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# The Maximum Independent Vertex Energy of a Graph

Research Article

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**Abstract:** In a graph $G = (V, E)$, A set $I \subseteq V$ is an independent vertex set if no two vertices in $I$ are adjacent. The number of vertices in a maximum independent set in a graph $G$ is the independence number (or vertex independence number) of $G$ and is denoted by $\beta(G)$. In this paper, we study the maximum independent vertex energy, denoted by $E_I(G)$, of a graph $G$. We are compute the maximum independent energies of complete graph, complete bipartite graph, star graph, cocktail party graph and Friendship graph. Upper and lower bounds for $E_I(G)$ are established.

**MSC:** 05C50, 05C99.

**Keywords:** Independent set, independence number, maximum Independent matrix, maximum Independent eigenvalues, maximum Independent energy.



## 1. Introduction

In this paper, by a graph $G = (V, E)$ we mean a simple graph that is finite, have no loops no multiple and directed edges. We denoted by $n = |V|$ and $m = |E|$ to the number of vertices and edges of $G$, respectively. For a vertex $v \in V$, the open neighborhood of $v$ in $G$, denoted $N(v)$, is the set of all vertices that are adjacent to $v$ and the closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. The degree of vertex $v$ in $G$, denoted by $d(v)$, is the number of its neighbors in $G$. We denoted by $\Delta$ and $\delta$ the maximum and minimum degree among the vertices of G , respectively. We denoted by $\lceil x \rceil$ to the smallest integer number greater than or equals to $x$ and $\lfloor x \rfloor$ to the greatest integer number smaller than or equals to $x$. For more terminologies and notations in graph theory do not define here, we refer the reader to book Harary F. [8].

A set $I \subseteq V$ is independent if no two vertices in $I$ are adjacent. Independent sets of vertices in graphs is one of the most commonly studied concepts in graph theory. The independent sets of maximum cardinality are called maximum independent sets and these are the independent sets that have received the most attention. The number of vertices in a maximum independent set in a graph $G$ is the independence number (or vertex independence number ) of $G$ and is denoted by $\beta(G)$. There are also certain independent sets of minimum cardinality that are of interest. Ordinarily, a graph contains many independent sets. An independent set of vertices that is not properly contained in any other independent set of vertices is a maximal independent set of vertices. The minimum number of vertices in a maximal independent set is denoted by $i(G)$. This parameter is also called the independent domination number as it is a smallest cardinality of an independent

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set of vertices that dominate all vertices of $G$. The concept of energy of a graph was introduced by I. Gutman [6] in the year 1978. Let $G$ be a graph with $n$ vertices and $m$ edges and let $A = (a_{ij})$ be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of A, assumed in non increasing order, are the eigenvalues of the graph $G$. As $A$ is real symmetric, the eigenvalues of $G$ are real with sum equal to zero. Let $\lambda_1, \lambda_2, ..., \lambda_t$ for $t \leq n$ be the distinct eigenvalues of $G$ with multiplicity $m_1, m_2, ..., m_t$, respectively, the multiset of eigenvalues of $A(G)$ is called the spectrum of $G$ and denoted by

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_t \\ m_1 & m_2 & ... & m_t \end{pmatrix}.$$

The energy $E(G)$ of $G$ is defined to be the sum of the absolute values of the eigenvalues of $G$, i.e.

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

For more details on the mathematical aspects of the theory of graph energy see [2, 7, 12]. The basic properties including various upper and lower bounds for energy of a graph have been established in [13, 14], and it has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [4, 5]. Recently C. Adiga et al [1] defined the minimum covering energy, $E_C(G)$ of a graph which depends on its particular minimum cover $C$. Further, minimum dominating energy, Laplacian minimum dominating energy and minimum dominating distance energy of a graph $G$ can be found in [9–11]. Motivated by these papers, we study the maximum independent energy $E_I(G)$ of a graph $G$. We compute maximum independent energies of some standard graphs. Some properties of characteristic polynomial of a maximum independent matrix of a graph $G$ are obtained. Upper and lower bounds for $E_I(G)$ are established. It is possible that the maximum independent energy that we are considering in this paper may be have some applications in chemistry as well as in other areas.

