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Study of Common Fixed Point Theorems in Fuzzy Normed Spaces

Research Article

Trapti Awasthi[1]* and S.S.Rajput[1]

1 Department of Mathematics, Shri Satya Sai University of Technology, Sehore, M.P., India.

2 Department of Mathematics, Govt. Mahatma Gandhi Mahavidhyalaya, Itarsi, M.P., India.

Abstract: In this paper, we prove some common fixed point theorems in fuzzy normed spaces also we give an example of fuzzy normed space which is not a fuzzy metric space. By using common property (E.A.) we prove some common fixed point theorem in fuzzy normed spaces which are generalizations of many known results of ([7, 12, 18–20, 22, 27]) and cited their in.

MSC: 47H10, 54H25.

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1. Introduction and Preliminaries

In 1965, Zadeh [28] introduced the well-known concept of a fuzzy set in his seminal paper. In the last two decades there has been a tremendous development and growth in fuzzy mathematics. In recent years, many authors have proved fixed point and common fixed point theorems in fuzzy metric spaces. To mention a few, we cite [2, 12, 16, 21, 23, 24, 27]. As patterned in Jungck [8], a metrical common fixed point theorem generally involves conditions on commutatively, continuity, completeness together with a suitable condition on containment of ranges of involved mappings by an appropriate contraction condition. Thus, researches in this domain are aimed at weakening one or more of these conditions.

Definition 1.1. *Let X be any set. A fuzzy set in X is a function with domain X and values in $[0,1]$.*

Definition 1.2. *A binary operation $* : [0,1]\times[0,1]\to[0,1]$ is a continuous $t-$norm if it satisfies the following conditions:*

(i) $*$ *is associative and commutative,*

(ii) $*$ *is continuous,*

(iii) $a * 1 = a$ *for every* $a\in[0,1]$,

(iv) $a * b \le c * d$ *if* $a\le c$ *and* $b\le d$ *for all* $a,b,c,d\in[0,1]$.

Definition 1.3. *A triplet $(X,\|.\|,*)$ is a Fuzzy normed space whenever X is an arbitrary set, $*$ is a continuous $t-$norm and satisfying, for every $x,y,z\in X$ and $s,t>0$, the following conditions:*

* *E-mail: traptiawasthimath@gmail.com*

(i) $\|x-y,t\| = 0 \quad when \quad t = 0,$

(ii) $\|x-y,t\| > 0,$

(iii) $\|x-y,t\| = 1$ *iff* $x = y,$

(iv) $\|x-y,t\| = \|y-x,t\|,$

(v) $\|\lambda(x-y),t\| = \|x-y,\frac{t}{|\lambda|}\|$ *for* $\lambda \neq 0$

(vi) $\|x-y,t\| * \|y-z,s\| \leq \|x-z,t+s\|,$

(vii) $\|x-y,\cdot\| : (0,+\infty) \to [0,1]$ *is continuous.*

Note that, $\|x-y,t\|$ can be realized as the measure of nearness between x and y with respect to t. It is known that $\|x-y,\cdot\|$ is nondecreasing for all $x,y \in X$. Let $(X,\|.\|,*)$ be a fuzzy normed space. For $t > 0$, the open ball $B(x,r,t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by $\|x-r,t\| = \{y \in X : \|x-y,t\| > 1-r\}$. Now, the collection $\{\|x-r,t\| : x \in X, 0 < r < 1, t > 0\}$ is a neighborhood system for a topology τ on X induced by the fuzzy normed M. This topology is Hausdorff and first countable.

Definition 1.4. *A sequence* $\{x_n\}$ *in* X *converges to* x *if and only if for each* $\varepsilon > 0$ *and each* $t > 0$ *there exists* $n_0 \in N$ *such that* $\|x_n - x,t\| > 1-\varepsilon$ *for all* $n \geq n_0$.

Example 1.5. *Let* (X,d) *be a usual metric space. We define* $a*b = ab$ *for all* $a,b \in [0,1]$ *and* $M(x,y,t) = \frac{t}{t+|x-y|}$ *for every* $(x,y,t) \in X \times (0,+\infty)$*, then* $(X,\|.\|,*)$ *is a fuzzy metric space. The fuzzy metric space* $(X,\|.\|,*)$ *is complete if and only if the metric space* $(X,\|.\|)$ *is complete.*

Example 1.6. *Suppose* $X = (0,1)$ *then* $(X,\|.\|)$ *is a normed space where* $\|.\|$ *is usual norm. We define* $a*b = ab$ *for all* $a,b \in [0,1]$ *and* $\|x-y,t\| = \frac{t}{t+\|x-y\|}$ *for every* $(x,y,t) \in X \times (0,+\infty)$*, then* $(X,\|.\|,*)$ *is a fuzzy normed space. Here we observe that fuzzy normed space* $(X,\|.\|,*)$ *is complete if and only if the normed space* $(X,\|.\|)$ *is complete.*

In other words, a fuzzy normed space is automatically a fuzzy metric space, but a metric space may have not be a fuzzy normed space.

