International Journal of Mathematics And its Applications
Volume 3, Issue 4–F (2015), 1–28.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Locally Linear Convex Maps and H-derivation

Research Article

Franco Fineschi[1]*

1 Department DISAG, University of Siena, Piazza S.Francesco, 53100 Siena, Italy.

Abstract: Linear convex maps are considered. The linearity of a map is related to a point. The space of functions with this property and the analytic form is obtained. A new polynomial for a function improves the convergence.

Keywords: Linearity, convex, h-derivation.

1. Introduction

A map $\phi : \Re^n \to \Re^m$ is said to be affine, see [9], when $\phi(\lambda x_1 + (1-\lambda)x_2) = \lambda\phi(x_1) + (1-\lambda)\phi(x_2)$ for all $x_1, x_2 \in \Re^n$ and all $\lambda \in \Re$. If $0 \le \lambda \le 1$, then ϕ is said a linear convex (l.c.) map. Applications of l.c. maps are in game theory and convex analysis , see [4] or [2]. Some algebraic properties of the class of the affine and l.c. maps are considered. In order to show a complete description, some propositions, without proofs, are recalled from the paper [2].

It is possible to reduce the linearity of a map to the neighborhood of a fixed point. This new definition allows to considerate a wide class, really a linear space, $Lc(b)$, of maps which satisfy this property. The analytic form of these functions is obtained as solution of a first order PDE. As an important obtained result, the space of the continuous linear functionals on $\Re^n$ is a subspace of $Lc(b)$, this opens the way to many extensions of known properties. The study of the topological properties of the l.c. maps, with respect to a point, is only started because of dimensiononal limit of the paper.

By l.c. maps a wider definition of differentiability is obtained. Functions, not differentiable at a point, may be l.c. differentiable at the same point. The l.c. maps have a geometrical meaning as cones. The derivatives in a Taylor's polynomial are multilinear functions so that the Taylor's formula may be written by cones.

A new definition for derivatives allows to consider a new development for functions, denoted by h-polynomial. Pointwise and mean square convergence of the h-polynomial are studied in order to improve the known developments. Applications of the new derivatives are considered in complex analysis.

2. Multilinear Convex Maps

Definition 2.1. *Let A be a subset of $\Re^n$ and let $C \subset \Re^m$, a k-linear convex mapping $\phi : A^k \to C$, for $a_i \in A$, is defined by $\phi(a_1, \dots, a_i, \dots, a_k) = \phi(a_1, \dots, \sum_{i=1}^r \lambda_i b_i, \dots, a_k) = \sum_{i=1}^r \lambda_i \phi(a_1, \dots, b_i, \dots, a_k)$, where $\lambda_i \ge 0$, $\sum_{i=1}^r \lambda_i = 1$, $a_i = \sum_{i=1}^r \lambda_i b_i$, and $b_i \in \Re^n$.*

* *E-mail: franco.fineschi@unisi.it*

Note that if a vector b_j, in the convex combination $a_i = \sum_{i=1}^{r} \lambda_i b_i$, is not at A, then $\phi(a_1, \dots, b_j, \dots, a_k)$ is not defined.

Proposition 2.2. *Let $\phi : (\Re^n)^k \to C$ be a k-linear map, then the restriction of ϕ to the bounded subset $A^k \subset (\Re^n)^k$ with $A = \{(a_{1i}, \dots, a_{ji}, \dots, a_{ni}) : \ s_{ji} \leq a_{ji} \leq r_{ji}, \ j = 1, \dots, n\}$ is not k-linear, instead the restricted map ϕ is k-linear convex.*

Proof. Let $\frac{s_{ji}}{2} < v_{ji} < w_{ji} < r_{ji}, \ j = 1, \dots, n,$, then there exist $\phi((a_1, \dots, v_i, \dots, a_k)$ and $\phi(a_1, \dots, w_i, \dots, a_k)$ even if $\phi(a_1, \dots, v_i + w_i, \dots, a_k)$ does not exist. So ϕ is not k-linear. Instead, with $\lambda \in [0, 1]$,

$$\phi(a_1, \dots, \lambda v_i + (1-\lambda) w_i, \dots, a_k) = \lambda \phi(a_1, \dots, v_i, \dots, a_k) + (1-\lambda)\phi(a_1, \dots, w_i, \dots, a_k)$$

and ϕ is k-linear convex. □

Example 2.3. *Consider the function $f(x, y) = 2xy \qquad 0 \leq x \leq a, \qquad 0 \leq y \leq b,$ let $\frac{a}{2} < x_1 < x_2 < a$, then $f(x_1, y) = 2x_1 y$ and $f(x_2, y) = 2x_2 y$, even if $f(x_1 + x_2, y)$ does not exist, so $f(x, y)$ is not a bilinear function. Whereas, for $0 \leq \lambda \leq 1$,*

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x_2, y) &= 2(\lambda x_1 + (1-\lambda)x_2)y \\ &= 2\lambda x_1 y + 2(1-\lambda)x_2 y \\ &= \lambda f(x_1, y) + (1-\lambda) f(x_2, y) \end{aligned}$$

that is, $f(x, y)$ is a convex linear function of each variable separately.

Some elementary properties of the k-linear convex maps follow. Let X be a convex subset of $\Re^n$, $\forall a_i \in X, \ \alpha \in [0, 1]$, $\alpha a_i + (1-\alpha)\underline{0} = \alpha a_i \in X$. In particular $\alpha a_i \in X$. Moreover

$$\begin{aligned} \Phi(a_1, \dots, \alpha a_i, \dots, a_k) &= \Phi(a_1, \dots, \alpha a_i + (1-\alpha)\underline{0}, \dots, a_k) \\ &= \alpha \Phi(a_1, \dots, a_i, \dots, a_k) + (1-\alpha)\Phi(a_1, \dots, \underline{0}, \dots, a_k) \end{aligned} \tag{1}$$

where $\underline{0}, a_1, \dots, a_k$ are vectors in X.

Proposition 2.4. *Let $\lambda \in \Re^+$ $(\lambda \in \Re^-)$ and $\lambda x \in X$, then $\alpha x \in X$ $(-\alpha x \in X)$, for $\alpha \in [0, 1]$.*

Proof. If $0 < \beta < 1$ satisfies $\alpha = \beta \cdot \lambda$, then $\alpha x = \beta(\lambda x) + (1-\beta)\underline{0}$, so $\alpha x \in X$. If $-\alpha = \beta \cdot \lambda$ then $-\alpha x = \beta(\lambda x) + (1-\beta)\underline{0}$ and $-\alpha x \in X$. □

Proposition 2.5. *Let $\phi : A^k \to C$ be a k-linear convex map and let ϕ defined on the vectors $\underline{0}, a_1, \dots, a_i, \dots, a_k$. Then, with $\lambda \in [0, 1]$,*

(i) $\phi(a_1, \dots, \lambda a_i, \dots, a_k) + \phi(a_1, \dots, (1-\lambda)a_i, \dots, a_k) = \phi(a_1, \dots, a_i, \dots, a_k) + \phi(a_1, \dots, \underline{0}, \dots, a_k)$.

(ii) $2\phi(a_1, \dots, \frac{1}{2}a_i, \dots, a_k) = \phi(a_1, \dots, a_i, \dots, a_k) + \phi(a_1, \dots, \underline{0}, \dots, a_k)$.

(iii) $\phi(a_1, \dots, \underline{0}, \dots, a_k) = \frac{1}{2}((\phi(a_1, \dots, a_i, \dots, a_k) + \phi(a_1, \dots, -a_i, \dots, a_k)$.

Proof. (i)

$$\begin{aligned} \phi(a_1, \dots, \lambda a_i, \dots, a_k) &= \phi(a_1, \dots, \lambda a_i + (1-\lambda)\underline{0}, \dots, a_k) \\ &= \lambda \phi(a_1, \dots, a_i, \dots, a_k) + (1-\lambda)\phi(a_1, \dots, \underline{0}, \dots, a_k) \\ \phi(a_1, \dots, (1-\lambda)a_i, \dots, a_k) &= \phi(a_1, \dots, (1-\lambda)a_i + \lambda\underline{0}, \dots, a_k) \\ &= (1-\lambda)\phi(a_1, \dots, a_i, \dots, a_k) + \lambda\phi(a_1, \dots, \underline{0}, \dots, a_k) \end{aligned}$$

(i) is obtained summing the two relations.

(ii) The (i) for $\lambda = \frac{1}{2}$.

(iii)

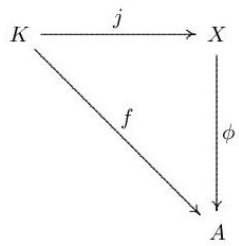
$$\begin{aligned}\phi(a_1, \dots, \underline{0}, \dots, a_k) &= \phi(a_1, \dots, (\frac{1}{2}a_i + \frac{1}{2}(-a_i)), \dots, a_k) \\ &= \frac{1}{2}\phi(a_1, \dots, a_i, \dots, a_k) + \frac{1}{2}\phi(a_1, \dots, -a_i, \dots, a_k)\end{aligned}$$

□

3. Free Convex Sets

A first application of k-linear convex maps is the definition of a convex free set. This concept is useful in order to define algebraic structures as free modules, vector spaces and so on.

Definition 3.1. *Let K be a subset of a convex set X in $\Re^n$ and let $j : K \to X$ be the insertion of K in X. Denote by A a subset of $\Re^m$, then X is free over K if, for every function $f : K \to A$, an unique linear convex mapping $\phi : X \to A$ exists such that $\phi \circ j = f$, as in the following commutative diagram*



The next proposition, recalled from [2], extends the 1 and defines a linear convex mapping if an its argument is outside the body.

Proposition 3.2. *Let $x_1, \dots, x_i, \dots, x_k$ be vectors in X and $\delta \in \Re$, then a linear convex mapping $\phi : X^k \to A$ satisfies*

$$\phi(x_1, \dots, \delta x_i, \dots, x_k) = \delta\phi(x_1, \dots, x_i, \dots, x_k) + (1 - \delta)\phi(x_1, \dots, \underline{0}, \dots, x_k) \tag{2}$$

Theorem 3.3. *Let $\phi : X^k \to Y$ be a k-linear convex function and X a convex set with $\underline{0} \in X$, then $\phi(a_1, \dots, a_k)$ may be expressed by a linear combination of $\phi(x_{j_1}, \dots, x_{j_k})$, where $x_{j_i} \in X$ span the vectors $a_i \in X$.*

In the n-dimensional vector space $\Re^n$, denote by S_n the convex hull of the vectors $\{e_1, \dots, e_n\}$ of the standard basis. S_n is a compact, connected, convex set and its elements may be expressed by convex combinations of the unit vectors $\{e_1, \dots, e_n\}$.

Theorem 3.4. *The set S_n is free over the standard basis $\{e_1, \dots, e_n\}$ of $\Re^n$.*

By the Fenchel-Bunt' theorem, any element a of a compact, connected, convex set A is expressed as a convex combination of the sequence $a_1, \dots, a_n$ of vectors of A, that is $a = \xi_1 a_1 + \cdots + \xi_n a_n$. By the theorem 3.4 exists an unique linear convex function ϕ such that $\phi(\xi_1 e_1 + \cdots + \xi_n e_n) = \sum \xi_i a_i = a = \sum \xi_i \phi(e_i)$ so, any element $a \in A$ may be expressed as a convex combination of the vectors $\phi(e_i), \dots, \phi(e_n)$. In other words, any $a \in A$ determines a linear convex function ϕ such that $\sum \xi_i \phi(e_i) = a$.

Example 3.5. *Let A be a convex, connected set in $\Re^2$. If $a = \xi_1 a_1 + \xi_2 a_2$, $\xi_i \geq 0$, $\sum \xi_i = 1$, $a_i = (a_{i1}, a_{i2})$ is an element of A, then, by the theorem 3.4, it follows $a = \phi(\xi_1 e_1 + \xi_2 e_2) = \xi_1 a_1 + \xi_2 a_2 = \xi_1 \phi(e_1) + \xi_2 \phi(e_2)$, where $\phi : S_2 \to A$ is linear convex. This implies $\phi(e_1) = a_1$, $\phi(e_2) = a_2$, and so*

$$\phi(x) = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} (x) \qquad x \in S_2$$

4. Affine and Linear Convex Maps with Respect to a Fixed Point

For an affine or linear convex (l.c.) map, the differentiability condition is showed by the next expression.

Proposition 4.1. *An affine or l. c. map $\phi : \Re^n \to C$, defined in some neighborhood of a, is differentiable at a if*

$$\phi(a) = \phi(b) - (b-a) \cdot \nabla\phi(a) - \|b-a\|\epsilon(t(b-a)) \tag{3}$$

where $a + t(b-a)$ is a point in the neighborhood of a, $0 < t < 1$, and $\nabla\phi(a)$ the gradient vector. The function $\epsilon(t(b-a)) \to 0$ as $t \to 0$.

Proof. By the differentiability condition is $\phi(a + t(b-a)) - \phi(a) = t(b-a) \cdot \nabla\phi(a) + \|t(b-a)\|\epsilon(t(b-a))$ by the convex linearity of ϕ

$$\begin{aligned} \phi((1-t)a + tb) - \phi(a) &= t(b-a) \cdot \nabla\phi(a) + \|t(b-a)\|\epsilon(t(b-a)) \\ (1-t)\phi(a) + t\phi(b) - \phi(a) &= t(b-a) \cdot \nabla\phi(a) + t\|b-a\|\epsilon(t(b-a)) \\ -\phi(a) + \phi(b) &= (b-a) \cdot \nabla\phi(a) + \|b-a\|\epsilon(t(b-a)) \end{aligned}$$

that is, the 3. □

The aim of the next definition is to reduce the linearity of a map to a neighborhood of a fixed point, that is, the property becomes local.

