\right]$. Which gives

$$ab + bc + ca = r^2 + s^2 + 4Rr \tag{1}$$

Now let us consider a cubic polynomial equation in the variable t whose roots are a, b and c as $t^3 + A_1t^2 + A_2t + A_3 = 0$, where A_1, A_2 and A_3 are coefficients such that

$$A_1 = -(a+b+c), A_2 = -(ab+bc+ca) \text{ and } A_3 = -(abc)$$

Now we know by Newtons Identities,

$$\begin{aligned} &S_1 + A_1 = 0 \\ &S_2 + S_1A_1 + 2A_2 = 0 \\ &S_3 + S_2A_1 + S_1A_2 + 3A_3 = 0 \text{ where } S_k = a^k + b^k + c^k,\ k \in W \end{aligned}$$

And by some computation we obtain

$$S_2 = a^2 + b^2 + c^2 = 2\left(s^2 - r^2 - 4Rr\right) \tag{2}$$

In the similar manner we can obtain $S_3 = a^3 + b^3 + c^3 = 2s\left(s^2 - 3r^2 - 6Rr\right)$. Hence

$$a^3 + b^3 + c^3 + abc = 2s\left(s^2 - 3r^2 - 4Rr\right) \tag{3}$$

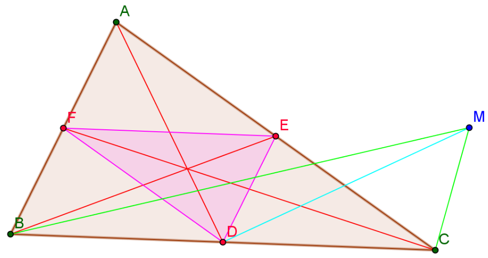
And now for an arbitrary real a, b and c, we have $(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$. It implies that $a^2+b^2+c^2 \geq ab+bc+ca$. Now using (1) and (2), above inequality can be rewritten as $2s^2 - 2r^2 - 8Rr \geq r^2 + s^2 + 4Rr$. Which gives our desired inequality,

$$s^2 \geq 3\left(r^2 + 4Rr\right) \tag{4}$$

and the equality holds when $a = b = c$. □

Theorem 2.7. *If S_P is the Spieker center of ABC and M be any point in the plane of triangle then*

$$S_PM^2 = \frac{2(b+c)AM^2 + 2(c+a)BM^2 + 2(a+b)CM^2 - \left(a^3+b^3+c^3+abc\right)}{4(a+b+c)}$$



Proof. Let D, E and F are the midpoints of sides BC, CA and AB respectively of triangle ABC and M be any point in the plane of triangle. It is clear that DM is cevian for triangle BMC, so by Stewart's theorem,

$$DM^2 = \frac{BD.CM^2}{BC} + \frac{CD.BM^2}{BC} - BD.DC = \frac{CM^2}{2} + \frac{BM^2}{2} - \frac{a^2}{4} \qquad \text{(since BD = CD = BC/2 = a/2)}$$

So $4a\,DM^2 = 2a\,CM^2 + 2a\,BM^2 - a^3$. Similarly,

$$4b\,EM^2 = 2b\,CM^2 + 2b\,AM^2 - b^3$$

$$4c\,FM^2 = 2c\,AM^2 + 2c\,BM^2 - c^3$$

Since D,E and F are the midpoints of sides of the triangle, we have $DE = c/2, EF = a/2, FD = b/2$ i.e., $\triangle DEF$ is medial triangle whose sides are $a/2$, $b/2$ and $c/2$ we know that The In center of medial triangle DEF is Spieker center of $\triangle ABC$ [4]. Hence by Lemma 2.5, we have

$$S_PM^2 = \frac{\left(\frac{a}{2}\right)DM^2 + \left(\frac{b}{2}\right)EM^2 + \left(\frac{c}{2}\right)FM^2 - \left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)}{\left(\frac{a}{2}+\frac{b}{2}+\frac{c}{2}\right)}$$

(by replacing I = SP and $a = \left(\frac{a}{2}\right)$, $b = \left(\frac{b}{2}\right)$, $c = \left(\frac{c}{2}\right)$ and $A = D$, $B = E$, $C = F$ in Lemma 2.5). It follows that

$$S_PM^2 = \frac{4a\,DM^2 + 4b\,EM^2 + 4c\,FM^2 - abc}{(a+b+c)}$$

Therefore

$$S_PM^2 = \frac{2(b+c)AM^2 + 2(c+a)BM^2 + 2(a+b)CM^2 - \left(a^3+b^3+c^3+abc\right)}{4(a+b+c)}$$

This completes the proof of main theorem related to the speiker center. □

Corollary 2.8. *Let M = Circum center = S, so $AM = AS = BM = BS = CM = CS = R$ = Circumradius. Hence*

$$S_PS^2 = \frac{4R^2(a+b+c) - \left(a^3+b^3+c^3+abc\right)}{4(a+b+c)}$$

Now let us prove the Euler's Inequality

3. Main Result

Theorem 3.1 (Euler's Inequality). *If R is the Circumradius and r is the Inradius of triangle ABC then and the equality holds when the triangle is equilateral.*

Proof. We have by corollary of theorem, $S_PS^2 = \frac{4R^2(a+b+c)-\left(a^3+b^3+c^3+abc\right)}{4(a+b+c)}$. Since the square of any real is non neagative, we have $S_PS^2 \geq 0$ and it is clear that the equality holds when S_P, S coincides with each other, this can be happen when the triangle is equilateral. That is $\frac{4R^2(a+b+c)-\left(a^3+b^3+c^3+abc\right)}{4(a+b+c)} \geq 0$. It follows that $R^2 \geq \frac{a^3+b^3+c^3+abc}{4(a+b+c)}$. This can be rewritten as $4R^2 \geq \left(s^2 - 3r^2 - 4Rr\right) \geq 12Rr - 4Rr \geq 8Rr$ (Using (3) and (4) in Lemma 2.6) Which gives our desired Eulers Inequality, $R \geq 2r$. Hence proved. □

Remark 3.2. *The Euler's Inequality can also be proved directly by using Lemma 2.5. Since by Lemma 2.5, for any point M, we have*

$$IM^2 = \frac{a\,AM^2 + b\,BM^2 + c\,CM^2 - abc}{a+b+c}.$$

Now if we fix M= Circumcenter = S, then by using the fact that $AM = BM = CM = AS = BS = CS = R$= Circumradius, we have $IS^2 = R^2 - \frac{abc}{a+b+c}$ and it is well known that $\frac{abc}{a+b+c} = 2Rr$. So $IS^2 = R^2 - 2Rr$, which is desired Eulers triangle formula. Now, Since the square of any real is non neagative, we have $IS^2 \geq 0$ and it is clear that the equality holds when the triangle is equilateral. It implies our desired Euler's Inequality, $R \geq 2r$.

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