Example 1.7. *Let* $X = [0,1]$ *or any other finite set. Define* $\|.\| : X \to R^+$ *as follows:*

$$\|x-y\| = \begin{cases} 0 & if \quad x = y \\ 1 & otherwise \end{cases}$$

and fuzzy metric space defined as

$$\|x-y,t\| = \frac{t}{t+\|x-y\|}$$

Then $\|x-y,t\|$ *is a fuzzy metric on* X*. This is an example of a fuzzy metric space that is not a fuzzy normed space.*

We observe that in Example-1.5,1.6,1.7, every fuzzy normed space is a fuzzy metric space but converse may not be true.

Remark 1.8. *Let* p *be a real number satisfying* $p \geq 1$*, and define* $\|x\|_p$ *on* R^n_+ *by*

$$|x|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}.$$

This is called the l_p *norm. Note that the Euclidean norm is the* l_2*-norm, the city block norm is the* l_1*-norm, and the* sup*-norm is the* l_∞*-norm.*

Example 1.9. *Let X be a normed space with euclidean norm R^2 and C be the unit circle $\{x \in X; \|x\| = 1\}$. This is another example of a metric space that is not a normed space X is a metric space, using the metric defined from $\|.\|$, and therefore, according to the above remark, so is C, so it is not a normed space.*

Beside this Sessa [17] introduced the notion of weakly commuting mappings which was further enlarged by Jungck [9] by defining compatible mappings. After this, there came a host of such definitions which are scattered throughout the recent literature whose survey and illustration (up to 2001) is available in Murthy [15]. Here we enlist the only those weak commutatively conditions which are relevant to presentation.

Definition 1.10. *A pair of self-mappings (f, g) defined on a fuzzy normed space $(X, \|.\|, *)$ is said to be compatible (or asymptotically commuting) if for all $t > 0$,*

$$\lim_{n \to +\infty} \|fgx_n - gfx_n, t\| = 1,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = z$, for some $z \in X$. Also the pair (f, g) is called non-compatible, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = z$, but either $\lim_{n\to+\infty} \|fgx_n - gfx_n, t\| \neq 1$ or the limit does not exists.

Definition 1.11. *A pair of self-mappings (f, g) defined on a fuzzy normed space $(X, \|.\|, *)$ is said to satisfy the property (E.A.) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = z$ for some $z \in X$.*

Clearly compatible as well as non-compatible pairs satisfy the property (E.A.).

Definition 1.12. *Two pairs of self mappings (A, S) and (B, T) defined on a fuzzy normed space $(X, \|.\|, *)$ are said to share common property (E.A.) if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = z$ for some $z \in X$.*

For more on properties (E.A.) and common (E.A.), one can consult [1] and [2] respectively.

Definition 1.13. *Two self mappings f and g on a fuzzy normed space $(X, \|.\|, *)$ are called weakly compatible if they commute at their point of coincidence, that is, $fx = gx$ implies $fgx = gfx$.*

Definition 1.14. *Two finite families of self mappings $\{A_i\}$ and $\{B_j\}$ are said to be pairwise commuting if:*

(i) $A_iA_j = A_jA_i \quad i, j \in \{1, 2, \ldots, m\}$,

(ii) $B_iB_j = B_jB_i \quad i, j \in \{1, 2, \ldots, n\}$,

(iii) $A_iB_j = B_jA_i \quad i \in \{1, 2, \ldots, m\}$ *and* $j \in \{1, 2, \ldots, n\}$,

The following definitions will be utilized to state various results in Section 3.