## 2. The Maximum Independent Vertex Energy of a Graph

Let $G$ be a graph of order $n$ with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G)$. Let $I$ be an independent set in $G$. The independence number $\beta(G)$ of $G$ is the cardinality of a largest independent set in $G$. Any independent set $S$ in $G$ with cardinality equals to $\beta(G)$ is called a maximum independent set of $G$. The maximum independent matrix of $G$ is the $n \times n$ matrix, denoted by $A_I(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E; \\ 1, & \text{if } i = j \text{ and } v_i \in I; \\ 0, & \text{othewise.} \end{cases}$$

The characteristic polynomial of $A_I(G)$ is denoted by

$$f_n(G, \lambda) = det\left(\lambda I - A_I(G)\right).$$

The maximum independent eigenvalues of a graph $G$ are the eigenvalues of $A_I(G)$. Since $A_I(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. The maximum independent energy of $G$ is defined as:

$$E_I(G) = \sum_{i=1}^{n} |\lambda_i|.$$

We first compute the maximum independent energy of a graph in Figure 1

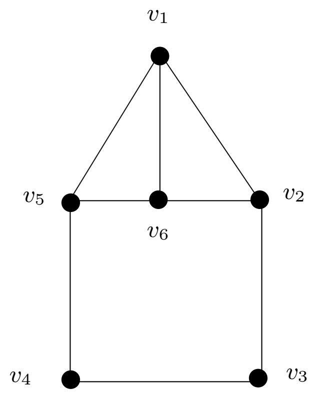


Figure 1

Let $G$ be a graph in Figure 1, with vertices set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Then $G$ has more than one set as a maximum independent set. for example, $I_1 = \{v_1, v_3\}$. Then

$$A_{GD_1}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{I_1}(G)$ is

$$f_n(G, \lambda) = \lambda^6 - 2\lambda^5 - 7\lambda^4 + 7\lambda^3 + 13\lambda^2 - 4\lambda - 4.$$

Hence, the maximum independent eigenvalues are $\lambda_1 \approx 3.1547$, $\lambda_2 \approx 1.6524$, $\lambda_3 \approx 0.6947$, $\lambda_4 \approx -0.4983$, $\lambda_5 \approx -1.3061$, $\lambda_6 \approx -1.6973$. Therefore the maximum independent energy of $G$ is

$$E_{I_1}(G) \approx 9.0036.$$

But if we take another maximum independent set of $G$, namely $I_2 = \{v_2, v_5\}$, we get that

$$A_{GD_2}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{N_2}(G)$ is

$$f_n(G, \lambda) = \lambda^6 - 2\lambda^5 - 7\lambda^4 + 6\lambda^3 + 14\lambda^2 - 4\lambda - 8.$$

The maximum independent eigenvalues are $\lambda_1 \approx 3.2361,\ \lambda_2 \approx 1.4142,\ \lambda_3 \approx 1,\ \lambda_4 = -1,\ \lambda_5 = -1.2361,\ \lambda_6 \approx -1.4142$. Therefore the maximum independent energy of $G$ is

$$E_{I_2}(G) \approx 9.3006.$$

The examples above illustrate that the maximum independent energy of a graph $G$ depends on the choice of the maximum independent set. i.e. the maximum independent energy is not a graph invariant.

In the following section, we introduce some properties of characteristic polynomial of maximum independent matrix of a graph $G$.

**Theorem 2.1.** *Let $G$ be a graph of order $n$, size $m$, independence number $\beta(G)$ and let*

$$f_n(G,\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + ... + c_n$$

*be the characteristic polynomial of the maximum independent matrix of a graph $G$. Then*

1. $c_0 = 1$.
2. $c_1 = -\beta(G)$
3. $c_2 = \binom{\beta(G)}{2} - m$

*Proof.* 1. From the definition of $f_n(G,\lambda)$.

2. Since the sum of diagonal elements of $A_I(G)$ is equal to $\beta(G)$. The sum of determinants of all $1 \times 1$ principal submatrices of $A_I(G)$ is the trace of $A_I(G)$, which evidently is equal to $\beta(G)$. Thus, $(-1)^1 c_1 = |I| = \beta(G)$.

3. $(-1)^2 c_2$ is equal to the sum of determinants of all $2 \times 2$ principal submatrices of $A_I(G)$ , that is

$$\begin{aligned} c_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1\le i<j\le n} (a_{ii}a_{jj} - a_{ij}a_{ji}) \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} a_{ij}^2 \\ &= \binom{\beta(G)}{2} - m. \end{aligned}$$

□

**Theorem 2.2.** *Let $G$ be a graph of order $n$. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of $A_I(G)$. Then*

*(i)* $\sum_{i}^{n} \lambda_i = \beta(G)$.

*(ii)* $\sum_{i}^{n} \lambda_i^2 = \beta(G) + 2m$.

*Proof.* $(i)$ Since the sum of the eigenvalues of $A_I(G)$ is the trace of $A_I(G)$, then

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = \beta(G)$$

.

$(ii)$ Similarly the sum of squares eigenvalues of $A_I(G)$ is the trace of $(A_I(G))^2$. Then

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + \sum_{i\neq j}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i<j}^{n} a_{ij}^2 \\ &= \beta(G) + 2m.\end{aligned}$$

□

Bapat and S. Pati [3], proved that if the graph energy is a rational number then it is an even integer. Similar result for maximum independent energy is given in the following theorem.