Definition 4.2. *Let A be a subset of $\Re^n$ and let $C \subset \Re^m$, a k-affine mapping $\phi : A^k \to C$ with respect to the fixed point $b = (b_1, \ldots, b_n)$, for $a_i, b \in A$, is defined by $\phi(a_1, \ldots, (1-\lambda)a_i + \lambda b, \ldots, a_k) = (1-\lambda)\phi(a_1, \ldots, a_i, \ldots, a_k) + \lambda\phi(a_1, \ldots, b, \ldots, a_k)$, where $\lambda \in \Re$. A k-linear convex mapping $\phi : A^k \to C$, with respect to the fixed point b, for $a_i, b \in A$, is defined by $\phi(a_1, \ldots, (1-\lambda)a_i + \lambda b, \ldots, a_k) = (1-\lambda)\phi(a_1, \ldots, a_i, \ldots, a_k) + \lambda\phi(a_1, \ldots, b, \ldots, a_k)$, where $0 \leq \lambda \leq 1$.*

The line segment connecting the points a_i and b can be represented in the parametric form $a_i + \lambda(b - a_i) = (1-\lambda)a_i + \lambda b$, $0 \leq \lambda \leq 1$. So the definition 4.2 imposes the linearity for any direction at b, that is, ϕ is linear in a neighborhood of b. By the above definition, the affinity and the convex linearity is restricted to an arbitrary fixed point, nevertheless a wide class of maps exists satisfying this property.

Example 4.3. *Consider the function $f(x,y) = (x - b_1)((\frac{y-b_2}{x-b_1})^2 + k) \qquad b_1, b_2, k \in \Re$ then, it is an affine or l.c. function with respect to the point (b_1, b_2). In fact*

$$\begin{aligned} f((1-t)x + tb_1, (1-t)y + tb_2) &= (1-t)\frac{(k(x-b_1)^2 + (y-b_2)^2)}{x - b_1} \\ &= (1-t)f(x,y) + tf(b_1, b_2) \end{aligned}$$

Observe that $f(x,y)$ is not a linear or l.c. function.

The affine and l. c. maps, with respect to a fixed point, satisfy an analytic property that characterizes themselves. Let E, F be normed vector spaces, and let $d = b - x$ be a direction at a fixed point $b \in E$. The directional derivative of $\phi : E \to F$ in that direction is denoted by $D\phi(x)(d)$, see, for example, [5], then

Theorem 4.4. *Let $\phi : U \subseteq E \to F$ be an affine or l. c. map, with respect to the point b, of class C^p in the open U , with $\|b - x\| = 1$, satisfies the relations*

(i)

$$\phi(x) = \phi(b) - D\phi(x)\,(b - x) \tag{4}$$

where the $D\phi(x)$ is the derivative of ϕ.

(ii)

$$D^k\phi(x)(b - x)^{(k)} = 0 \qquad\qquad k = 2, \ldots, p - 1 \tag{5}$$

where $D^k\phi(x)$ is the k-th derivative of ϕ at the point x.

(iii) $D\phi(x)b - D\phi(x)x = \phi(b) - \phi(x)$.

Proof. (i) With $0 < t < 1$ and by the convex linearity

$$\begin{aligned} D\phi(x)d &= \lim_{t\to 0} \frac{1}{t}(\phi(x + t(b - x)) - \phi(x) \\ &= \lim_{t\to 0} \frac{1}{t}(\phi((1 - t)x + t\,b) - \phi(x) \\ &= \lim_{t\to 0} \frac{1}{t}(-t\phi(x) + t\phi(b)) \\ &= -\phi(x) + \phi(b) \end{aligned}$$

(ii) The Taylor's formula of ϕ is $\phi(b) = \phi(x) + \frac{1}{1!}D\phi(x)(b-x) + \cdots + \frac{1}{(p-1)!}D^{p-1}\phi(x)(b-x)^{(p-1)} + \theta(b-x)$, where $(b-x)^{(k)}$ denotes the k-tuple $(b - x, \ldots, b - x)$. Comparing 4 and the Taylor's formula the (ii) follows.
(iii) It is well known, see [5], that the derivative mapping $Df(x) : E \to F$ is linear. □

Later it is showed that a l.c. function with respect to a point has a non null Hessian matrix even if it satisfies 5. The following example shows that a linear function satisfies the 4.

Example 4.5. *The function $\phi(x, y) = k(x, y)$, with $k \in \Re$, is linear on $\Re^2$ and*

$$\begin{aligned} \phi(b_1, b_2) = k(b_1, b_2) &= \phi(x, y) + \phi_x(x, y)(b_1 - x) + \phi_y(x, y)(b_2 - y) \\ &= k(x, y) + k(1, 0)(b_1 - x) + k(0, 1)(b_2 - y) \\ &= k(x, y) + (kb_1 - kx, 0) + (0, kb_2 - ky) \\ &= k(b_1, b_2) \end{aligned}$$

5. Real-valued Affine and Linear Convex Functions of a Real Variable

For functions of one real variable, the theorem 4.4 becomes the following proposition

Proposition 5.1. *The affine and l. c. derivable function $f : A \subset \Re \to \Re$, with respect to a fixed point $x_0 \in A$, satisfies*

$$f(x) = f(x_0) + f'(x)(x - x_0) \tag{6}$$

Proof. By the convex linearity, with $0 < t < 1$,

$$\begin{aligned} f'(x)(x_0 - x) &= \lim_{t\to 0} \frac{1}{t}(f(x + t(x_0 - x)) - f(x)) \\ &= \lim_{t\to 0} \frac{1}{t}(f((1-t)x + tx_0) - f(x)) \\ &= \lim_{t\to 0} \frac{1}{t}((1-t)f(x) + tf(x_0) - f(x)) \\ &= \lim_{t\to 0} \frac{1}{t}(-tf(x) + tf(x_0)) \\ &= -f(x) + f(x_0) \end{aligned}$$

□

By $x_0 = x + h$, the 6 may be written as $f(x+h) - f(x) = f'(x)h$, that is, the affine and l.c. functions , of one variable, with respect to a point, satisfy $\Delta f(x) = d\, f(x)$. The relation 6 is a simple ODE and its solution is

$$f(x) = f(x_0) + (x - x_0)k \tag{7}$$

with k an arbitrary real constant. The relation 7 characterizes the affine and l.c. functions, so these coincide with the affine and l.c. functions with respect to a point.

6. Affine and l. c. Functions of Two Variables, with Respect to a Point

The relation 4 is a very useful tool in order to determinate the wide class of the affine and l. c. maps with respect to a point . The simplest set of these maps is obtained by two variable functions. For a differentiable function $f : \Re^2 \to \Re$, the 4 becomes

$$f(x_1, x_2) = f(b_1, b_2) - (b_1 - x_1)f_{x_1}(x_1, x_2) - (b_2 - x_2)f_{x_2}(x_1, x_2) \tag{8}$$

the 8 is a first order PDE, see, for example, [7], and the general integral may be written as

$$\psi\left(\frac{f(x_1, x_2) - f(b_1, b_2)}{x_1 - b_1}, \frac{f(x_1, x_2) - f(b_1, b_2)}{x_2 - b_2}\right) = 0 \tag{9}$$

where ψ is an arbitrary function. Another form for the solution is

$$f(x_1, x_2) = f(b_1, b_2) + (x_1 - b_1)\psi\left(\frac{x_2 - b_2}{x_1 - b_1}\right) \tag{10}$$

and again ψ is an arbitrary function. The solutions 9 or 10 are linear convex functions with respect to the arbitrary point (b_1, b_2). This means that the solutions satisfy the relation of affine or convex linearity for every combination written in the form $\lambda_1(x_1, x_2) + \lambda_2(b_1, b_2)$, $\forall x_1, x_2, b_1, b_2 \in \Re$, $\lambda_1 + \lambda_2 = 1$ and (b_1, b_2) is a critical point of $f(x_1, x_2)$.

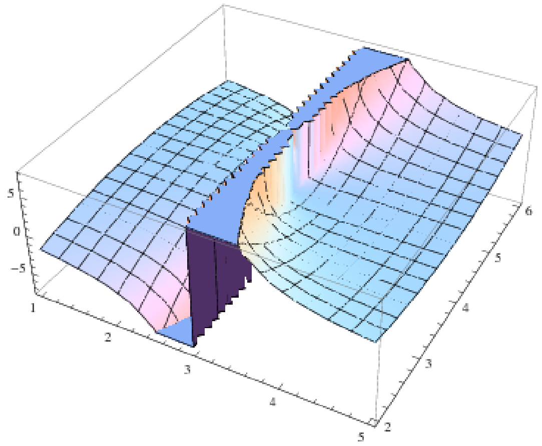
Example 6.1. *Choose the function* ψ *as* $\frac{f(x_1,x_2)-f(b_1,b_2)}{x_1-b_1} + \frac{f(x_1,x_2)-f(b_1,b_2)}{x_2-b_2} + 1 = 0$ *expressing with respect to* $f(x_1, x_2)$*, a solution of 8 is*

$$f(x_1, x_2) = f(b_1, b_2) - \frac{(x_1 - b_1)(x_2 - b_2)}{x_1 + x_2 - (b_1 + b_2)} \tag{11}$$

It is straightforward to verify the linear convexity of 11, that is $f((1-t)(x_1, x_2) + t(b_1, b_2)) = f(b_1, b_2) - \frac{(1-t)(x_1-b_1)(x_2-b_2)}{x_1+x_2-(b_1+b_2)}$ *is equal to* $(1-t)f(x_1, x_2) + tf(b_1, b_2)$.

Example 6.2. *Using the solution of 8 in the form 10, choose as a solution* $f(x_1,x_2) = f(b_1,b_2) + (x_1 - b_1)e^{\frac{x_2-b_2}{x_1-b_1}}$ *then* $f((1-t)(x_1,x_2) + t(b_1,b_2)) = f(b_1,b_2) + (1-t)(x_1-b_1)e^{\frac{x_2-b_2}{x_1-b_1}}$ *is equal to* $(1-t)f(x_1,x_2) + tf(b_1,b_2)$.

Example 6.3. *The graph of the l.c. function* $f(x_1,x_2) = \frac{(x_2-4)^2}{x_1-3}$ *with respect to the point* $(3,4)$ *is*



(Computer-generated graph).

The following proposition proves (ii) of 4.4 for two variable functions .

Proposition 6.4. *The affine or l.c. function set, with respect to the point* (b_1,b_2), $f(x_1,x_2) = f(b_1,b_2) + (x_1-b_1)\psi(\frac{x_2-b_2}{x_1-b_1})$, *with* ψ *an arbitrary, twice differentiable, function, satisfies*

$$(b_1 - x_1, b_2 - x_2)^T H(x_1,x_2)(b_1 - x_1, b_2 - x_2) = 0 \tag{12}$$

where H *is the Hessian matrix of* f.

Proof. By

$$f_{x_1} = \psi(\frac{x_2-b_2}{x_1-b_1}) + (\frac{b_2-x_2}{x_1-b_1})\psi'(\frac{x_2-b_2}{x_1-b_1}), \qquad f_{x_2} = \psi'(\frac{x_2-b_2}{x_1-b_1}) \text{ and}$$

$$f_{x_1x_1} = \frac{(x_2-b_2)^2}{(x_1-b_1)^3}\psi''(\frac{x_2-b_2}{x_1-b_1}), \qquad f_{x_1x_2} = \frac{-x_2+b_2}{(x_1-b_1)^2}\psi''(\frac{x_2-b_2}{x_1-b_1})$$

$$f_{x_2x_2} = \frac{1}{x_1-b_1}\psi''(\frac{x_2-b_2}{x_1-b_1}) \qquad H(x_1,x_2) = \frac{1}{x_1-b_1}\psi''(\frac{x_2-b_2}{x_1-b_1})\begin{pmatrix} (\frac{x_2-b_2}{x_1-b_1})^2 & \frac{-x_2+b_2}{x_1-b_1} \\ \frac{-x_2+b_2}{x_1-b_1} & 1 \end{pmatrix}$$

it follows 12 . □

Observe that the relation $(b_1 - x_1, b_2 - x_2)^T H(x_1,x_2)(b_1 - x_1, b_2 - x_2) = 0$ is true for every $\psi''(\frac{x_2-b_2}{x_1-b_1})$.

7. Affine and l. c. Functions of n Variables, with Respect to a Point

For a function $f : \Re^3 \to \Re$, the 4 becomes

$$f(x_1,x_2,x_3) = f(b_1,b_2,b_3) - (b_1 - x_1)f_{x_1}(x_1,x_2,x_3) - (b_2 - x_2)f_{x_2}(x_1,x_2,x_3) - (b_3 - x_3)f_{x_3}(x_1,x_2,x_3) \tag{13}$$

the 13 is a first order PDE, and the solution may be written as

$$f(x_1,x_2,x_3) = f(b_1,b_2,b_3) + (x_1 - b_1)\psi(\frac{x_2-b_2}{x_1-b_1}, \frac{x_3-b_3}{x_1-b_1}) \tag{14}$$

where ψ is an arbitrary function.

Example 7.1. *The function* $f(x_1, x_2, x_3) = \frac{(x_2-b_2)(x_3-b_3)}{x_1-b_1}$ *is l.c. with respect to the point* (b_1, b_2, b_3)*,* $f((1-t)(x_1, x_2, x_3) + t(b_1, b_2, b_3)) = (x_1 - b_1)^{-1}((1-t)(x_2 - b_2)(x_3 - b_3))$ *is equal to* $(1-t)f(x_1, x_2, x_3) + tf(b_1, b_2, b_3)$

A result similar to the proposition 6.4 can be proved.

More in general, let $f : U \subset \Re^n \to \Re$ be a differentiable affine or l.c. function with respect to the point b, U an open set, then the 4 is

$$f(x) = f(b) - \nabla f(x)\,(b - x) \tag{15}$$

where $x = x_1, \ldots, x_n, b = b_1, \ldots, b_n \in U$.