Definition 1.15. *Let $(X, \|.\|, *)$ be a fuzzy normed space and $f, g : X \to X$ be a pair of maps. The map f is called a fuzzy contraction with respect to g if there exists an upper semi continuous function $r : [0, +\infty) \to [0, +\infty)$ with $r(\tau) < \tau$ for every $\tau > 0$ such that*

$$\frac{1}{\|fx - fy, t\|} - 1 \leq r\left(\frac{1}{m(f, g, x, y, t)} - 1\right)$$

for every $x, y \in X$ and each $t > 0$, where

$$m(f, g, x, y, t) = \min\{\|gx - gy, t\|, \|fx - gx, t\|, \|fy - gy, t\|\}.$$

Definition 1.16. *Let $(X, \|.\|, *)$ be a fuzzy normed space and $f, g : X \to X$ be a pair of maps. The map f is called a fuzzy $k-$contraction with respect to g if there exists $k \in (0, 1)$, such that*

$$\frac{1}{\|fx - fy, t\|} - 1 \leq k \left(\frac{1}{m(f, g, x, y, t)} - 1 \right),$$

where $m(f, g, x, y, t) = \min\{\|gx - gy, t\|, \|fx - gx, t\|, \|fy - gy, t\|\}$, for every $x, y \in X$ and each $t > 0$.

Definition 1.17. *Let A, B, S and T be four self mappings of a fuzzy normed space $(X, \|.\|, *)$. Then the mappings A and B are called a generalized fuzzy contraction with respect to S and T if there exists an upper semi-continuous function $r : [0, +\infty) \to [0, +\infty)$, with $r(\tau) < \tau$ for every $\tau > 0$ such that, for each $x, y \in X$ and $t > 0$,*

$$\frac{1}{\|Ax - By, t\|} - 1 \leq r \left(\frac{1}{\min\{\|Sx - Ty, t\|, \|Ax - Sx, t\|, \|By - Ty, t\|\}} - 1 \right). \tag{1}$$

2. Main Results

Now, we state and prove our main theorem as follows.

Theorem 2.1. *Let A, B, S and T be self mappings of a fuzzy normed space $(X, \|.\|, *)$ such that the mappings A and B are generalized fuzzy contraction with respect to mappings S and T. Suppose that the pairs (A, S) and (B, T) share the common property (E.A.) and $S(X)$ and $T(X)$ are closed subsets of X. Then the pair (A, S) as well as (B, T) have a point of coincidence each. Further, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.*

Proof. Since the pairs (A, S) and (B, T) share the common property (E.A.), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that, for some $z \in X$,

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = z.$$

Since $S(X)$ is a closed subset of X, therefore $\lim_{n\to+\infty} Sx_n = z \in S(X)$ and henceforth there exists a point $u \in X$ such that $Su = z$. Now we assert that $Au = Su$. If not, then by (1), we have

$$\frac{1}{\|Au - By_n, t\|} - 1 \leq r \left(\frac{1}{\min\{\|Su - Ty_n, t\|, \|Au - Su - t\|, \|By_n - Ty_n, t\|\}} - 1 \right)$$

which on making $n \to +\infty$, for every $t > 0$, reduces to

$$\frac{1}{\|Au - z, t\|} - 1 \leq r(\frac{1}{\min\{\|Au - z, t\|\}} - 1)$$

that is a contradiction yielding thereby $Au = Su$. Therefore, u is a coincidence point of the pair (A, S). If $T(X)$ is a closed subset of X, then $\lim_{n\to+\infty} Ty_n = z \in T(X)$. Therefore, there exists a point $w \in X$ such that $Tw = z$. Now we assert that $Bw = Tw$. If not, then according with (1), we have

$$\frac{1}{\|Ax_n - Bw, t\|} - 1 \leq r \left(\frac{1}{\min\{\|Sx_n - Tw, t\|, \|Ax_n - Sx_n, t\|, \|Bw - Tw, t\|\}} - 1 \right)$$

which on making $n \to +\infty$, for every $t > 0$, reduces to

$$\frac{1}{\|z - Bw, t\|} - 1 \leq r \left(\frac{1}{\min\{\|z - Bw, t\|\}} - 1 \right)$$

which is a contradiction as earlier. It follows that $Bw = Tw$ which shows that w is a point of coincidence of the pair (B, T). Since the pair (A, S) is weakly compatible and $Au = Su$, hence $Az = ASu = SAu = Sz$. Now, we assert that z is a common fixed point of the pair (A, S). Suppose that $Az \neq z$, then using again (1) we have, for all $t > 0$,

$$\frac{1}{\|Az - Bw, t\|} - 1 \leq r\left(\frac{1}{\min\{\|Az - Bw, t\|\}} - 1\right)$$

implying thereby $Az = Bw = z$. Finally, using the notion of weak compatibility of the pair (B, T) together with (1), we get $Bz = z = Tz$. Hence, z is a common fixed point of both the pairs (A, S) and (B, T). Uniqueness of the common fixed point z is an easy consequence of condition (1). □

The following example is utilized to highlight the utility of Theorem 2.1 over earlier relevant results.