**Theorem 2.3.** *Let $G$ be a graph with a independent number $\beta(G)$. If the maximum independent energy $E_I(G)$ of $G$ is a rational number, then*

$$E_I(G) \equiv \beta(G) \pmod 2.$$

*Proof.* Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the maximum independent eigenvalues of a graph $G$ of which $\lambda_1, \lambda_2, ..., \lambda_r$ are positive and the rest are non-positive, then

$$\begin{aligned}\sum_{i=1}^{n} |\lambda_i| &= (\lambda_1 + \lambda_2 + ... + \lambda_r) - (\lambda_{r+1} + \lambda_{r+2} + ... + \lambda_n). \\ &= 2(\lambda_1 + \lambda_2 + ... + \lambda_r) - (\lambda_1 + \lambda_2 + ... + \lambda_n). \\ &= 2q - \beta(G).\ Where\ q = \lambda_1 + \lambda_2 + ... + \lambda_r.\end{aligned}$$

Since $\lambda_1, \lambda_2, ..., \lambda_r$ are algebraic integers, so is their sum. Hence $(\lambda_1 + \lambda_2 + ... + \lambda_r)$ must be an integer if $E_I(G)$ is rational. Hence the theorem is hold. □

# 3. The Maximum Independent Vertex Energy of Some Graphs

In this section, we investigate the exact values of the maximum independent energy of some standard graphs.

**Theorem 3.1.** *For $n \geq 2$, the maximum independent energy of the complete graph $K_n$, is equal to $(n-2) + \sqrt{n^2 - 2n + 5}$.*

*Proof.* For the complete graph $K_n$ the independence number is $\beta(K_n) = 1$. Hence, for $K_n$ the maximum independent matrix is same as minimum dominating matrix [11], therefore the maximum independent energy of the complete $K_n$ graph is equal to the minimum dominating energy of $K_n$. i.e. $E_I(K_n) = (n-2) + \sqrt{n^2 - 2n + 5}$. □

**Theorem 3.2.** *For $n \geq 2$, the maximum independent energy of a star graph $K_{1,n-1}$ is equal to $(n-2) + \sqrt{4n-3}$.*

*Proof.* Let $K_{1,n-1}$ be a star graph with vertex set $V = \{v_0, v_1, v_2, \cdots, v_{n-1}\}$, where $v_0$ is the central vertex. Then $\beta(K_{n-1}) = n-1$, where the maximum independent set of $K_{1,n-1}$ is $V - \{v_0\}$. Hence, for $K_{1,n-1}$ the maximum independent matrix is

$$A_I(K_{1,n-1}) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n}$$

The characteristic polynomial of $A_I(K_{1,n-1})$ is

$$f_n(K_{1,n-1}, \lambda) = \begin{vmatrix} \lambda & -1 & -1 & \cdots & -1 \\ -1 & \lambda-1 & 0 & \cdots & 0 \\ -1 & 0 & \lambda-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda-1 \end{vmatrix}_{n\times n}$$

$$= (\lambda-1)^{n-2}(\lambda^2 - \lambda - (n-1))$$

and the maximum independent spectrum of $K_{1,n-1}$ is

$$MI\ Spec(K_{1,n-1}) = \begin{pmatrix} 1 & \frac{1+\sqrt{4n-3}}{2} & \frac{1-\sqrt{4n-3}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

Therefore the maximum independent energy of star graph is $E_I(K_{1,n-1}) = (n-2) + \sqrt{4n-3}$. □

**Theorem 3.3.** *For the complete bipartite graph $K_{r,s}$, for $s \geq r$, the maximum independent energy is equal to $(s-1) + \sqrt{4rs+1}$.*

*Proof.* For the complete bipartite graph $K_{r,s}$ with vertex set $V = (V_1, V_2)$ where $V_1$ and $V_2$ are the partite sets of its, $V_1 = \{v_1, v_2, \cdots, v_r\}$ and $V_2 = \{u_1, u_2, \cdots, u_s\}$. The independent number is $\beta(K_{r,s}) = maxr, s = s$, where the maximum independent set is $I = V_2$. Hence, for $K_{r,s}$ the maximum independent matrix is

$$A_I(K_{r,s}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 1 \end{pmatrix}_{(r+s)\times(r+s)}$$