Theorem 7.2. *The set* $Lc(b)(\Re^n, \Re)$ *of the affine or l.c. functions, with respect to* b*, is given by*

$$f(x) = f(b) + (x_1 - b_1)\psi(\frac{x_2 - b_2}{x_1 - b_1}, \ldots, \frac{x_n - b_n}{x_1 - b_1}) \tag{16}$$

where ψ *is an arbitrary differentiable function.*

Proof. The set $Lc(b)$ is the solution of the PDE 15, in fact, being

$$f_{x_1} = \psi(z_2, \ldots, z_n) - z_2\psi_{z_2}(z_2, \ldots, z_n) - \cdots - z_n\psi_{z_n}(z_2, \ldots, z_n)$$
$$f_{x_2} = \psi_{z_2}(z_2, \ldots, z_n),\ f_{x_3} = \psi_{z_3}(z_2, \ldots, z_n), \ldots, f_{x_n} = \psi_{z_n}(z_2, \ldots, z_n)$$

where $z_i = \frac{x_i - b_i}{x_1 - b_1}$, and replacing in equation 15 it follows the identity

$$\begin{aligned} f(x) - f(b) &= (x_1 - b_1)(\psi(z_2, \ldots, z_n) - z_2\psi_{z_2}(z_2, \ldots, z_n) - \cdots - z_n\psi_{z_n}(z_2, \ldots, z_n)) \\ &\quad + (x_2 - b_2)\psi_{z_2}(z_2, \ldots, z_n) + \cdots + (x_n - b_n)\psi_{z_n}(z_2, \ldots, z_n) \\ &= (x_1 - b_1)\psi(z_2, \ldots, z_n) \end{aligned}$$

The solution 16 is in $Lc(b)$, indeed

$$\begin{aligned} f((1-t)x + tb) &= f((1-t)x_1 + tb_1, \ldots, (1-t)x_n + tb_n) \\ &= f(b) + ((1-t)x_1 + tb_1 - b_1)\psi\left(\frac{(1-t)x_2 + tb_2 - b_2}{(1-t)x_1 + tb_1}, \ldots, \frac{(1-t)x_n + tb_n - b_n}{(1-t)x_1 + tb_1}\right) \\ &= f(b) + (1-t)(x_1 - b_1)\psi\left(\frac{x_2 - b_2}{x_1 - b_1}, \ldots, \frac{x_n - b_n}{x_1 - b_1}\right) \\ &= (1-t)(f(b) + (x_1 - b_1)\psi\left(\frac{x_2 - b_2}{x_1 - b_1}, \ldots, \frac{x_n - b_n}{x_1 - b_1}\right) + t\,f(b) \\ &= (1-t)f(x) + t\,f(b) \end{aligned}$$

□

By the $f(1-t)x + tb) = tf(b) + (1-t)(x_1 - b_1)\psi(\frac{x_2-b_2}{x_1-b_1}, \ldots, \frac{x_n-b_n}{x_1-b_1})$, setting $t = 0$, it follows $f(x) = (x_1 - b_1)\psi(\frac{x_2-b_2}{x_1-b_1}, \ldots, \frac{x_n-b_n}{x_1-b_1})$.

Proposition 7.3. *The set* $Lc(b)$ *is a linear space.*

Proof. Let $\phi_1, \phi_2 \in Lc(b)$, then $\phi = \phi_1 + \phi_2 \in Lc(b)$, indeed

$$\begin{aligned}\phi((1-t)x + t\,b) &= \phi_1((1-t)x + t\,b) + \phi_2((1-t)x + t\,b) \\ &= (1-t)\phi_1(x) + t\,\phi_1(b) + (1-t)\phi_2(x) + t\,\phi_2(b) \\ &= (1-t)(\phi_1(x) + \phi_2(x)) + t(\phi_1(x) + \phi_2(x)) \\ &= (1-t)\phi(x) + t\,\phi(b)\end{aligned}$$

If $\lambda \in \Re,\ \phi \in Lc(b)$, then $\lambda\phi((1-t)x + tb) = \lambda(1-t)\phi(x) + \lambda t\phi(b) = (1-t)(\lambda\phi(x)) + t(\lambda\phi(b))$ that is $\lambda\phi \in Lc(b)$. □

Proposition 7.4. *Let $E,\ F$ be normed linear spaces. If $D\phi(x) : E \to F$, with $\phi \in Lc(b)(E, F)$, is injective, then ϕ is injective too.*

Proof. Let $x_1, x_2 \in E$, with $x_1 \neq x_2$. It is $D\phi(x)(x_1) = \phi(x_1) - \phi(0)$ and $D\phi(x)(x_2) = \phi(x_2) - \phi(0)$ and subtracting $D\phi(x)(x_1) - D\phi(x)(x_2) = \phi(x_1) - \phi(x_2)$. Then $D\phi(x)(x_1) \neq D\phi(x)(x_2)$ implies $\phi(x_1) \neq \phi(x_2)$. □

Proposition 7.5. *Let $\phi_1 \in L_c(b)$ and $\phi_2 \in L_c(\phi_1(b))$, then $\phi_2 \circ \phi_1 \in L_c(\phi_2 \circ \phi_1(b))$.*

Proof.

$$\begin{aligned}\phi_2 \circ \phi_1((1-t)x + tb) &= \phi_2((1-t)\phi_1(x) + t\phi_1(b)) \\ &= (1-t)\phi_2 \circ \phi_1(x) + t\phi_2 \circ \phi_1(b)\end{aligned}$$

□

8. Some Topological Properties of Lc(b)

Let E, F be normed vector spaces. The continuity definition of a map $f : E \to F$ at a point $x_0 \in E$ may be rewritten as $\forall\epsilon > 0,\ \forall x \in I(x_0, \delta) \subset E,\ \exists t_\epsilon,\ 0 < t_\epsilon \leq 1,$ such that $0 < t < t_\epsilon$ implies $|f(x_0 + t(x - x_0)) - f(x_0)| < \epsilon$. If the map f is l.c. with respect to the point x_0, then

$$\begin{aligned}|f(x_0 + t(x - x_0)) - f(x_0)| &= |(1-t)f(x_0) + tf(x) - f(x_0)| = |tf(x) - tf(x_0)| \\ &= t|f(x_0) - f(x)| = t|Df(x)(x_0 - x)| < \epsilon\end{aligned}$$

that is, by restricting enough the open ball $I(x_0, \delta)$, the derivative is close to zero. The dual space of $\Re^n$, that is the space of the continuous linear functionals on $\Re^n$, is denoted by $L(\Re^n, \Re)$, see [5]. The link with the space $L_c(b)(\Re^n, \Re)$ is the following property

Proposition 8.1. *$L(\Re^{n-1}, \Re)$ is a subspace of $L_c(b)$.*

Proof. It is immediate, for any $\lambda \in L(\Re^n, \Re)$, by $\lambda((1-t)x + tb) = (1-t)\lambda(x) + t\lambda(b)$. □

Moreover, for any $\lambda \in L(\Re^{n-1}, \Re)$, let $\lambda(b_2, \ldots, b_n) = k$ with $k \in \Re$, then

$$\begin{aligned}\lambda(x_2, \ldots, x_n) &= k - \lambda(b_2, \ldots, b_n) + \lambda(x_2, \ldots, x_n) \\ &= k + (x_1 - b_1)\lambda(\frac{x_2 - b_2}{x_1 - b_1}, \ldots, \frac{x_n - b_n}{x_1 - b_1})\end{aligned}$$

So the functional λ may have the form of the elements of $L_c(b)$. It is known, see [5], that a linear map $\lambda : E \to F$ is continuous if and only if there exists $C > 0$ such that $|\lambda x| \le C|x|$ for all $x \in E$. The following proposition extends a similar property to the l.c. maps with respect to a point.

Proposition 8.2. *The l.c. map $\phi : E \to F$, with respect to the point b, is continuous if and only if there exists $C > 0$ such that $|D\phi(x)x| \le C|x|$, for all $x \in E$.*

Proof. Let $\phi \in Lc(b)$. By the equation 4, it follows $D\phi(x)(b - x_0) = \phi(b) - \phi(x_0)$ and subtracting with the 4 it is $D\phi(x)(x - x_0) = \phi(x) - \phi(x_0)$. For $|x - x_0| < \delta$, with $x, x_0 \in E$, it is $|D\phi(x)(x - x_0)| = |\phi(x) - \phi(x_0)| \le C|x - x_0| < C\delta < \epsilon$, where $\delta < \frac{\epsilon}{C}$ then $\phi(x)$ is continuous at x_0.
Conversely, by the continuity of ϕ, there exists δ such that, for $|x - x_0| \le \delta$, it follows $|\phi x - \phi x_0| = |D\phi(x)(x - x_0)| < \epsilon < 1$. Then $|D\phi(x)(\frac{\delta(x-x_0)}{|x-x_0|})| = |\frac{\delta}{|x-x_0|} D\phi(x)(x - x_0)| < 1$ for all $x - x_0 \in E$, with $|x - x_0| \le \delta$, namely $|D\phi(x)x| \le C|x|$. □

Definition 8.3. *Let $\phi \in Lc(b)$, then $|\phi|$, the norm of ϕ, is defined by $|\phi| = |D\phi(x)|$, where $|D\phi(x)|$ is the usual norm of the linear map $D\phi(x)$ with $|D\phi(x)| \le C|x|$, $C > 0$.*

Proposition 8.4. *If $\phi_1 \in Lc(b)$ and $\phi_2 \in Lc(\phi_1(b))$, then $|\phi_2 \circ \phi_1| \le |(D\phi_2(x)| \, |\phi_1| \, |x|$.*

Proof. $|\phi_2 \circ \phi_1(x)| = |D\phi_2(x)(\phi_1(x))| \le |D\phi_2(x)| \, |\phi_1(x)| \le |D\phi_2(x)| \, |\phi_1| \, |x|$. □

Let $\phi : F \to G$, with $\phi \in Lc(b)$ and F a subspace of E. Since $\phi(x) = \phi(0) + D\phi(x)(x)$, with $D\phi(x)x : E \to G$, then there exists the extension of ϕ to the space E, defined by the same $\phi(x) = \phi(0) + D\phi(x)x$. Let $E^{\star\star} = Lc(b)(Lc(b)(E, \Re), \Re)$ be the double dual space of E with respect to the space $Lc(b)$. Functions $\Phi_x : E^\star \to \Re$ are defined by $\Phi_x(\phi) = \phi(x)$ for any $\phi \in E^\star$, $x \in E$.

Proposition 8.5. *The map of $E \to E^{\star\star}$ defined by $x \mapsto \Phi_x$ is linear , injective and norm preserving, that is $|x| = |\Phi_x|$.*

Proof. Let $x_1, x_2 \in E$ with $x_1 \ne x_2$, so $x_1 - x_2 \ne 0$. By the Hahn-Banach theorem there exists $D\phi(x) \in L(E, \Re)$, with $\phi \in Lc(E, \Re)$, such that $D\phi(x)(x_1 - x_2) \ne 0$, then $D\phi(x)(x_1) \ne D\phi(x)(x_2)$ so $D\phi(x)$ is injective. By the proposition 7.4 also ϕx is injective, that is $\phi x_1 \ne \phi x_2$, this implies that the map $x \mapsto \Phi_x$ is injective. By $|\phi x| \le |\phi| \, |x|$ and $|\Phi_x(\phi)| \le |\Phi_x| \, |\phi|$, since $|\phi(x)| = |\Phi_x(\phi)|$ it follows $|\phi| \, |x| = |\Phi_x| \, |\phi|$ and $|\Phi_x| = |x|$. □

The Linear Extension Theorem, see [5], for a linear map $\lambda : F \to G$, where E is a normed vector space, F a subspace of E and G a Banach space, proves that there exists a unique extension of λ to a continuous linear map $\bar{\lambda} : \bar{F} \to G$. Where $\bar{F}$ is the closure of F and λ, $\bar{\lambda}$ have the same norm. The next theorem is a similar result for the space $Lc(b)$.

Theorem 8.6. *Let $\phi : F \to G$, with $\phi \in Lc(b)$, E a normed vector space and F a subspace of E, G a Banach space. The norm of ϕ is C. $\bar{F}$ denotes the closure of F in E. Then there exists a unique extension of ϕ to a continuous $\bar{\phi} : \bar{F} \to G$, with $\bar{\phi} \in Lc(b)$, and $\bar{\phi}$ has the same norm C.*

Proof. Uniqueness. Suppose $x = \lim x_n$, with $x \in \bar{F}$ and $x_n \in F$. By the continuity of ϕx it follows $\lim \phi(x_n) = \bar{\phi} x \in G$, in fact G is complete. So

$$\begin{cases} \bar{\phi} x = \phi x \text{ if } x \in F \\ \bar{\phi} x = \bar{\phi} x \text{ if } x \in \bar{F} \end{cases}$$

is an extension of ϕ. If δ is again an extension, it follows

$$\begin{cases} \delta x = \phi x \text{ if } x \in F \\ \delta x = \bar{\phi} x \text{ if } x \in \bar{F} \end{cases}$$

so $\delta = \bar{\phi}$. Existence. Suppose $x = \lim x_n$, with $x \in \bar{F},\ x_n \in F,\ \phi \in Lc(b),\ b \in F$. Then

$$|\phi(x_n) - \phi(x_m)| = |\phi b + D\phi(x)(x_n - b) - \phi b - D\phi(x)(x_m - b)|$$
$$= |D\phi(x)(x_n - b) - D\phi(x)(x_m - b)| = |D\phi(x)(x_n - x_m)| \le C|x_n - x_m|$$

so $\{\phi(x_n)\}$ is a Cauchy sequence in the Banach space G. Denote $\lim \phi(x_n) = \bar{\phi}x$. It is immediate that $\bar{\phi}x$ is independent of the sequence $x_n \to x$. If $x \in F$ and $x = \lim x$, then $\bar{\phi}x = \phi x$. This implies $\phi b = \bar{\phi} b$ because $b \in F$ and $\bar{\phi}$ is an extension of ϕ. Now one must prove that $\bar{\phi} \in Lc(b)$.

$$\bar{\phi}((1-t)x + t\,b) = \lim \phi((1-t)x_n + t\,b) = \lim((1-t)\phi(x_n) + t\phi b)$$
$$= (1-t)\lim \phi(x_n) + t\phi b = (1-t)\bar{\phi}x + t\bar{\phi}b$$

so $\bar{\phi}$ is l.c. with respect to b. The norm is a continuous function, then $|\bar{\phi}x| = \lim |\phi(x_n)|$ and by $|\phi(x_n)| \le C|x_n|$, it is $|\bar{\phi}x| = \lim |\phi x_n| \le C|\lim x_n| = C|x|$, hence $|\bar{\phi}| = |\phi|$. □

9. The Function $\psi(\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1})$

The $x_1\psi(\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1})$, with ψ an arbitrary C^n function, is l.c. with respect to the point zero. Then it holds $x_1\psi = x_1\frac{\partial x_1\psi}{\partial x_1} + \dots + x_n\frac{\partial x_1\psi}{\partial x_n}$ and this implies

$$x_1\frac{\partial \psi}{\partial x_1} + \dots + x_n\frac{\partial \psi}{\partial x_n} = 0 \tag{17}$$

this is a known result by the Euler's theorem, since the function ψ is homogeneous of degree 0, that is $\psi(t\,x) = \psi(x),\ t > 0$. The next theorem states a stronger property of ψ.