Example 2.2. *Let $X = [2, 20]$ and $(X, M, *)$ be a fuzzy normed space defined as*

$$\|x - y, t\| = \frac{t}{t + \|x - y\|} \quad \text{if } t > 0 \text{ and } x, y \in X.$$

Define $A, B, S, T : X \to X$ by

$$Ax = \begin{cases} 2 & \text{if } x = 2 \\ 3 & \text{if } x > 2 \end{cases}, \quad Sx = \begin{cases} 2 & \text{if } x = 2 \\ 6 & \text{if } x > 2 \end{cases},$$

$$Bx = \begin{cases} 2 & \text{if } x = 2 \\ 6 & \text{if } 2 < x \leq 5 \\ 3 & \text{if } x > 5 \end{cases}, \quad Tx = \begin{cases} 2 & \text{if } x = 2 \\ 18 & \text{if } 2 < x \leq 5 \\ 12 & \text{if } x > 5 \end{cases}.$$

Then, A, B, S and T satisfy all the conditions of the Theorem 2.1 with $r(\tau) = k\tau$, where $k \in (4/9, 1)$ and have a unique common fixed point $x = 2$ which also remains a point of discontinuity. Moreover, it can be seen that $A(X) = \{2, 3\} \not\subset \{2, 12, 18\} = T(X)$ and $B(X) = \{2, 3, 6\} \not\subset \{2, 6\} = S(X)$. Here, it is worth noting that none of the earlier theorems (with rare possible exceptions) can be used in the context of this example as most of earlier theorems require conditions on the containment of range of one mapping into the range of other.

In the foregoing theorem, if we set $r(\tau) = k\tau$, $k \in (0, 1)$, and $\|x - y, t\| = \frac{t}{t + \|x - y\|}$, then, we get the following result which improves and generalizes the result of Jungck ([8] (Corollary 3.2)) in metric space.

Corollary 2.3. *Let A, B, S and T be self mappings of a metric space (X, d) such that,*

$$d(Ax, By) \leq k \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\},$$

for every $x, y \in X$, $k \in (0, 1)$. Suppose that the pairs (A, S) and (B, T) share the common property (E.A.), and $S(X)$ and $T(X)$ are closed subsets of X. Then the pair (A, S) as well as (B, T) have a point of coincidence each. Further, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

By choosing A, B, S and T suitably, one can deduce corollaries for a pair as well as for two different trios of mappings. For sake of brevity we deduce, by setting $A = B$ and $S = T$, a corollary for a pair of mapping is as follows,

Corollary 2.4. *Let (A, S) be a pair of self mappings of a fuzzy normed space $(X, \|.\|, *)$ such that (A, S) satisfies the property (E.A.), A is a fuzzy contraction with respect to S, and $S(X)$ is a closed subset of X. Then the pair (A, S) has a point of coincidence, whereas the pair (A, S) has a unique common fixed point provided it is weakly compatible.*

Now, we know that A fuzzy k-contraction with respect to S implies A fuzzy contraction with respect to S. Thus, we get the following corollary,

Corollary 2.5. *Let A and S be self mappings of a fuzzy normed space $(X, \|.\|, *)$ such that, the pair (A, S) enjoys the property (E.A.), A is a fuzzy $k-$contraction with respect to S, and $S(X)$ is a closed subset of X. Then, the pair (A, S) has a point of coincidence. Further, A and S have a unique common fixed point provided the pair (A, S) is weakly compatible.*

3. Implicit functions and Common fixed point

We recall the following two implicit functions defined and studied in [23] and [13] respectively. Firstly following Singh and Jain [23], let Φ be the set of all real continuous function $\phi : [0, 1]^4 \to \mathbb{R}$, non decreasing in first argument and satisfying the following conditions:

(i) For $u, v \geq 0$, $\phi(u, v, u, v) \geq 0$ or $\phi(u, v, v, u) \geq 0$ implies that $u \geq v$.

(ii) $\phi(u, u, 1, 1) \geq 0$ implies that $u \geq 1$.

Example 3.1. *Define $\phi(t_1, t_2, t_3, t_4) = 15t_1 - 13t_2 + 5t_3 - 7t_4$. Then $\phi \in \Phi$.*

Secondly following Imdad and Javid [13], let Ψ denote the family of all continuous functions $F : [0, 1]^4 \to \mathbb{R}$ satisfying the following conditions:

- F_1 : For every $u > 0, v \geq 0$ with $F(u, v, u, v) \geq 0$ or $F(u, v, v, u) \geq 0$, we have $u > v$.
- F_2 : $F(u, u, 1, 1) < 0$, for each $u > 0$.