The characteristic polynomial of $A_I(K_{r,s})$ is

$$f_n(K_{r,s},\lambda)=\begin{vmatrix}\lambda & 0 & \cdots & 0 & -1 & -1 & \cdots & -1\\ 0 & \lambda & \cdots & 0 & -1 & -1 & \cdots & -1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda & -1 & -1 & \cdots & -1\\ -1 & -1 & \cdots & -1 & \lambda-1 & 0 & \cdots & 0\\ -1 & -1 & \cdots & -1 & 0 & \lambda-1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & \lambda-1\end{vmatrix}_{(r+s)\times(r+s)}$$

$$=\lambda^{r-1}(\lambda-1)^{s-1}(\lambda^2-\lambda-rs).$$

and the maximum independent spectrum of $K_{r,s}$ is

$$MI\ Spec(K_{r,s})=\begin{pmatrix}0 & 1 & \frac{1+\sqrt{4rs+1}}{2} & \frac{1-\sqrt{4rs+1}}{2}\\ r-1 & s-1 & 1 & 1\end{pmatrix}$$

Therefore the maximum independent energy is $E_I(K_{r,s})=(s-1)+\sqrt{4rs+1}$. □

**Definition 3.4.** *The cocktail party graph, denoted by $K_{2\times p}$, is a graph having vertex set $V(K_{2\times p})=\bigcup_{i=1}^{p}\{u_i,v_i\}$ and edge set $E(K_{2\times p})=\{u_iu_j,v_iv_j,u_iv_j,v_iu_j:\ 1\le i<j\le p\}$. i.e. $n=2p$, $m=\frac{p^2-3p}{2}$ and for ever $v\in V(K_{2\times p})$, $d(v)=2p-2$.*

**Theorem 3.5.** *For the cocktail party graph $K_{2\times p}$ of order $2p$, for $p\ge 3$, the maximum independent energy is equal to*

$$(2p-3)+\sqrt{4n^2-4n+9}.$$

*Proof.* For cocktail party graphs $K_{2\times p}$ with vertex set $V=\bigcup_{i=1}^{p}\{u_i,v_i\}$, the maximum independent set is is same as minimum dominating matrix [11], therefore the maximum independent energy is equal to minimum dominating energy. i.e. $E_I(K_{2\times p})=(2n-1)+\sqrt{4n^2-4n+9}$. □

# 4. Bounds for Maximum Independent Vertex Energy of a Graph

In this section we shall investigate with some bounds for maximum independent energy of a graph.

**Theorem 4.1.** *Let $G$ be a graph of order $n$ and size $m$. Then*

$$\sqrt{2m+\beta(G)}\le E_I(G)\le\sqrt{n(2m+\beta(G))}$$

*Proof.* Consider the Couchy-Schwartiz inequality

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2\le\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right).$$

By choose $a_i = 1$ and $b_i = |\lambda_i|$, we get

$$\begin{aligned}\Big(E_I(G)\Big)^2 = \Big(\sum_{i=1}^{n}|\lambda_i|\Big)^2 &\le \Big(\sum_{i=1}^{n}1\Big)\Big(\sum_{i=1}^{n}\lambda_i^2\Big)\\ &\le n(2m+|I|)\\ &\le n(2m+\beta(G)).\end{aligned}$$

Therefore, the upper bound is hold. For the lower bound, since

$$\left(\sum_{i=1}^{n}|\lambda_i|\right)^2 \ge \sum_{i=1}^{n}\lambda_i^2.$$

Then

$$(E_I(G))^2 \ge \sum_{i=1}^{n}\lambda_i^2 = 2m + |I| = 2m + \beta(G).$$

Therefore.

$$E_I(G) \ge \sqrt{2m+\beta(G)}.$$

□

Similar to McClellands [14] bounds for energy of a graph, bounds for $E_I(G)$ are given in the following theorem.

**Theorem 4.2.** *Let $G$ be a graph of order and size $n$ and $m$, respectively. If $P = det(A_I(G))$, then*

$$E_I(G) \ge \sqrt{2m+\beta(G)+n(n-1)P^{2/n}}.$$

*Proof.* Since

$$(E_I(G))^2 = \left(\sum_{i=1}^{n}|\lambda_i|\right)^2 = \left(\sum_{i=1}^{n}|\lambda_i|\right)\left(\sum_{i=1}^{n}|\lambda_i|\right) = \sum_{i=1}^{n}|\lambda_i|^2 + 2\sum_{i\neq j}|\lambda_i||\lambda_j|.$$

Employing the inequality between the arithmetic and geometric means, we get

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| \ge \left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{1/[n(n-1)]}.$$

Thus

$$\begin{aligned}(E_I(G))^2 &\ge \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{1/[n(n-1)]}\\ &\ge \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\left(\prod_{i=j}|\lambda_i|^{2(n-1)}\right)^{1/[n(n-1)]}\\ &= \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\left|\prod_{i\neq j}\lambda_i\right|^{2/n}\\ &= 2m+\beta(G)+n(n-1)P^{2/n}.\end{aligned}$$

This completes the proof. □

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