Theorem 9.1. *Let $\psi : \Re^n \to \Re$ be the homogeneous function of class C^p, defined by $x \to \psi(\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1})$, with arbitrary ψ. Then*

$$D^k\psi(x)\,x^{(k)} = 0 \qquad\qquad k = 1, \dots, p \tag{18}$$

Proof. Denote by $\psi^{(0,0,\dots,i,\dots,0)}(x)$ the partial derivative with respect to the i-th variable.

$$\begin{aligned} D\psi(x)\,x = &(-\frac{x_2}{x_1}\psi^{(1,0,\dots,0)} - \frac{x_3}{x_1}\psi^{(0,1,\dots,0)} - \cdots \\ &- \frac{x_n}{x_1}\psi^{(0,0,\dots,1)}) + (\frac{1}{x_1}\psi^{(1,0,\dots,0)})x_2 + (\frac{1}{x_1}\psi^{(0,1,\dots,0)})x_3 + \dots + (\frac{1}{x_1}\psi^{(0,0,\dots,1)})x_n \\ &= 0 \end{aligned}$$

and $D^p\psi(x)\,x^{(p)} = D(D^{p-1}\psi(x)\,x^{(p-1)})x$, so, by induction, it follows 18 □

10. Linear Convex Differentiability

The l.c. maps allow an extension of the differentiability' s definition .

Definition 10.1. *Let U open in E, and $b \in U$. Let $f : U \to F$ be a map. Then f is linear convex differentiable at b if there exists a continuous l.c. $\phi \in Lc(b)$, defined for all sufficiently small h in E, such that $\lim_{h\to 0}\frac{1}{|h|}(f(b+h) - f(b) - \phi(b+h)) = 0$*

Proposition 10.2. *If f is l.c. differentiable at b, then it has derivative for every direction at b.*

Proof. Suppose $h = t(x-b)$, $x - b \in U$ and observing that $\phi(b + t(x-b)) = t\phi(x)$ it follows

$$\begin{aligned} Df(b)(x-b) &= \lim_{t\to 0} \frac{1}{|t|}(f(b+t(x-b)) - f(b) - \phi(b+t(x-b)) \\ &= \lim_{t\to 0} \frac{1}{|t|}(f(b+t(x-b)) - f(b) - t\phi(x)) \\ &= \lim_{t\to 0} \frac{1}{|t|}(f(b+t(x-b)) - f(b)) = \phi(x) \end{aligned}$$

□

In particular if $x - b = e_i$, $i = 1, \dots, n$, it is $Df(b)e_i = \psi(b + e_i)$. The definition 10.1 becomes especially useful if a map is not differentiable at a point.

Example 10.3. *The function*

$$f(x,y) = \begin{cases} \frac{x^2 y(y+1)}{x^2+y^2} & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0) \end{cases}$$

is not differentiable at the point $(0,0)$*, in fact, with* $d = (d_1, d_2)$*,*

$$\begin{aligned} Df(0,0)d &= \lim_{t\to 0} \frac{1}{t}(f(0+td) - f(0)) \\ &= \frac{d_1^2 d_2}{d_1^2 + d_2^2} \\ &\neq Df(0,0)e_1 d_1 + Df(0,0)e_2 d_2 \\ &= 0\, d_1 + 0\, d_2 \end{aligned}$$

In order to $f(x,y)$ *is l.c. differentiable at* $(0,0)$*, a* $\phi(x,y) \in Lc(0,0)$ *has to exist such that* $Df(0,0)((x,y)-(0,0)) = \phi(x,y)$*, that is*

$$\begin{aligned} \phi(x,y) &= \frac{1}{t}(f(0+t(x-0)) - f(0)) \\ &= \frac{1}{t}(f(t(x),t(y)) - f(0)) \\ &= \frac{1}{t}(f(t(x),t(y))) \\ &= \frac{x^2 y}{x^2+y^2} \end{aligned}$$

Then $\phi(x,y) = x\frac{xy}{x^2+y^2} = x\frac{\frac{y}{x}}{1+\frac{y^2}{x^2}} = x\psi(\frac{y}{x})$*. So the* $f(x,y)$ *is a l.c. differentiable function at* $(0,0)$*. Observe that* $f(x,y) \notin Lc(0,0)$*.*

Proposition 10.4. *The continuous functions of the space* $Lc(b)$ *are l.c. differentiable at* b*.*

Proof. By $f \in Lc(b)$,

$$\begin{aligned} Df(b)(x-b) &= \lim_{t\to 0} \frac{1}{|t|}(f(b+t(x-b)) - f(b)) \\ &= \lim_{t\to 0} \frac{1}{|t|}(f(b(1-t)+tx)) - f(b) - \phi(b+t(x-b)) \\ &= \lim_{t\to 0} \frac{1}{|t|}((1-t)f(b) + tf(x) - f(b)) \\ &= f(x) - f(b) \end{aligned}$$

and by $f \in Lc(b)$ it follows $\phi(x) = f(x) - f(b) \in Lc(b)$, so f is l.c. differentiable at b. □

Example 10.5. *The function*

$$f(x,y) = \begin{cases} (x-1)\sin(\frac{x-1}{y-1}+1) & \text{if } (x,y) \neq (1,1) \\ 0 & \text{if } (x,y) = (1,1) \end{cases}$$

is not differentiable at the point $(1,1)$*, in fact, with* $d = (d_1, d_2)$*,* $Df(1,1)d = d_1 \sin(\frac{d_1}{d_2}+1)$ *but* $Df(1,1)e_1$ *is indeterminate. Since* $Df(1,1)((x,y)-(1,1)) = (x-1)\sin(\frac{x-1}{y-1}+1) = (x-1)\psi(\frac{x-1}{y-1}) = \phi(x,y)$*. So the function* $f(x,y)$ *is l.c. differentiable at* $(1,1)$*.*

Proposition 10.6. *If* f *is differentiable at* b *then* f *is l.c. differentiable at the same point.*

Proof. In the definition 10.1, if f is differentiable at b then $\phi(b+h) = Df(b)(x-b)$ is linear, so $Df(b)(x-b) = \phi(x)$ and f is l.c. differentiable. □

Example 10.7. *Let the function* $f(x,y) = y\log((x+1)^3 y)$*, with* $x > -1$*,* $y > 0$*, be differentiable at* $b = (b_1, b_2)$*. The* f *is l.c. differentiable at* b *if there exists* $\phi \in Lc(b)$ *such that* $Df(b_1,b_2)((x,y)-(b_1,b_2)) = \phi(x,y)$*, that is* $Df(b_1,b_2)((x,y)-(b_1,b_2)) = \lim_{t\to 0}\frac{1}{t}(f((b_1,b_2)+t(x-b_1,y-b_2)) - f(b_1,b_2)$*.*

$$\begin{aligned}
&= \lim_{t\to 0}\frac{1}{t}((b_1+t\,x+t\,b_1)\log\frac{b_1+t\,x+t\,b_1}{b_2+t\,y+t\,b_2} - b_1\log\frac{b_1}{b_2} \\
&= \log((1+b_1)^3 b_2)(y-b_2) + \frac{1}{1+b_1}(y+b_1(y-4b_2)+(-1+3x)b_2) \\
&= (x-b_1)(\frac{1}{1+b_1}(3b_2+\frac{y-b_2}{x-b_1}((1+b_1)\log((1+b_1)^3 b_2)+1+b_1))) \\
&= (x-b_1)\psi(\frac{y-b_2}{x-b_1}) \\
&= \phi(x,y)
\end{aligned}$$

then $\phi(x,y) \in Lc(b)$ *and* f *is l.c. differentiable at* b*.*

Proposition 10.8. *If* $f(x)$ *is l.c.differentiable at* b*, then it is continuous at* b*.*

Proof. By the l.c. differentiability $\lim_{t\to 0}\frac{1}{t}(f(b+t(x-b)-f(b)) = \phi(x)$ with $\phi(x)$ a bounded function . Set $\frac{1}{|t|}(f(b+t(x-b)-f(b)) = \phi(x)+\theta(t)$, then $\lim_{t\to 0}(f(b+t(x-b)-f(b)) = \lim_{t\to 0}(|t|\phi(x)+|t|\theta(t)) = 0$, so $\lim_{t\to 0} f(b+t(x-b)) = f(b)$. □

11. Cones and Derivatives

Recall the known cone's definition, and apply this to $\phi^{(\alpha)}(x) = (x_1-b_1)^\alpha \psi_\alpha((\frac{x_2-b_2}{x_1-b_1})^\alpha, \ldots, (\frac{x_n-b_n}{x_1-b_1})^\alpha) = 0$, where $\alpha \in \Re - \{0\}$, ψ is an arbitrary function, with $\phi^{(\alpha)}(a) = 0$. Then $\phi^{(\alpha)}(x)$ is a cone if the straight line $r = a + t(a-b)$, $t \in \Re$, joining the two points $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, is completely contained in $\phi^{(\alpha)}(x)$. Since

$$\begin{aligned}
\phi^{(\alpha)}(a+t(a-b))) &= (a_1+t(a_1-b_1))^\alpha \psi((\frac{a_2+t(a_2-b_2)-b_2}{a_1+t(a_1-b_1)-b_1})^\alpha, \ldots, (\frac{a_n+t(a_n-b_n)-b_n}{a_1+t(a_1-b_1)-b_1})^\alpha) \\
&= (1+t)^\alpha (a_1-b_1)^\alpha \psi((\frac{a_2-b_2}{a_1-b_1})^\alpha, \ldots, (\frac{a_n-b_n}{a_1-b_1})^\alpha) \\
&= (1+t)^\alpha \phi^{(\alpha)}(a) = 0
\end{aligned}$$

then the line r is in $\phi^{(\alpha)}(x)$. The Taylor's formula may be written by the cones $\phi^{(i)}(x)$, $i \in N - \{0\}$

Proposition 11.1. *Let* $f : U \subseteq \Re^n \to \Re$ *be a function of class* C^p *in the open* U*, with* $\|x-b\| = 1$*, then its Taylor's formula may be set in the form*

$$f(x) = f(b) + \frac{1}{1!}\phi^{(1)}(x) + \cdots + \frac{1}{(p-1)!}\phi^{(p-1)}(x) + \theta(x-b). \tag{19}$$

Proof. Since $D^i f(b)(x-b)^{(i)}$ is multilinear, then $D^i f(b)(x-b)^{(i)} = \phi^{(i)}(x) = (x_1-b_1)^i \psi_i((\frac{x_2-b_2}{x_1-b_1})^i, \ldots, (\frac{x_n-b_n}{x_1-b_1})^i)$ with $i = 1, \ldots, p-1$. □

Example 11.2. *The function $f(x,y) = x^4 + (y-2)^3$, with respect to the point $b = (b_1, b_2)$, has the Taylor's formula*

$$x^4 + (y-2)^3 = b_1^4 + (b_2-2)^3 + \frac{1}{1!}(x-b_1)(4b_1^3 + 3(\frac{y-b_2}{x-b_1})(b_2-2)^2)$$
$$+ \frac{1}{2!}(x-b_1)^2 6(2b_1^2 + (\frac{y-b_2}{x-b_1})^2(b_2-2)) + \frac{1}{3!}(x-b_1)^3 6(4b_1 + (\frac{y-b_2}{x-b_1})^3) + \frac{1}{4!}(x-b_1)^4\, 24$$

Let $c = x + t(x-b)$, with $t \in \Re$, be a point on the straight line connecting x and b, then

Proposition 11.3. *Let $f : U \subseteq \Re^n \to \Re$ be a function of class C^p in the open U, it holds*

$$f(c) = f(b) + (1+t)\phi^{(1)}(x) + \frac{1}{2!}(1+t)^2\phi^{(2)}(x) + \cdots + \frac{1}{(p-1)!}(1+t)^{p-1}\phi^{(p-1)}(x) + \theta_1(x-b). \tag{20}$$

Proof. The function $\phi^{(\alpha)}(x) = (x_1-b_1)^\alpha \psi_\alpha((\frac{x_2-b_2}{x_1-b_1})^\alpha, \ldots, (\frac{x_n-b_n}{x_1-b_1})^\alpha)$ satisfies

$$\phi^{(\alpha)}(c) = \phi^{(\alpha)}((1+t)x_1 - t\,b_1, \ldots, (1+t)x_n - t\,b_n)$$
$$= (1+t)^\alpha (x_1-b_1)^\alpha \psi_\alpha((\frac{(1+t)(x_2-b_2)}{(1+t)(x_1-b_1)})^\alpha + \cdots + (\frac{(1+t)(x_n-b_n)}{(1+t)(x_1-b_1)})^\alpha)$$
$$= (1+t)^\alpha \phi^{(\alpha)}(x)$$

so, substituting for 19, it follows the 20. □

The derivatives of a function f may be expressed by the cones $\phi^{(i)}$ of the Taylor's formula 19.

Proposition 11.4. *Let $f : U \subseteq \Re^n \to \Re$ be a function of class C^p in the open U, then*

$$1\frac{1}{i!}D^i f(x)(x-b)^{(i)} = \frac{1}{i!}\binom{i}{i}\phi^{(i)}(x) + \frac{1}{(i+1)!}\binom{i+1}{i}\phi^{(i+1)}(x) + \cdots + \frac{1}{(p-1)!}\binom{p-1}{i}\phi^{(p-1)}(x) \tag{21}$$

for $i = 1, \ldots, p-1$ and $n \geq 2$.