The following examples are essentially contained in [13].

Example 3.2. *Define $F : [0, 1]^4 \to \mathbb{R}$ as $F(t_1, t_2, t_3, t_4) = t_1 - \phi(\min\{t_2, t_3, t_4\})$, where $\phi : [0, 1] \to [0, 1]$ is a continuous function such that $\phi(s) > s$ for $0 < s < 1$. Then $F \in \Psi$.*

Example 3.3. *Define $F : [0, 1]^4 \to \mathbb{R}$ as $F(t_1, t_2, t_3, t_4) = t_1 - k \min\{t_2, t_3, t_4\}$, where $k > 1$.*

Example 3.4. *Define $F : [0, 1]^4 \to \mathbb{R}$ as $F(t_1, t_2, t_3, t_4) = t_1 - kt_2 - \min\{t_3, t_4\}$, where $k > 0$.*

Example 3.5. *Define $F : [0, 1]^4 \to \mathbb{R}$ as $F(t_1, t_2, t_3, t_4) = t_1 - at_2 - bt_3 - ct_4$ where $a > 1$ and $b, c \geq 0$ $(b, c \neq 1)$.*

Example 3.6. *Define $F(: [0, 1]^4 \to \mathbb{R}$ as $F(t_1, t_2, t_3, t_4) = t_1 - at_2 - b(t_3 + t_4)$ where $a > 1$ and $0 \leq b < 1$.*

Example 3.7. *Define $F : [0, 1]^4 \to \mathbb{R}$ as $F(t_1, t_2, t_3, t_4) = t_1^3 - kt_2t_3t_4$ where $k > 1$.*

Before proving our results, it may be noted that above mentioned classes of functions Φ and Ψ are independent classes as the implicit function $F(t_1, t_2, t_3, t_4) = t_1 - k \min\{t_2, t_3, t_4\}$, where $k > 1$ (belonging to Ψ) does not belongs to Φ as $F(u, u, 1, 1) \leq 0$ implies $u < 0$, whereas implicit function $\phi(t_1, t_2, t_3, t_4) = 15t_1 - 13t_2 + 5t_3 - 7t_4$ (belonging to Φ) does not belongs to Ψ as $F(u, v, u, v) = 0$ implies $u = v$ instead of $u > v$. The following lemma interrelates the property (E.A.) with the common property (E.A.).

Lemma 3.8. *Let A, B, S and T be self mappings of a fuzzy normed space $(X, \|.\|, *)$. Assume that, for all distinct $x, y \in X$ and $t > 0$, there exists $F \in \Psi$ with*

$$F(\|Ax - By, t\|, \|Sx - Ty, t), \|Sx - Ax, t\|, \|By - Ty, t\|) \geq 0. \tag{2}$$

Suppose that pair (A, S) and (or) (B, T) satisfies the property (E.A.), and $A(X) \subset T(X)$ and (or) $B(X) \subset S(X)$. Then the pairs (A, S) and (B, T) share the common property (E.A.).

Proof. If the pair (A, S) enjoys the property (E.A.), then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = z$ for some $z \in X$. Since $A(X) \subset T(X)$, hence for each x_n there exists y_n in X such that $Ax_n = Ty_n$, henceforth $\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Ty_n = z$. Thus, we have $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$. Now, we assert that $By_n \to z$. We note that $By_n \to z$ if and only if $\|Ax_n, By_n, t) \to 1$. Assume that $By_n \nrightarrow z$, then there exists a subsequence of $\{By_n\}$, say $\{By_{n_k}\}$, such that $\|Ax_{n_k} - By_{n_k}, t\| \to u < 1$. By condition (2), we have

$$F(\|Ax_{n_k} - By_{n_k}, t\|, \|Sx_{n_k} - Ty_{n_k}, t\|, \|Sx_{n_k} - Ax_{n_k}, t\|, \|By_{n_k} - Ty_{n_k}, t\|) \geq 0,$$

which on making $k \to +\infty$, reduces to

$$F(u, 1, 1, u) \geq 0.$$

Implying thereby $u > 1$, a contradiction. Hence $\lim_{n\to+\infty} By_n = z$ which shows that the pairs (A, S) and (B, T) share the common property (E.A.). □

With a view to generalize some fixed point theorems contained in Imdad and Ali [12, 13] we prove the following fixed point theorem which in turn generalizes several previously known results due to Chugh and Kumar [3], Turkoglu et al. [25], Vasuki [26] and some others.

Theorem 3.9. *Let A, B, S and T be self mappings of a fuzzy normed space $(X, \|.\|, *)$. Assume that, for all distinct $x, y \in X$ and $t > 0$, there exists $F \in \Psi$ with*

$$F(\|Ax - By, t\|, \|Sx - Ty, t\|, \|Sx - Ax, t\|, \|By - Ty, t\|) \geq 0. \tag{3}$$

Suppose that the pairs (A, S) and (B, T) share the common property (E.A.) and $S(X)$ and $T(X)$ are closed subsets of X. Then the pair (A, S) as well as (B, T) have a point of coincidence each. Further, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. Since the pairs (A, S) and (B, T) share the common property (E.A.), then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = z,$$

for some $z \in X$. Since $S(X)$ is a closed subset of X, then $\lim_{n\to+\infty} Sx_n = z \in S(X)$. Therefore, there exists a point $u \in X$ such that $Su = z$. Now we assert that $Au = Su$. If not, then by (3) we have

$$F(\|Au - By_n, t\|, \|Su - Ty_n, t\|, \|Su - Au, t\|, \|By_n - Ty_n, t\|) \geq 0$$

which on making $n \to +\infty$ reduces to

$$F(\|Au - z, t\|, \|Su - z, t\|, \|Su - Au, t\|, \|z - z, t\|) \geq 0,$$

that is a contradiction to F_1. Hence $Au = Su$. Therefore, u is a point of coincidence of the pair (A, S). Since $T(X)$ is a closed subset of X, then $\lim_{n\to+\infty} Ty_n = z \in T(X)$. Therefore, there exists a point $w \in X$ such that $Tw = z$. Now, we assert that $Bw = Tw$. If not, then again using (3), we have

$$F(\|Ax_n - Bw, t\|, \|Sx_n - Tw, t\|, \|Sx_n - Ax_n, t\|, \|Bw - Tw, t\|) \geq 0.$$

On making $n \to +\infty$, it reduces to

$$F(\|z - Bw, t\|, \|z - z, t\|, \|z - z, t\|, \|Bw - Tw, t\|) \geq 0,$$

implying thereby, $\|Tw - Bw, t\| > 1$, a contradiction. Hence $Tw = Bw$, which shows that w is a point of coincidence of the pair (B, T). Since the pair (A, S) is weakly compatible and $Au = Su$, we deduce that $Az = ASu = SAu = Sz$. Now, we assert that z is a common fixed point of the pair (A, S). Suppose $Az \neq z$. Then using (3), we have

$$F(\|Az - Bw, t\|, \|Sz - Tw, t\|, \|Sz - Az, t\|, \|Bw - Tw, t\|) \geq 0$$

that is $F(\|Az, z, t), \|Az - z, t\|, 1, 1) \geq 0$ which contradicts F_2. Hence $Az = z$. Now, using the notion of the weak compatibility of the pair (B, T) and inequality (3), we get $Bz = z = Tz$. Hence z is a common fixed point of both the pairs (A, S) and (B, T). Uniqueness of z is an easy consequence of (3). □

Example 3.10. *In the setting of Example 1, retain the same mappings A, B, S and T and define $F : [0,1]^4 \to \mathbb{R}$ as $F(t_1, t_2, t_3, t_4) = t_1 - \phi(\min\{t_2, t_3, t_4\})$ with $\phi(r) = \sqrt{r}$. Then, A, B, S and T satisfy all the conditions of the Theorem 3.9 and have a unique common fixed point $x = 2$ which also remains a point of discontinuity.*

Corollary 3.11. *The conclusions of Theorem 3.9 remain true if for all distinct $x, y \in X$, condition (3) is replaced by one of the following conditions:*

(i) $\|Ax - By, t\| \geq \phi(\min\{\|Sx - Ty, t\|, \|Sx - Ax, t\|, \|By - Ty, t\|\})$ *where* $\phi : [0,1] \to [0,1]$ *is a continuous function such that* $\phi(s) > s$ *for all* $0 < s < 1$.

(ii) $\|Ax - By, t\| \geq k(\min\{\|Sx - Ty, t\|, \|Sx - Ax, t\|, \|By - Ty, t\|\})$ *where* $k > 1$.

(iii) $\|Ax - By, t\| \geq k\|Sx - Ty, t\| + \min\{\|Sx - Ax, t\|, \|By - Ty, t\|\})$ *where* $k > 0$.

(iv) $\|Ax - By, t\| \geq a\|Sx - Ty, t\| + b\|Sx - Ax, t\| + c\|By - Ty, t\|$ *where* $a > 1$ *and* $b, c \geq 0$ $(b, c \neq 1)$.