Proof. By the Taylor's formula, it is

$$1f(x + t(x-b)) = f(x) + t\,Df(x)(x-b) + \frac{t^2}{2!}D^2 f(x)(x-b)^{(2)} + \cdots + \frac{t^{p-1}}{(p-1)!}D^{p-1}f(x)(x-b)^{(p-1)} + \theta_2(x-b) \tag{22}$$

comparing the right sides of 20 and 22, it follows

$$f(x) = f(b) + \frac{1}{1!}\phi^{(1)}(x) + \cdots + \frac{1}{(p-1)!}\phi^{(p-1)}(x) + \theta_1(x-b)$$
$$+ (\frac{t}{1!}\phi^{(1)}(x) + \frac{2t}{2!}\phi^{(2)}(x) + \cdots + \frac{(p-1)t}{(p-1)!}\phi^{(p-1)}(x)) - t\,Df(x)(x-b) + \theta_1(x-b))$$
$$+ (\frac{t^2}{2!}\binom{2}{2}\phi^{(2)}(x) + \frac{t^2}{3!}\binom{3}{2}\phi^{(3)}(x) + \cdots + \frac{t^2}{(p-1)!}\binom{p-1}{2}\phi^{(p-1)}(x)$$
$$- \frac{t^2}{2!}Df(x)(x-b)^{(2)} + \theta_1(x-b))$$
$$+ \quad \cdots \quad \cdots \quad \cdots$$
$$+ (\frac{t^i}{i!}\binom{i}{i}\phi^{(i)}(x) + \frac{t^i}{(i+1)!}\binom{i+1}{i}\phi^{(i+1)}(x) + \cdots + \frac{t^i}{(p-1)!}\binom{p-1}{i}\phi^{(p-1)}(x)$$
$$- \frac{t^i}{i!}D^i f(x)(x-b)^{(i)} + \theta_1(x-b))$$
$$+ \quad \cdots \quad \cdots \quad \cdots$$
$$+ \frac{t^{p-1}}{(p-1)!}\binom{p-1}{p-1}\phi^{(p-1)}(x) - \frac{t^{p-1}}{(p-1)!}D^{p-1}f(x)(x-b)^{(p-1)} + \theta_1(x-b) - \theta_2(x-b)$$

then the 22. □

A corollary of the Proposition 11.4 is

Proposition 11.5. *Let $f : U \subseteq \Re^n \to \Re$ be a function of class C^p in the open U, then*

$$\frac{1}{i!}D^i f(x)(x-b)^{(i)} = \frac{1}{i!}\binom{i}{i}D^i f(b)(x-b)^{(i)} + \frac{1}{(i+1)!}\binom{i+1}{i}D^{i+1}f(x)(x-b)^{(i+1)}$$
$$\cdots + \frac{1}{(p-1)!}\binom{p-1}{i}D^{p-1}f(b)(x-b)^{(p-1)}$$

for $i = 1, \dots, p-1$ and $x, b \in U$.

Proof. Immediate by $\phi^{(i)}(x) = D^i f(b)(x-b)^{(i)}$ if $f : U \subseteq \Re^n \to \Re$. ☐

The derivatives of the functions $\phi^{(i)}(x)$ satisfy some properties

Proposition 11.6. *Let $\phi^{(i)} : \Re^n \to \Re$ be a function defined by $\phi^{(i)}(x) = (x_1-b_1)^i \psi((\frac{x_2-b_2}{x_1-b_1})^i, \dots, (\frac{x_n-b_n}{x_1-b_1})^i)$ for an arbitrary C^k differentiable ψ, with $x, b \in \Re^n$, then*

(i) $D\phi^{(i)}(x)(x-b) = i\phi^{(i)}(x) \quad i = 1, 2, \dots$

(ii) $D^k\phi^{(i)}(x)(x-b)^{(i)} = i(i-1)\cdots(i-(k-1))\,\phi^{(i)}(x) \quad 0 < k \le i$

(iii) $D^k\phi^{(i)}(x)(x-b)^{(k)} = 0 \quad k = i+1, \dots.$

Proof. (i) By the $\phi^{(i)}(x + t(x-b)) = (1+t)^i\phi^{(i)}$ it follows

$$\begin{aligned}
D\phi^{(i)}(x)(x-b) &= \lim_{t\to 0}\frac{1}{t}(\phi^{(i)}(x+t(x-b)) - \phi^{(i)}(x)) \\
&= \lim_{t\to 0}\frac{1}{t}((1+t)^i\phi^{(i)}(x) - \phi^{(i)}(x)) \\
&= \lim_{t\to 0}\frac{1}{t}((1 + it + \frac{i(i-1)}{2}t^2 + \cdots + t^i)\phi^{(i)}(x) - \phi^{(i)}(x)) \\
&= \lim_{t\to 0}\frac{1}{t}((it + \frac{i(i-1)}{2}t^2 + \cdots + t^i)\phi^{(i)}(x)) \\
&= i\phi^{(i)}(x)
\end{aligned}$$

(ii) by induction on k. (iii)For $i = 1$. The first step is to prove $D^2\phi^{(1)}(x)(x-b)^{(2)} = 0$. Denote $\psi((\frac{x_2-b_2}{x_1-b_1}), \dots, (\frac{x_n-b_n}{x_1-b_1}))$ by $\psi(z)$ and $\psi^{(1,0,\dots,0)}(z)$ be the partial derivative with respect to the first variable, then

$$\begin{aligned}
D^2\phi^{(1)}(x)(x-b)^{(2)} &= \phi^{(1)}_{x_1x_1})(x)(x_1-b_1)^2 + \cdots + \phi^{(1)}_{x_1x_n}(x)(x_1-b_1)(x_n-b_n) + \\
&\quad \cdots + \phi^{(1)}_{x_nx_n}(x)(x_n-b_n)^2 \\
&= \psi^{(2,0,\dots,0)}(z)(\frac{(x_2-b_2)^2}{x_1-b_1} - 2\frac{(x_2-b_2)^2}{x_1-b_1} + \frac{(x_2-b_2)^2}{x_1-b_1}) + \\
&\quad \dots + \psi^{(0,0,\dots,2)}(z)(\frac{(x_n-b_n)^2}{x_1-b_1} - 2\frac{(x_n-b_n)^2}{x_1-b_1} + \frac{(x_n-b_n)^2}{x_1-b_1}) \\
&\quad + 4\psi^{(1,1,\dots,0)}(z)(\frac{(x_2-b_2)(x_3-b_3)}{x_1-b_1} - \frac{(x_2-b_2)(x_3-b_3)}{x_1-b_1}) + \\
&\quad \cdots + 4\psi^{(0,\dots,1,1)}(z)(\frac{(x_{n-1}-b_{n-1})(x_n-b_n)}{x_1-b_1} - \frac{(x_{n-1}-b_{n-1})(x_n-b_n)}{x_1-b_1}) \\
&= 0
\end{aligned}$$

Now, by induction, $D^{k-1}\phi(x)(x-b) = 0$ so $D^k\phi(x)(x-b) = D(D^{k-1}\phi(x)(x-b)) = 0$. ☐

12. h-derivatives

Let $f : U \subseteq \Re^2 \to \Re$ be a function of class C^p in the open U and suppose the point $c = (b + t(x - b)) \in U$, with $t \in \Re$, then the function $g(t) = f(b + t(x - b))$ is defined and it is known that $g^n(t) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k,k)}(b + t(x - b))(x_1 - b_1)^{(n-k)}(x_2 - b_2)^k$, where $f^{(n-k,k)}$ denote the partial derivative $\frac{\partial^n f}{\partial x_1^{n-k} \partial x_2^k}$. In general, if $f : U \subseteq \Re^r \to \Re$, then $g^n(t) = \sum_{k_1,\dots,k_r=n} \binom{n}{k_1,\dots,k_r} f^{(k_1,\dots,k_r)}(b + t(x - b)) \prod_{i=1}^{r}(x_i - b_i)^{k_i}$, where $\binom{n}{k_1,\dots,k_r} = \frac{n!}{\prod_{i=1}^{r} k_i!}$ and $\sum_{k_1,\dots,k_r=n}$ denote the sum over all subsets of nonnegative integer indices k_1 through k_r such that the sum of all k_i is n. By a similar way, for the function f, it is possible to define new "derivatives". In the simplest case of a differentiable $f : \Re \to \Re$, the first derivative may be defined by the finite $\lim_{t\to 0} \frac{f(x+k(t))-f(x)}{t}$. This limit has value $f'(x)$ if $k(t)$ is a differentiable function with $\lim_{t\to 0} k(t) = 0$ and $\lim_{t\to 0} k'(t) = 1$. Among the functions with this property, the next definition chooses $k(t) = e^t - 1$.

Definition 12.1.

(i) The h-derivative of the function $f : U \subseteq \Re^r \to \Re$, $r \geq 2$, of class C^p, at the point b, is defined by $H^n f(b)(b - x)^{(n)} = (\frac{d}{dt})^n h(0) \qquad n = 1, 2, \dots, p$, where $h(t) = f(x + e^t(b - x))$ and $(x + e^t(b - x)) \in U$.

(ii) In the special case $f : U \subseteq \Re \to \Re$, the h-derivative at the point x is $H^n f(x) = (\frac{d}{dt})^n h(0) \qquad n = 1, 2, \dots, p$, where $h(t) = f(x - 1 + e^t)$, with $(x - 1 + e^t) \in U$.

Next example stresses (i) as a particular case of (ii).

Example 12.2. *By the (i), for $f(x)$ at a point $b = (b_1, b_2)$, setting $(x - b) = e_1 = (1, 0)$, is $H^4 f(b)e_1^{(4)} = f'(b) + 7f^{(2)}(b) + 6f^{(3)}(b) + f^{(4)}(b)$. By the (ii), with $f : \Re \to \Re$ $H^4 f(x) = f'(x) + 7f^{(2)}(x) + 6f^{(3)}(x) + f^{(4)}(x)$. Then, the derivatives are equal.*

Example 12.3. *For the elementary function x^α,*

$$H\,x^\alpha = \alpha x^{\alpha-1}, \qquad H^2 x^\alpha = \alpha(\alpha-1)x^{\alpha-2} + \alpha x^{\alpha-1}$$
$$H^3 x^\alpha = \alpha(\alpha-1)(\alpha-2)x^{\alpha-2} + 3\alpha(\alpha-1)x^{\alpha-2} + \alpha x^{\alpha-1}$$

that is, $H^n x^\alpha$ is a polynomial with n addend and degree $\alpha - 1$.

It is immediate that

(i) $h'(0) = D_b f(b)(b - x) = -D\,f(b)(x - b) = -g'(0)$

(ii) $h''(0) = D_b^2 f(b)(b - x)^{(2)} = D^2 f(b)(b - x)^{(2)} + D\,f(b)(b - x) = -(g^2(0) + g'(0))$

where D_b denotes the derivative with respect to the vector variable b. The higher n-th derivative will be denoted by $h^n(0) = D_b^n f(b)(b - x)^{(n)} = H^n f(b)(b - x)^{(n)}$.

Proposition 12.4. *Let $\psi : \Re^n \to \Re$ be the homogeneous function of class C^p, defined by $x \to \psi(\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1})$, with arbitrary ψ. Then $H^k\psi(x)\,x^{(k)} = 0 \qquad k = 1, \dots$.*

Proof. By the Theorem 9.1

$$H^k\psi(x)\,x^{(k)} = \frac{d}{dt}\psi(\frac{x_2 + e^t x_2}{x_1 + e^t x_1}, \dots, \frac{x_n + e^t x_n}{x_1 + e^t x_1})_{t=0} = D^k\psi(x)x^{(k)} = 0$$

□

Proposition 12.5. *Let $f : U \subseteq \Re^r \to \Re$ be a function of class C^n in the open U, then*

(i) $D_b g^n(t)(x-b) = (1-t)g^{n+1}(t) - n\, g^n(t) \qquad c = b + t(x-b) \in U$

(ii) $D_t(D_b\, g^{n-1}(t)(x-b)) = D_b g^n(t)(x-b)$

(iii) $D^n f(b)(x-b)^{(n)} = \phi^{(n)}(x) = g^n(0) = D_b\phi^{(n-1)}(x)(x-b) + (n-1)\phi^{(n-1)}(x)$

where $n = 1, \ldots, p$ and $\phi^{(0)}(x) = f(b)$.

Proof. (i) In order to reduce the proof, only two variables x_1, x_2 are considered

$$
\begin{aligned}
D_b g^n(t)(x-b) &= D_b(\sum_{k=0}^{n} \binom{n}{k} f^{(n-k,k)}(c)(x_1-b_1)^{n-k}(x_2-b_2)^k)(x-b) \\
&= (x_1-b_1)\sum_{k=0}^{n}(\binom{n}{k}(-(n-k)(x_1-b_1)^{n-k-1}(x_2-b_2)^k f^{(n-k,k)}(c)) \\
&+ \binom{n}{k} f^{(n-k+1,k)}(c)(1-t)(x_1-b_1)^{n-k}(x_2-b_2)^k) \\
&+ (x_2-b_2)\sum_{k=0}^{n}(\binom{n}{k}(-k(x_2-b_2)^{k-1}(x_1-b_1)^{n-k} f^{(n-k,k)}(c))+ \\
&+ \binom{n}{k} f^{(n-k,k+1)}(c)(1-t)(x_1-b_1)^{n-k}(x_2-b_2)^k) \\
&= -\sum_{k=0}^{n}(\binom{n}{k}((n-k)(x_1-b_1)^{n-k}(x_2-b_2)^k f^{(n-k,k)}(c)) \\
&+ \binom{n}{k} k(x_1-b_1)^{n-k}(x_2-b_2)^k f^{(n-k,k)}(c)) \\
&+ \sum_{k=0}^{n}(\binom{n}{k} f^{(n-k+1,k)}(c)(x_1-b_1)^{n-k-1}(x_2-b_2)^k) \\
&+ \binom{n}{k} f^{(n-k,k+1)}(c)(1-t)(x_1-b_1)^{n-k}(x_2-b_2)^{k+1})(1-t) \\
&= -\sum_{k=0}^{n}(n\binom{n}{k}(x_1-b_1)^{n-k}(x_2-b_2)^k f^{(n-k,k)}(c)) \\
&+ \sum_{k=0}^{n}\binom{n}{k}((x_1-b_1)^{n-k+1}(x_2-b_2)^k f^{(n-k+1,k)}(c) \\
&+ (x_1-b_1)^{n-k}(x_2-b_2)^{k+1} f^{(n-k,k+1)}(c))(1-t) \\
&= -ng^n(t) + (\sum_{k=0}^{n+1}\binom{n+1}{k}(x_1-b_1)^{n+1-k}(x_2-b_2)^k f^{(n+1-k,k)}(c))(1-t) \\
&= (1-t)g^{n+1}(t) - n\, g^n(t)
\end{aligned}
$$

(ii) By (i) $D_b g^{n-1}(t)(x-b) = (1-t)g^n(t) - (n-1)g^{n-1}(t)$, then

$$
\begin{aligned}
D_t(D_b g^{n-1}(t)(x-b)) &= D_t((1-t)g^n(t) - (n-1)g^{n-1}(t)) \\
&= -g^n(t) + (1-t)g^{n+1}(t) - (n-1)g^n(t) \\
&= (1-t)g^{n+1}(t) - n\, g^n(t) \\
&= D_b g^n(t)(x-b)
\end{aligned}
$$

(iii) It is a special case of (i) by $t = 0$. □

Recall the known functions

Definition 12.6. *The k-th elementary symmetric function on the n numbers $\lambda_1, \dots, \lambda_n$ is* $S_k(\lambda_1, \dots, \lambda_n) = \sum_{1 \le i_1 < \dots < i_k \le n} \Pi_{j=1}^k \lambda_{i_j}$ *the sum of all $\binom{n}{k}$ k-fold products of distinct items from $\lambda_1, \dots, \lambda_n$.*

Particular cases are $S_1(\lambda_1, \dots, \lambda_n) = \lambda_1 + \dots + \lambda_n$ and $S_n(\lambda_1, \dots, \lambda_n) = \lambda_1 \cdots \lambda_n$.