(v) $\|Ax - By, t\| \geq a\|Sx - Ty, t\| + b[\|Sx - Ax, t\| + \|By - Ty, t\|]$ *where* $a > 1$ *and* $0 \leq b < 1$.

(vi) $\|Ax - By, t\| \geq k\|Sx - Ty, t\|\|Sx - Ax, t\|\|By - Ty, t\|$ *where* $k > 1$.

Proof. The proof of various corollaries corresponding to contraction conditions (i)-(vi) follows from Theorem 3.9 and Examples 3-8. □

Remark 3.12. *Corollary 3.11 corresponding to condition (i) is a result due to Imdad and Ali [12], whereas Corollary 3.11 corresponding to various conditions presents a sharpened form of Corollary 2 of Imdad and Ali [13]. Similarly to this corollary, one can also deduce generalized versions of certain results contained in [3, 22, 26].*

The following theorem generalizes a theorem contained in Singh and Jain [23].

Theorem 3.13. *Let A, B, S and T be self mappings of a fuzzy normed space $(X, \|.\|, *)$. Assume that, for all distinct $x, y \in X$, $k \in (0,1)$ and $t > 0$, there exists $\phi \in \Phi$ with*

$$\phi(\|Ax - By, kt\|, \|Sx - Ty, t\|, \|Ax - Sx, t\|, \|By - Ty, kt\|) \geq 0. \tag{4}$$

$$\phi(\|Ax - By, kt\|, \|Sx - Ty, t\|, \|Ax - Sx, kt\|, \|By - Ty, t\|) \geq 0. \tag{5}$$

Suppose that the pairs (A, S) and (B, T) enjoy the common property (E.A.) and $S(X)$ and $T(X)$ are closed subsets of X. Then the pairs (A, S) and (B, T) have a point of coincidence each. Further, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. The proof of this theorem can be completed on the lines of the proof of Theorem 3.9, hence details are omitted. □

Example 3.14. *In the setting of Example 1, we define $\phi(t_1, t_2, t_3, t_4) = 15t_1 - 13t_2 + 5t_3 - 7t_4$, besides retaining the rest of the example as it stands. Then all the conditions of Theorem 3.13 with $k \in (1/4, 1)$ are satisfied.*

Notice that 2 is the unique common fixed point of A, B, S and T but this example cannot be covered by Theorem 3.1 due to Singh and Jain [23] as $A(X) = \{2, 3\} \not\subset \{2, 12, 18\} = T(X)$ and $B(X) = \{2, 3, 6\} \not\subset \{2, 6\} = S(X)$. This example cannot also be covered by Theorem 3.9 of this paper as $\phi(u, u, 1, 1) = 2(u - 1)$ implies $\phi(1, 1, 1, 1) = 0$ which contradicts F_2. Now, we state (without proof) the following result.

Theorem 3.15. *Let $\{A_1, A_2, \dots, A_m\}$, $\{B_1, B_2, \dots, B_n\}$, $\{S_1, S_2, \dots, S_p\}$ and $\{T_1, T_2, \dots, T_q\}$ be four finite families of self mappings of a fuzzy normed space $(X, \|.\|, *)$ such that the mappings $A = A_1A_2 \cdots A_m$, $B = B_1B_2 \cdots B_n$, $S = S_1S_2 \cdots S_p$ and $T = T_1T_2 \cdots T_q$ satisfy condition (3). Suppose that the pairs (A, S) and (B, T) share the common property (E.A.), and $S(X)$ as well as $T(X)$ are closed subsets of X. Then the pairs (A, S) and (B, T) have a point of coincidence each. Further, provided the pairs of families $(\{A_i\}, \{S_k\})$ and $(\{B_r\}, \{T_t\})$ commute pairwise, where $i \in \{1, \dots, m\}, k \in \{1, \dots, n\}, r \in \{1, \dots, p\}$ and $t \in \{1, \dots, q\}$, then A_i, S_k, B_r and T_t, have a unique common fixed point.*

Proof. The proof of this theorem can be completed on the lines of Theorem 3.1. due to Imdad et al. [14], hence details are avoided. □

By setting $A = A_1 = A_2 = \cdots = A_m, B = B_1 = B_2 = \cdots = B_n, S = S_1 = S_2 = \cdots = S_p$ and $T = T_1 = T_2 = \cdots = T_q$ in Theorem 3.15, one can deduce the following result for certain iterates of mappings which is a partial generalization of Theorem 3.9.

Corollary 3.16. *Let A, B, S and T be four self mappings of a fuzzy normed space $(X, \|.\|, *)$ such that A^m, B^n, S^p and T^q satisfy the condition (3). Suppose that the pairs (A^m, S^p) and (B^n, T^q) share the common property (E.A.) and $S^p(X)$ as well as $T^q(X)$ are closed subsets of X. Then the pairs (A^m, S^p) and (B^n, T^q) have a point of coincidence each. Further, A, B, S and T have a unique common fixed point provided the pairs (A, S) and (B, T) commute pairwise.*

Remark 3.17. *Results similar to Corollary 4 as well as Corollary 5 can be outlined in respect of Theorem 3.13, Theorem 3.15 and Corollary 5. But due to the repetition, details are avoided.*

Now, we conclude this note by deriving the following results of integral type.

Corollary 3.18. *Let A, B, S and T be four self mappings of a fuzzy normed space $(X, \|.\|, *)$. Assume that there exist a Lebesgue integrable function $\varphi : \mathbb{R} \to \mathbb{R}$ and a function $\phi : [0, 1]^4 \to \mathbb{R}$ such that*

$$\int_0^{\phi(u,1,u,1)} \varphi(s)ds \geq 0, \quad \int_0^{\phi(u,1,1,u)} \varphi(s)ds \geq 0 \ \text{or} \ \int_0^{\phi(u,u,1,1)} \varphi(s)ds \geq 0 \tag{6}$$

implies u=1. Suppose that the pairs (A, S) and (B, T) share the common property (E.A.), and $S(X)$ and $T(X)$ are closed subsets of X. If for $t > 0$

$$\int_0^{\phi(\|Ax-By,t\|,\|Sx-Ty,t\|,\|Sx-Ax,t\|,\|By-Ty,t\|)} \varphi(s)ds \geq 0 \ \text{for all distinct} \ x, y \in X, \tag{7}$$

then the pairs (A, S) and (B, T) have a point of coincidence each. Further, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. Since the pairs (A, S) and (B, T) share the common property (E.A.), then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = z,$$

for some $z \in X$. Since $S(X)$ is a closed subset of X, then $\lim_{n\to+\infty} Sx_n = z \in S(X)$. Therefore, there exists a point $u \in X$ such that $Su = z$. Now we assert that $Au = Su$. If not, by (7) we have

$$\int_0^{\phi(\|Au-By_n,t\|,\|Su-Ty_n,t\|,\|Su-Au,t\|,\|By_n-Ty_n,t\|)} \varphi(s)ds \geq 0.$$

On making $n \to +\infty$, it reduces to

$$\int_0^{\phi(\|Au-z,t\|,1,\|z-Au,t\|,1)} \varphi(s)ds \geq 0,$$

which implies $\|Au - z, t\| = 1$ and so $Au = z$. Being $T(X)$ a closed subset of X, repeating the same argument, we deduce that there exists a point $w \in X$ such that $Bw = Tw$. Since the pair (A, S) is weakly compatible and $Au = Su$, we deduce that $Az = ASu = SAu = Sz$. Now, we assert that z is a common fixed point of the pair (A, S). Suppose $Az \neq z$. Then using (7), with $x = z$ and $y = w$, we have

$$\int_0^{\phi(\|Az-z,t\|,\ \|Az-z,t\|,1,1)} \varphi(s)ds \geq 0$$

that implies $\|Az - z, t\| = 1$. Hence $Az = z$. Similarly we prove that $Bz = Tz = z$ and so z is a common fixed point of A, B, S and T. Uniqueness of z is a consequence of condition (7). □

Corollary 3.19. *Let A, B, S and T be four self mappings of a fuzzy normed space $(X, \|.\|, *)$. Assume that there exist a Lebesgue integrable function $\varphi : \mathbb{R} \to \mathbb{R}_+$ and a function $\phi : [0, 1]^4 \to \mathbb{R}$, where $\phi \in \Phi$, such that*

$$\int_0^{\phi(\|Ax-By,t\|,\|Sx-Ty,t\|,\|Sx-Ax,t\|,\|By-Ty,t\|)} \varphi(s)ds \geq 0 \text{ for all } x, y \in X \text{ and } t > 0, \tag{8}$$

$$\int_0^{\phi(u,u,1,1)} \varphi(s)ds \geq 0 \text{ for all } u \in (0, 1). \tag{9}$$

Suppose that the pairs (A, S) and (B, T) enjoy the common property (E.A.), and $S(X)$ and $T(X)$ are closed subsets of X. Then the pairs (A, S) and (B, T) have a point of coincidence each. Further, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. The proof is the same of Corollary 3.18, so details are omitted. □

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