Proposition 12.7. *Let $f : U \subseteq \Re^r \to \Re$ be a function of class C^n in the open U, then*

$$\begin{aligned}
D^n f(b)(x-b)^{(n)} &= D_b^n f(b)(x-b)^{(n)} + S_1(1, \dots, n-1) D_b^{n-1} f(b)(x-b)^{(n-1)} \\
&\quad + S_2(1, \dots, n-1) D_b^{n-2} f(b)(x-b)^{(n-2)} + \dots + S_{n-1}(1, \dots, n-1) D_b f(b)(x-b) \\
&= (D_b f(b)(x-b) + 1) \circ (D_b f(b)(x-b) + 2) \circ \dots \circ (D_b f(b)(x-b) + n - 1) + D_b^n f(b)(x-b)^{(n)}
\end{aligned}$$

where $D_b f(b)(x-b) \circ D_b f(b)(x-b) = D_b^2 f(b)(x-b)^{(2)}$.

Proof. If $n = 1$ it immediate that $Df(b)(x-b) = D_b f(b)(x-b)$. By induction and using (iii) of Proposition 12.4

$$\begin{aligned}
D^{n+1} f(b)(x-b)^{(n+1)} &= g^{(n+1)}(0) = n\, g^n(0) + D_b g^n(0)(x-b) \\
&= n(D_b^n f(b)(x-b)^{(n)} + S_1(1, \dots, n-1) D_b^{n-1} f(b)(x-b)^{(n-1)} + \cdots \\
&\cdots + S_{n-1}(1, \dots, n-1) D_b f(b)(x-b)) + D_b(D_b^n f(b)(x-b)^{(n)} \\
&+ S_1(1, \dots, n-1) D_b^{n-1} f(b)(x-b)^{(n-1)} + \cdots \\
&\cdots + S_{n-1}(1, \dots, n-1) D_b f(b)(x-b))(x-b) \\
&= D_b^{n+1} f(b)(x-b)^{(n+1)} + (n + S_1(1, \dots, n-1)) D_b^n f(b)(x-b)^{(n)} \\
&+ (n\, S_1(1, \dots, n-1) + S_2(1, \dots, n-1)) D_b^{n-1} f(b)(x-b)^{(n-1)} + \cdots \\
&\cdots + (n\, S_{n-2}(1, \dots, n-1) + S_{n-1}(1, \dots, n-1)) D_b^2 f(b)(x-b)^{(2)} \\
&+ n\, S_{n-1}(1, \dots, n-1) D_b f(b)(x-b) \\
&= D_b^{n+1} f(b)(x-b)^{(n+1)} + S_1(1, \dots, n) D_b^n f(b)(x-b)^{(n)} \\
&+ S_2(1, \dots, n) D_b^{n-1} f(b)(x-b)^{(n-1)} + \dots + S_n(1, \dots, n) D_b f(b)(x-b)
\end{aligned}$$

□

By the h-derivatives the Taylor's formula has the following form

Proposition 12.8. *Let $f : U \subseteq \Re^r \to \Re$ be a function of class C^n in the open U and $b + t(x-b) \in U$, then*

$$\begin{aligned}
f(x) &= f(b) + (\frac{1}{1!} + \sum_{i=1}^{n-1} \frac{1}{(i+1)!} S_i(1, \dots, i)) D_b f(b)(x-b) \\
&\quad + (\frac{1}{2!} + \sum_{i=1}^{n-2} \frac{1}{(i+2)!} S_i(1, \dots, i, i+1)) D_b^2 f(b)(x-b)^{(2)} + \cdots \\
&\quad \cdots + (\frac{1}{j!} + \sum_{i=1}^{n-j} \frac{1}{(i+j)!} S_i(1, \dots, i, i+1, \dots, i+j-1)) D_b^j f(b)(x-b)^{(j)} + \cdots \\
&\quad \cdots + \frac{1}{n!} D_b^n f(b)(x-b)^{(n)} + \theta(x-b)
\end{aligned}$$

Proof. In the Taylor's formula

$$f(x) = f(b) + \frac{1}{1!} D\, f(b)(x-b) + \cdots + \frac{1}{n!} D^n f(b)(x-b)^{(n)} + \theta(x-b)$$

by the Proposition 12.7, replacing the derivatives

$$\begin{aligned} f(x) = f(b) + (S_0(0) + \frac{1}{2!} S_1(1) + \frac{1}{3!} S_2(1,2) + \cdots + \frac{1}{n!} S_{n-1}(1,\ldots,n-1)) D_b f(b)(x-b) \\ + (\frac{1}{2!} S_0(1) + \frac{1}{3!} S_1(1,2) + \frac{1}{4!} S_2(1,2,3) + \cdots + \frac{1}{n!} S_{n-2}(1,\ldots,n-1)) D_b^2 f(b)(x-b)^{(2)} + \cdots + \frac{1}{n!} D_b^n f(b)(x-b)^{(n)} \end{aligned}$$

the new formula follows. □

Example 12.9. *For $n = 2$ it is $f(x) = f(b) + \frac{3}{2} D_b f(b)(x-b) + \frac{1}{2} D_b^2 f(b)(x-b)^{(2)} + \theta(x-b)$ for $n = 4$*

$$f(x) = f(b) + \frac{25}{12} D_b f(b)(x-b) + \frac{35}{24} D_b^2 f(b)(x-b)^{(2)} + \frac{10}{24} D_b^3 f(b)(x-b)^{(3)} + \frac{1}{24} D_b^4 f(b)(x-b)^{(4)} + \theta(x-b)$$

13. New Polynomial for f(x)

In this section, by the h-derivatives, a polynomial of degree n for f about the point b, is obtained. The following is a known Lemma, see [6]

Proposition 13.1. *Let $f : U \subseteq \Re^r \to \Re$ be a function of class C^n in the open U, then a ξ exists, with $0 \le \xi \le 1$, such that $f(1) = \sum_{\nu=0}^{n-1} \frac{f^{(\nu)}(0)}{\nu!} + \frac{f^{(n)}(\xi)}{n!}$, where $f^{\nu}(t) = (\frac{d}{dt})^{\nu} f(t)$ and the closed unit interval $0 \le t \le 1$ in U.*

It follows

Theorem 13.2. *Let $f : U \subseteq \Re^r \to \Re$ be a function of class C^n in the open U, $t \in [0,1]$, and $(x + e^t(b-x)) \in U$, then*

$$f(x + e(b-x)) = f(b) + \sum_{\nu=1}^{n-1} \frac{H^{(\nu)} f(b)(b-x)^{(\nu)}}{\nu!} + \theta(b-x) \tag{23}$$

where $r \ge 2$, $H^{(\nu)} f(b)(b-x)^{(\nu)} = h^{(\nu)}(t)$ with $h(t) = f(x + e^t(b-x))$. For $r = 1$

$$f(x + \frac{1}{k}(e-1)) = f(x) + \sum_{\nu=1}^{n-1} \frac{h^{(\nu)}(0)}{\nu!} + \theta(x) \tag{24}$$

where $h(t) = f(x + \frac{1}{k}(e^t - 1))$, $k \in \{\Re - 0\}$ and $(x + \frac{1}{k}(e^t - 1)) \in U$.

Proof. By the lemma 13.1, applied to the function $h(t) = f(x + e^t(b-x))$ or, in the particular case, to the function $h(t) = f(x + \frac{1}{k}(e^t - 1))$, it is $h(1) = h(0) + \sum_{\nu=1}^{n-1} \frac{h^{(\nu)}(0)}{\nu!} + \frac{h^{(n)}(\xi)}{n!}$ then the polynomial 23 and 24. □

By the 24, for $t \to 0$,

$$\begin{aligned} f(x + \frac{1}{k}(e^t - 1)) = f(x) + t\frac{f'(x))}{k} + \frac{t^2}{2!}(\frac{f'(x)}{k} + \frac{f''(x))}{k^2} \\ + \frac{t^3}{3!}(\frac{f'(x)}{k} + 3\frac{f''(x)}{k^2} + \frac{f^{(3)}(x)}{k^3}) + \frac{t^4}{4!}(\frac{f'(x)}{k} + 7\frac{f''(x_0)}{k^2} + 6\frac{f^{(3)}(x)}{k^3} + \frac{f^{(4)}(x)}{k^4}) + \circ(t^4) \end{aligned}$$

or equivalently

$$\begin{aligned} f(x) = f(x_0) + \mu\frac{f'(x))}{k} + \frac{\mu^2}{2!}(\frac{f'(x)}{k} + \frac{f''(x))}{k^2}) \\ + \frac{\mu^3}{3!}(\frac{f'(x)}{k} + 3\frac{f''(x)}{k^2} + \frac{f^{(3)}(x)}{k^3}) + \frac{\mu^4}{4!}(\frac{f'(x)}{k} + 7\frac{f''(x_0)}{k^2} + 6\frac{f^{(3)}(x)}{k^3} + \frac{f^{(4)}(x)}{k^4})) + \circ(\mu^4) \end{aligned}$$

where $\mu = \log(k(x-x_0)-1)$. By a similar way, starting from $h(t) = f(x-1+k^t)$, $k>0$, the same development is obtained. It is $\log(k(x-x_0)-1) = O(k(x-x_0))$. Moreover, by the Leibtniz's formula

$$D^{(n)}r(t)s(t) = \sum_{i=0}^{n} \binom{n}{i} r^{(n-i)}s^{(k)},$$

suppose $r(t) = f'(x_0 + \frac{1}{k}(e^t-1))$ and $s(t) = e^t$, so

$$h^n(t) = \sum_{i=0}^{n-1} \binom{n-1}{i} (f')^{(n-1-i)}(e^t)^{(i)} = e^t \sum_{i=0}^{n-1} \binom{n-1}{i} (f')^{(n-1-i)}$$

and

$$h^n(0) = (\sum_{i=0}^{n-1} \binom{n-1}{i} (f')^{(n-1-i)}(t))_{t=0}$$

then the final form

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + (kf'(x_0) + f''(x_0))\frac{(x-x_0)^2}{2!} \\ &+ (k^2f'(x_0) + 3kf''(x_0) + f^{(3)}(x_0))\frac{(x-x_0)^3}{3!} + \cdots + (\sum_{i=0}^{n-1} \binom{n-1}{i} (f')^{(n-1-i)}(t))_{t=0}\frac{(x-x_0)^n}{n!} + \circ(x-x_0)^n \end{aligned} \tag{25}$$

Example 13.3. *Using the 25, with $x_0 = 0$, $k = 1$*

$$\begin{aligned} e^x &= e^0 + e^0x + (e^0+e^0)\frac{x^2}{2!} + (e^0 + 3e^0 + e^0)\frac{x^3}{3!} + (e^0 + 6e^0 + 7e^0 + e^0)\frac{x^4}{4!} + \circ(x^4) \\ &= 1 + x + x^2 + \frac{5}{6}x^3 + \frac{15}{24}x^4 + \circ(x^4) \end{aligned}$$

The pointwise convergence is slower with respect to the Taylor' development, this is due to the choice $k = 1$.

14. Pointwise Convergence

In Numerical Analysis and other applications, it is useful to know a development of a differentiable function f with pointwise convergence faster of the Taylor' formula. The aim of this section is to make a such representation for f. Let $h(t)$ be a differentiable function of class $C^{n+1}(U)$, where U is an open set with $t_0 \in U$. By the Taylor' formula it follows

$$h(t) = \sum_{i=0}^{n} \frac{h^{(i)}(t_0)}{i!} + E_{T,n}(t)$$

it known that the remainder $E_{T,n}(t)$ may be written in the form $E_{T,n}(t) = \frac{1}{n!}\int_{t_0}^{t}(t-v)^n h^{(n+1)}(v)\,dv$. The following known theorem, see [1] , estimates the remainder.

Proposition 14.1. *If $h^{(n+1)}(t)$ satisfies, in $(t_0-\delta, t_0+\delta)$, $\delta > 0$, the inequality $m \leq h^{(n+1)}(t) \leq M$, then , in the same interval, it is*

$$\begin{aligned} m\frac{(t-t_0)^{n+1}}{(n+1)!} &\leq E_{T,n}(t) \leq M\frac{(t-t_0)^{n+1}}{(n+1)!} \qquad \text{for } t > t_0 \\ m\frac{(t_0-t)^{n+1}}{(n+1)!} &\leq (-1)^{n+1}E_{T,n}(t) \leq M\frac{(t_0-t)^{n+1}}{(n+1)!} \qquad \text{for } t < t_0 \end{aligned} \tag{26}$$

In 25, the h-development of f , the remainder $E_{H,n}$ is given by

$$
\begin{aligned}
E_{H,n} &= f(x) - (f(x_0) + f'(x_0)(x - x_0) + (kf'(x_0) + f''(x_0))\frac{(x-x_0)^2}{2!} \\
&\quad + (k^2 f'(x_0) + 3kf''(x_0) + f^{(3)}(x_0))\frac{(x-x_0)^3}{3!} + \cdots + (\sum_{i=0}^{n-1}\binom{n-1}{i}(f')^{(n-1-i)}(t))_{t=0}\frac{(x-x_0)^n}{n!} \\
&= f(x) - (f(x_0) + f'(x_0)(x - x_0) + f''(x_0\frac{(x-x_0)^2}{2!} + f^{(3)}(x_0))\frac{(x-x_0)^3}{3!} + \cdots f^{(n)}(x_0))\frac{(x-x_0)^n}{n!}) \\
&\quad - (\sum_{i=2}^{n}\frac{k^i(x-x_0)^i}{i!}((\sum_{j=0}^{i-1}\binom{i-1}{j}(f')^{(i-1-j)}(t))_{t=0} - \frac{1}{k^i}f^{(i-1)}(x_0))) \\
&= E_{T,n} - (\sum_{i=2}^{n}\frac{k^i(x-x_0)^i}{i!}((\sum_{j=0}^{i-1}\binom{i-1}{j}(f')^{(i-1-j)}(t))_{t=0} - \frac{1}{k^i}f^{(i-1)}(x_0))) \\
&= E_{T,n} - r_n
\end{aligned}
$$

where r_n denotes the sum in right side. In order to determinate a value of k such that $0 < |E_{H,n}| < |E_{T,n}|$, consider the two cases

(i) $0 < E_{H,n} < E_{T,n}$, if $E_{T,n} > 0$. By the 26, it follows $0 < r_n < E_{T,n}$ and this inequality is satisfied substituting for the lower bound of $E_{T,n}$, that is

$$
\begin{cases}
0 < r_n < m\frac{(x-x_0)^{n+1}}{(n+1)!} & \text{if } x > x_0 \\
0 < (-1)^{n+1} r_n < m\frac{(x_0-x)^{n+1}}{(n+1)!} & \text{if } x < x_0
\end{cases} \tag{27}
$$

(ii) $E_{T,n} < E_{H,n} < 0$ if $E_{T,n} < 0$ in the same way, using the upper bound of $E_{T,n}$

$$
\begin{cases}
M\frac{(x-x_0)^{n+1}}{(n+1)!} < r_n < 0 & \text{if } x > x_0 \\
M\frac{(x_0-x)^{n+1}}{(n+1)!} < (-1)^{n+1} r_n < 0 & \text{if } x < x_0
\end{cases} \tag{28}
$$

If $n + 1$ is odd, then 27 and 28 became an unique inequality. The following examples show how to use these inequalities.

Example 14.2. *Consider the h-polynomial of degree two of $f(x) = \cos x$ about $x_0 = 2$. The third derivative is $\sin x$ and this satisfies the inequality $\frac{1}{2} < \sin x < 1$ on the interval $(1.8, 2.5)$. So the $E_{T,2}$' estimate is*

$$
\begin{cases}
\frac{1}{2}\frac{(x-2)^3}{3!} < E_{T,2} < 1\frac{(x-2)^3}{3!} & \textit{if } x > x_0, \textit{ where } E_{T,2} > 0 \\
\frac{1}{2}\frac{(2-x)^3}{3!} < (-1)^3 E_{T,2} < 1\frac{(x-2)^3}{3!} & \textit{if } x < x_0, \textit{ where } E_{T,2} < 0
\end{cases} \tag{29}
$$

(i) $E_{T,2} > 0$ for $x > 2$, then

$$
\begin{aligned}
0 < E_{H,2} &= (f(x) - f(x_0) - (x - x_0)f'(x_0) - \frac{(x-x_0)^2}{2!}f''(x_0)) - \frac{k(x-x_0)^2}{2}f'(x_0) \\
&= E_{T,2} - \frac{k(x-x_0)^2}{2}f'(x_0) < E_{T,2}
\end{aligned}
$$

that is $0 < \frac{k(x-x_0)^2}{2}f'(x_0) < E_{T,2}$. By the 29, it is $0 < k\frac{(x-2)^2}{2}f'(2) < \frac{1}{12}(x-2)^3$, so $k > \frac{x-2}{6(-\sin 2)}$. Considering together the inequalities $-\frac{x-2}{5.4558} < k < 0$. Then choose $k = -\frac{x-2}{6}$, the h-polynomial is

$$
\cos x \approx \cos 2 - (\sin 2)(x-2) + \frac{(x-2)^2}{2}(-\frac{x-2}{6}(-\sin 2) - \cos 2) \qquad \textit{for } x > 2
$$

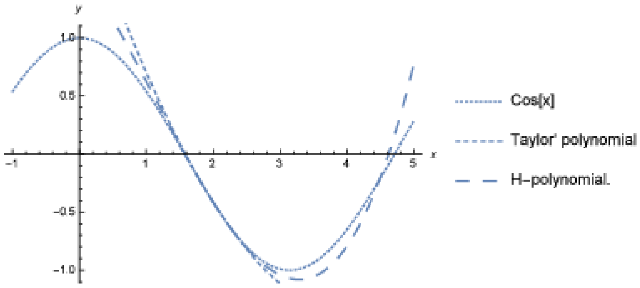
(ii) $E_{T,2} < 0$ for $x < 2$, then

$$
E_{T,2} < E_{H,2} = E_{T,2} - \frac{k(x-x_0)^2}{2}f'(x_0) < 0
$$

then $0 < -\frac{k(x-x_0)^2}{2} < -E_{T,2}$. *By the* 29, $0 < -\frac{k(x-x_0)^2}{2} < \frac{1}{12}(2-x)^3$ *and* $0 < k < -\frac{2-x}{6f'(x_0)}$, *then choose* $k = \frac{2-x}{6}$, *the h-polynomial is*

$$\cos x \approx \cos 2 - (\sin 2)(x-2) + \frac{(x-2)^2}{2}(\frac{2-x}{6}(-\sin 2) - \cos 2) \qquad \text{for } x < 2$$

The following graph immediately verifies that the h-polynomial is faster in pointwise convergence.



(Computer-generated graph).

Example 14.3. *Consider the h-polynomial of degree three of* $f(x) = x\sin x$ *about* $x_0 = 0$. *The fourth derivative is* $-4\cos x + x\sin x$ *and this satisfies the inequality* $-4 < -4\cos x + x\sin x < -1.3$ *on the interval* $(-1, 1)$. *So the* $E_{T,3}$*' estimate is*

$$\begin{cases} -4\frac{x^4}{4!} < E_{T,3}(x) < -1.3\frac{x^4}{4!} & \text{if } x > 0, \text{ where } E_{T,3} < 0 \\ -4\frac{(-x)^4}{4!} < (-1)^4 E_{T,3}(x) < -1.3\frac{(-x)^4}{4!} & \text{if } x < 0, \text{ where } E_{T,3} < 0 \end{cases} \tag{30}$$

that this

$$-\frac{1}{6}x^4 < E_{T,3}(x) < -\frac{1.3}{24}x^4 \qquad \text{for } x < 0 \text{ and } x > 0$$

Because of $E_{T,3} < 0$, *impose* $E_{T,3} < E_{H,3} < 0$ *and then*

$$E_{T,3} < \frac{k(x-x_0)^2}{6}((3 + k(x-x_0))f'(x_0) + 3(x-x_0)f''(x_0)) < 0 \tag{31}$$

(i) For $x > 2$, *by the second inequality of* 31 *it is*

$$k((3 + k(x-x_0))f'(x_0) + 3(x-x_0)f''(x_0)) < 0$$

in the example $6kx < 0$ *then* $k < 0$ *the first inequality of* 31, *using the upper bound of* $E_{T,3}$ *becomes*

$$\frac{k(x-x_0)^2}{6}((3 + k(x-x_0))f'(x_0) + 3(x-x_0)f''(x_0)) > \frac{-1.3x^4}{24}$$

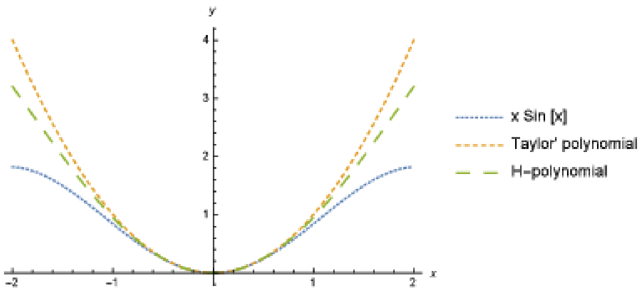
that is

$$4k((3 + k(x-x_0))(f'(x_0) + 3(x-x_0)f''(x_0)) + 1.3x^2 > 0$$

in the example

$$24kx + 1.3x^2 > 0 \qquad \text{then} \qquad k > -0.054167x$$

so k *has to satisfy* $-0.054167 < k < 0$, *choose* $k = -0.05x$.



(Computer-generated graph)

(ii) For $x < 0$, by the second inequality of 31, in the same way of (i), it follows

$$6 + 6kx < 0 \qquad \textit{then} \qquad k > -\frac{1}{x}$$

by the first inequality, in the same way of (i),

$$24kx + 1.3x^2 > 0 \qquad \textit{then} \qquad k < -0.05417x \tag{32}$$

Again choose $k = -0.05x$ then the h-polynomial is

$$x \sin x \approx x^2 - 0.05x^4$$

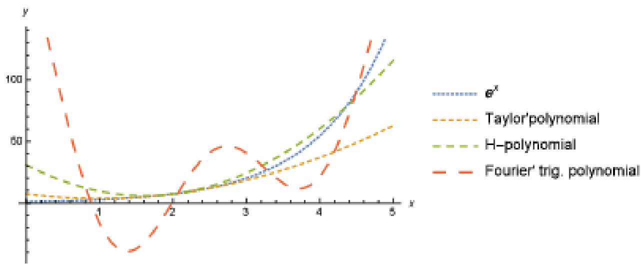
The graph above verifies the pointwise convergence of the h-polynomial.

15. Convergence in Square Mean

For an integrable function $f(x)$, with the h-polynomial H_n, the square error E_n in the interval (a, b), is defined by

$$E_n = \int_a^b (f(x) - H_n)^2 \, dx$$

It is possible to minimize E_n suitably choosing the k value in the h-polynomial. Next example shows this algorithm and confronts the result with the Taylor and Fourier polynomials.



(Computer-generated graph)

Example 15.1. *Consider $f(x) = e^x$ and its H-polynomial to order two about the point $x_0 = 2$. The square error, in the interval $(0, 5)$, is*

$$E_2 = \int_a^b (f(x) - (f(x_0 + f'(x_0)(x - x_0) + \frac{(x - x_0)^2}{2}(kf'(x_0) + f''(x_0)))^2 dx$$

that is

$$E_2 = \int_0^5 (e^x - (e^2 + e^2(x-2) + \frac{(x-2)^2}{2}(ke^2 + e^2))^2 dx$$
$$= \frac{1}{4}(-2 + 2e^{10} + 8e^2(3+5k) - 4e^7(11+5k) + \frac{5}{3}e^4(152 + 133k + 33k^2))$$

Find the minimum of $E_2(k)$

$$D_k E_2(k) = \frac{1}{4}(40e^2 - 20e^7 + \frac{5}{3}e^4(133 + 66k)$$

and

$$D_k E_2(k) = 0 \qquad for \qquad k = \frac{1}{66e^2}(-24 - 133e^2 + 12e^5) = 1.5876$$

The h-polynomial, with $k = 1.5876$*, is*

$$e^2(1 + (x-2) + \frac{(x-2)^2}{2}(2.5876))$$

The Taylor' polynomial is

$$e^2(1 + (x-2) + \frac{(x-2)^2}{2})$$

The Fourier' trigonometric polynomial is

$$\frac{e^{2\pi} - 1}{2\pi} + \sum_{h=1}^{2} \frac{1}{\pi} \frac{e^{2\pi} - 1}{1 + h^2} \cos(hx) + \sum_{h=1}^{2} \frac{h}{\pi} \frac{-e^{2\pi} + 1}{1 + h^2} \sin(hx)$$

The graph above shows that the h-polynomial is mean square convergent, in the interval $(0, 5)$, better than the other polynomials. As a numerical check :

by the Taylor' polynomial, the square error is 2452

by the Fourier' polynomial, the square error is 14338

by the H- polynomial, the square error is 559.919

16. Partial h-derivatives

The partial derivatives, by the h-derivation, have the following

Definition 16.1. *Let* $f(x, y)$ *be a function on an open set* U *which possess continuous partial h-derivatives, denoted by* $H^{(\alpha_1,\alpha_2)} f(x, y)$*, then*

$$H^{(\alpha_1,\alpha_2)} f(x, y) = (\frac{d}{dt})^{\alpha_2}(((\frac{d}{dt})^{\alpha_1} h_1(t))_{t=0}(x, y - 1 + e^t))_{t=0}$$

where $h_1(t) = f(x - 1 + e^t, y)$ *and* $k = 1$*.*

The definition may be extended to functions with more variables.

Example 16.2.

$$H^{(2,1)} f(x, y) = (\frac{d}{dt})^1(((\frac{d}{dt})^2 f(x - 1 + e^t, y))_{t=0}(x, y - 1 + e^t))_{t=0}$$
$$= (\frac{d}{dt})^1(f^{(1,0)}(x, y - 1 + e^t) + f^{(2,0)}(x, y - 1 + e^t))_{t=0}$$
$$= f^{(1,1)}(x, y) + f^{(2,1)}(x, y)$$

With respect to the vector $b - x$, a new definition of partial derivatives is

Definition 16.3. *Let $f(x,y)$ be a function of class C^n on an open set U, then*

$$K^{(\alpha_1,\alpha_2)}f(b_1,b_2) = (\frac{d}{dt})^{\alpha_2}(((\frac{d}{dt})^{\alpha_1}k_1(0))(x, y+e^t(b_2-y)))_{t=0} \text{ where } k_1(t) = f(x+e^t(b_1-x), y).$$

Example 16.4.

$$\begin{aligned}K^{(2,1)}f(b_1,b_2) &= (\frac{d}{dt})^1(((\frac{d}{dt})^2 f(x+e^t(b_1-x), y))_{t=0}(x, y+e^t(b_2-y)))_{t=0}\\ &= (\frac{d}{dt})^1((b_1-x)f^{(1,0)}(x, y+e^t(b_2-y)) + (b_1-x)^2 f^{(2,0)}(x, y+e^t(b_2-y)))_{t=0}\\ &= (b_1-x)(b_2-y)f^{(1,1)}(b_1,b_2) + (b_1-x)^2(b_2-y)f^{(2,1)}(b_1,b_2)\end{aligned}$$

The h-derivative and the k-partials are related by the following statement

Proposition 16.5. *Let U be an open set in $\Re^2$ and let $f \in C^n(U)$. Then*

$$H^n f(b)(b-x)^{(n)} = \sum_{i=0}^{n}\binom{n}{i}K^{(n-i,i)}f(b)$$

with $x=(x_1,x_2),\ b=(b_1,b_2)\in U$.

Proof. It is immediate $H^1 f(b)(b-x) = K^{(1,0)}f(b) + K^{(0,1)}f(b)$ with $h(t) = f(x+e^t(b-x))$, by induction

$$\begin{aligned}H^{n+1}f(b)(b-x)^{(n+1)} &= \frac{d}{dt}((\frac{d}{dt})^n h(t))_{t=0}\\ &= \frac{d}{dt}(\sum_{i=0}^{n}\binom{n}{i}K^{(n-i,i)}f(x+e^t(b-x)))_{t=0}\\ &= \sum_{i=0}^{n}\binom{n}{i}(K^{(n-i+1,i)}f(b) + K^{(n-i,i+1)}f(b))\\ &= \sum_{i=0}^{n+1}\binom{n+1}{i}K^{(n+1-i,i)}f(b)\end{aligned}$$

□

The proposition may be extended to functions with more variables.

Example 16.6. *For $n=3$*

$$H^3 f(b)(b-x)^{(3)} = K^{(3,0)}f(b) + 3K^{(2,1)}f(b) + 3K^{(1,2)}f(b) + K^{(0,3)}f(b)$$

By Definition 16.1, it is immediate to verify the Schwarz's property, that is permissible to interchange the order of differentiation

Example 16.7.

$$\begin{aligned}&H^{(2,0)}f(x,y) = f^{(1,0)}(x,y) + f^{(2,0)}(x,y),\\ &(\frac{d}{dt}(f^{(1,0)}(x, y-1+e^t) + f^{(2,0)}(x, y-1+e^t))_{t=0} = f^{(1,1)}(x,y) + f^{(2,1)}(x,y) = H^{(2,1)}f(x,y)\end{aligned}$$

to the same result by

$$\begin{aligned}&H^{(0,1)}f(x,y) = f^{(0,1)}(x,y),\\ &((\frac{d}{dt})^2(f^{(0,1)}(x-1+e^t, y))_{t=0} = f^{(1,1)}(x,y) + f^{(2,1)}(x,y)\\ &= H^{(2,1)}f(x,y)\end{aligned}$$

17. Homogeneous Complex Functions

The definition of h-derivative for a complex function $f(z)$ may be rewritten in the form

Definition 17.1.

$$Hf(z) = h'(0) = \lim_{v\to 0} \frac{f(z+k(e^{t+v}-1)) - f(z)}{v}$$

where $h(t) = f(z+k(e^t-1))$, $t, v, k \in C$.

It is immediate that $f(z)$ is necessarily continuous. Indeed, by $h(t+v) - h(t) = k(h(t+v) - h(t))/v$, it follows

$$\begin{aligned}\lim_{v\to 0}(h(t+v) - h(t)) &= \lim_{v\to 0}(f(z+k(e^{t+v}-1)) - f(z+k(e^t-1)) \\ &= 0 \cdot h'(t) = 0\end{aligned}$$

so, for $t = 0$, $\lim_{v\to 0} f(z+k(e^v-1)) = f(z)$ and f is continuous at z. Let $h(t) = f(z+k(e^t-1))$ be differentiable at $t = 0$ and let $z = x+iy$, $t = t_1 + it_2$. By the Cauchy-Riemann equation, it follows

$$\begin{aligned}H^{(1,0)}f(z) &= (\frac{\partial h(t)}{\partial t_1})_{t=0} = k(\frac{\partial f(z+k(e^t-1))}{\partial x})_{t=0} = k\frac{\partial f(z)}{\partial x} = kf'(z) \\ H^{(0,1)}f(z) &= (\frac{\partial h(t)}{\partial t_2})_{t=0} = k(\frac{\partial f(z+(e^t-1))}{\partial y})_{t=0} = k\frac{\partial f(z)}{\partial y} = kif'(z)\end{aligned}$$

that is $Hf(z) = kf'(z)$ and $iH^{(1,0)}f(z) = H^{(0,1)}f(z)$.

Proposition 17.2. *Let $f(z)$ be analytic in a region Ω. Then*

$$H^{(2,0)}f(z) + H^{(0,2)}f(z) = 0 \tag{33}$$

Proof. By

$$\begin{aligned}H^{(2,0)}f(z) &= (\frac{\partial^2}{\partial t_1^2}f(z+k(e^{t_1+it_2}-1)))_{t=0} = kf'(z) + k^2f''(z) \qquad \text{and} \\ H^{(0,2)}f(z) &= (\frac{\partial^2}{\partial t_2^2}f(z+k(e^{t_1+it_2}-1)))_{t=0} = -kf'(z) - k^2f''(z)\end{aligned}$$

the 33. □

That is, the h-derivation satisfies the Laplace's equation.

Example 17.3. *Let $f(z) = 4xy - i(x-iy)^2$, then $H^{(2,0)}f(z) = kf'(z) + k^2f''(z) = -2ik^2 + k(-2i(x-iy) + 4y)$*

The theorem 9.1 has a version for complex homogeneous functions.

Proposition 17.4. *Let $\psi : C^2 \to C$ be the homogeneous function of class C^p, defined by $(z_1, z_2) \to \psi(\frac{z_2}{z_1})$, with $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then*

$$(i) \quad \begin{cases} D^k\psi(x_1,x_2)\ (z_1,z_2)^{(k)} = 0 & k = 1,\dots,p \\ D^k\psi(y_1,y_2)\ (z_1,z_2)^{(k)} = 0 & k = 1,\dots,p \end{cases}$$

$$(ii) \quad \begin{cases} \psi^{(n-1,1,0,0)}(\frac{z_2}{z_1}) + (-i)^n\psi^{(1,n-1,0,0)}(\frac{z_2}{z_1}) = 0 & \text{for } n \geq 2 \\ \psi^{(0,0,n-1,1)}(\frac{z_2}{z_1}) + (-i)^n\psi^{(0,0,1,n-1)}(\frac{z_2}{z_1}) = 0 & \text{for } n \geq 2 \end{cases}$$

where $\psi(\frac{z_2}{z_1}) = \psi(x_1,x_2) = \psi(y_1,y_2) = \psi(x_1,y_1,x_2,y_2)$.

Proof. (*i*) The partial derivatives with respect to x_1 and x_2 are

$$\psi^{(1,0,0,0)}(\frac{z_2}{z_1}) = \frac{1}{z_2}\psi'(\frac{z_2}{z_1}) \qquad \text{and} \qquad \psi^{(0,0,1,0)}(\frac{z_2}{z_1}) = -\frac{z_1}{z_2^2}\psi'(\frac{z_2}{z_1})$$

then $(\psi^{(1,0,0,0)}(\frac{z_2}{z_1}))\cdot z_1 + (\psi^{(0,0,1,0)}(\frac{z_2}{z_1}))\cdot z_2 = 0$. By induction $D^p\psi(x_1,x_2)\ (z_1,z_2)^{(p)} = D(D^{p-1}\psi(x_1,x_2)\ (z_1,z_2)^{(p-1)})(z_1,z_2) = 0$. The second of (*i*) is proved by the same way.

(*ii*) The partial derivatives with respect to x_1 and x_2 are

$$\psi^{(n-1,1,0,0)}(\frac{z_2}{z_1}) = \frac{i}{z_2^n}\psi^{(n)}(\frac{z_2}{z_1}) \qquad \text{and} \qquad \psi^{(1,n-1,0,0)}(\frac{z_2}{z_1}) = -\frac{(-i)^{1-n}}{z_2^n}\psi^{(n)}(\frac{z_2}{z_1})$$

summing the partial derivatives, the first of (*ii*) follows. By a same way, the second of (*ii*). □

18. Power Series

The Cauchy's Theorem has an extension by the h-derivation. Let $H(\Omega)$ be the ring of all holomorphic functions in the region Ω.

Proposition 18.1. *Let $h(t) = f(z+\frac{1}{k}(e^t-1)) \in H(\Omega)$ and γ in Ω represents a circle $a+re^{i\theta}$, $0\le\theta\le 2\pi$, then*

$$(i) \qquad \frac{h^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{h(t)}{(t-a)^{n+1}}dt \tag{34}$$

supposing $a = 0$

$$(ii) \qquad \frac{h^{(n)}(0)}{n!} = \frac{H^{(n)}f(z)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(z+\frac{1}{k}(e^t-1))}{t^{n+1}}dt \tag{35}$$

Proof. Immediate by the Cauchy's formula. □

Example 18.2. *The 35, for $n = 2$, $f(z) = z^2$, $r = 1$, $t = e^{i\theta}$ and $dt = ie^{i\theta}d\theta$, is*

$$\frac{1}{2}(\frac{2z}{k}+\frac{2}{k^2}) = \frac{H^2(z^2)}{2} = \frac{h^2(0)}{2} = \frac{1}{2\pi i}\int_\gamma \frac{(z+\frac{1}{k}(e^t-1))^2}{t^3}dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{(z+\frac{1}{k}(e^{e^{i\theta}}-1))^2}{(e^{i\theta})^2}d\theta = \frac{1}{2\pi}\cdot\frac{2\pi(1+kz)}{k^2} = \frac{z}{k}+\frac{1}{k^2}$$

The following result gives new power series for holomorphic functions, see [8]

Theorem 18.3. *Let $f(z) \in H(\Omega)$ be a holomorphic function in a region Ω with $z_0 \in \Omega$. Then f can be represented in Ω as the power series centered at z_0*

$$f(z) = \sum_{n\ge 0}\frac{h^{(n)}(0)}{n!}(z-z_0)^n = f(z_0) + \sum_{n\ge 1}\frac{1}{n!}(\sum_{k=0}^{n-1}\binom{n-1}{k}(f')^{(n-1-k)}(z_0))(z-z_0)^n \tag{36}$$

where $h(t) = f(z+\frac{1}{k}(e^t-1))$.

Proof. $h(t)$ is a holomorphic function at $t = 0$, indeed it is the composition of two holomorphic functions . So $h(t)$ is represented by the power series $h(t) = f(z+\frac{1}{k}(e^t-1)) = \sum_{n\ge 0}\frac{h^{(n)}(0)}{n!}(t)^n$. By a substitution, it is $f(z) = \sum_{n\ge 0}\frac{h^{(n)}(0)}{n!}\log^n(z+1-z_0)$, where log is a branch of the logarithm, and recalling $\log(z-1+z_0) = O(z-z_0)$ it follows the 36 □

The next proposition gives a relation for holomorphic functions at each point of a close disk centered at 0.

Theorem 18.4. *Let γ be the counterclockwise circle with radius r centered at 0 and $f(z+t)$ be holomorphic on γ and inside, then there exists c, with $0 < c < 2\pi$ such that*

$$f(z+re^{ic}) = \frac{re^{ic}f(re^{ic})}{re^{ic}-z} \qquad \text{for all } z \text{ inside } \gamma \tag{37}$$

Proof. By the Cauchy's integral formula it follows $f(z+w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z+t)}{t-w}dt$ then, for $w=0$, supposing $t = re^{i\theta}$, with $dt = ire^{i\theta}d\theta$, and $0 \leq \theta \leq 2\pi$

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\, d\theta$$

Again, by the Cauchy's integral formula, $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(v)}{v-z}dv$ with z inside γ , supposing $v = re^{i\theta}$

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(re^{i\theta})}{re^{i\theta}-z} ire^{i\theta}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(re^{i\theta})}{re^{i\theta}-z} re^{i\theta}\, d\theta \qquad z \text{ inside } \gamma \end{aligned}$$

comparing the two forms for $f(z)$

$$\int_0^{2\pi} (f(z+re^{i\theta}) - \frac{re^{i\theta}f(re^{i\theta})}{re^{i\theta}-z})\, d\theta = 0 \qquad z \text{ inside } \gamma$$

as the function in the integral is continuous, by the mean value theorem, there is at least one point c, with $0 < c < 2\pi$, such that 37. □

Example 18.5. *Suppose $f(z) = z^2$ and $r = 1$, the 37 becomes $(z+re^{ic})^2 = \frac{(re^{ic})^2}{re^{ic}-z}$. Solving the equation by e^{ic} it follows $e^{ic} = \frac{1}{2}z(1\pm\sqrt{5})$, then the identity $(z+\frac{1}{2}z(1\pm\sqrt{5})^2 = \frac{(\frac{1}{2}z(1\pm\sqrt{5})^2}{\frac{1}{2}z(1\pm\sqrt{5})-z}$.*

References

[1] T.M.Apostol, *Calculus*, J.Wiley&S., (1969).

[2] F.Fineschi and G.Quaranta, *A convex hull' characterization*, Pure and Applied Mathematics Journal, 3(2)(2014), 40-48.

[3] J.B.Hiriart-Urruty and C.Lemaréchal, *Fundamentals of Convex Analysis*, Springer-Verlag, (2001).

[4] S.Karlin, *Mathematical Methods and Theory in Games, Programming, and Economics - The Theory of Infinite Games*, Addison-Wesley, (1959).

[5] S.Lang, *Real and Functional Analysis*, Springer-Verlag, (1993).

[6] R.Narasimhan, *Analysis on Real and Complex Manifolds*, North-Holland Publishing Company, (1973).

[7] Y.Pinchover and J.Rubinstein, *An Introduction to Partial Differential Equations*, Cambridge, (2005).

[8] W.Rudin, *Real and Complex Analysis*, McGraw-Hill Book C., (1987).

[9] E.Snapper, R.J.Troyer, Metric Affine Geometry, Dover Publications, Inc. New York, (1989).