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The Kumaraswamy Flexible Weibull Extension

Research Article

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Abstract: This paper introduces a new continuous distribution which is a generalization of the flexible weibull extension, called the kumaraswamy flexible weibull extension distribution. We refer to the new distribution as KW-FWE for short. We obtain some of its mathematical properties, also some structural properties of this new distribution are studied. We provide an explicit expressions for the moments and the moment generating function. The method of maximum likelihood is used to evaluate the model parameters . We illustrate the usefulness of the proposed model by applications to real data.

MSC: 60E05, 62F10, 62F12, 62N02, 62N05.

Keywords: Flexible weibull extension; Kumaraswamy distribution; Moment generating function; Maximum likelihood estimation.

1. Introduction

The Weibull distribution, having exponential and Rayleigh as special cases, is one of the most commonly used distributions for modeling lifetime data and for modeling phenomenon with monotone failure rates. When modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its positively and negatively skewed density shapes. However, this distribution does not provide a good fit to data sets with bathtub shaped or upside down bathtub shaped (unimodal) failure rates. These cases are quite common in reliability and biological studies. Such bathtub hazard curves have nearly flat middle portions and the corresponding densities have a positive anti-mode. We can observe the unimodal hazard rates in course of a disease whose mortality reaches a peak after some finite period and then declines gradually. Many generalization of the Weibull distribution is proposed, such as the exponentiated Weibull distribution introduced by Mudholkar and Srivastava [16], the additive Weibull distribution proposed by Xie et al. [21], the modified Weibull extension distribution introduced by Xie et al. [22] and the new modified Weibull distribution introduced by Lai et al. [14]. Bebbington et al. [2] proposed a new two-parameter ageing distribution called the flexible Weibull extension distribution. This new distribution is shown to be quite flexible, being able to model both IFR and IFRA ageing classes. The cumulative distribution function of the flexible Weibull extension distribution is given as follows

$$F(x) = 1 - e^{-e^{\alpha x - \frac{\beta}{x}}}, \; x > 0, \; \alpha, \; \beta > 0,$$

and the corresponding density function is given in the form

$$f(x) = \left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}} e^{-e^{\alpha x - \frac{\beta}{x}}}, \; x > 0, \; \alpha, \; \beta > 0.$$

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El-Gohary et al. [8] introduced the three parameters exponentiated flexible Weibull extension distribution and discussed some of its properties. The aim of this paper is to introduce a new generalization of the flexible Weibull extension called the Kumaraswamy flexible Weibull extension. The paper by Kumaraswamy [12] introduced a two-parameter distribution on (0,1), so-called Kumaraswamy distribution. The Kumaraswamy cumulative distribution function (cdf) is defined in the form

$$G(x,a,b) = 1-(1-x^{a})^{b},\ x \in (0,1)\ ,\ a,\ b>0,$$

and the corresponding probability density function

$$g(x,a,b) = abx^{a-1}(1-x^{a})^{b-1},\ x \in (0,1)\ ,\ a,\ b>0.$$

It should be noted that the Kumaraswamy probability density function can be unimodal, increasing, decreasing or constant depending on the choice of the parameters a and b. Jones [11] explored the background and genesis of the Kumaraswamy distribution and more importantly, highlighted some advantages and disadvantages of the beta and Kumaraswamy distributions. For an arbitrary baseline cumulative function $G(x)$, of a random variable X, Cordeiro and de Castro [4] defined the Kumaraswamy-G (KW-G for short) distribution by

$$F(x) = 1-[1-G^{a}(x)]^{b}, \tag{1}$$

where $a>0$ and $b>0$ are two additional shape parameters. Since the cdf is quite tractable then KW-G distribution can be used quite effectively even if the data are censored. Correspondingly, the probability density function of KW-G distribution is given as follow

$$f(x) = abg(x)G^{a-1}(x)[1-G^{a}(x)]^{b-1}, \tag{2}$$

where $g(x) = \frac{d}{dx}G(x)$. The KW-G distribution has the same parameters of the $G(x)$ distribution plus two additional shape parameters $a>0$ and $b>0$. Also we note that the probability density function of the family given in (2) has many of the same properties of the class of Beta-G distributions introduced by Eugene et al. [7]. Using probability density function (2), many Kumaraswamy generalized distributions were proposed in recent years. Cordeiro et al. [5] introduced the Kumaraswamy Weibull distributon, Bourguignon et al. [3] introduced the Kumaraswamy Pareto distributon, Paranaba et al. [19] introduced the Kumaraswamy Burr XII distribution, Gomes et al. [10] introduced the Kumaraswamy generalized Rayleigh distribution, de Pascoa et al. [6] introduced the Kumaraswamy generalized gamma distribution etc.

This paper is organized as follows. In Section 2 we define the cumulative distribution function, probability density function, reliability function and hazard function of the Kumaraswamy flexible Weibull extension distribution. In Section 3 we study some different properties of Kumaraswamy flexible Weibull extension distribution include, the quantile function, the median, the mode, the moments and the moment generating function. Section 4 discusses the distribution of the order statistics for Kumaraswamy flexible Weibull extension distribution. Moreover, maximum likelihood estimation of the parameters is determined in Section 5. Also, an application of Kumaraswamy flexible Weibull extension distribution using a real data set is presented in Section 6.

2. The Kumaraswamy Flexible Weibull Extension Distribution

In this section, we introduce a new four parameters distribution called the Kumaraswamy flexible Weibull extension(KW-FWE) distribution with parameters a, α, β and b written in the form KW-FWE(Φ), where the vector Φ is defined by

$\Phi = (a, \alpha, \beta, b)$. The cumulative distribution function(cdf) of KW-FWE model is defined as follows

$$F(x) = 1 - \left[1 - \left[1 - e^{-e^{\alpha x - \frac{\beta}{x}}}\right]^a\right]^b, \ x > 0, \ a, \ b, \ \alpha, \ \beta > 0. \tag{3}$$

The probability density function(pdf) corresponding to (3) is given by

$$f(x) = ab\left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}} e^{-e^{\alpha x - \frac{\beta}{x}}} \left[1 - e^{-e^{\alpha x - \frac{\beta}{x}}}\right]^{a-1} \left[1 - \left[1 - e^{-e^{\alpha x - \frac{\beta}{x}}}\right]^a\right]^{b-1}, \tag{4}$$

where $x > 0, a, b, \alpha, \beta > 0$.

2.1. Survival and Hazard Functions

The survival function corresponding to the cdf of KW-FWE(Φ) given in (3), is obtained in the form

$$s(x) = \left[1 - \left[1 - e^{-e^{\alpha x - \frac{\beta}{x}}}\right]^a\right]^b. \tag{5}$$

The hazard function of KW-FWE(Φ) is defined in the form

$$h(x) = \frac{ab\left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}} e^{-e^{\alpha x - \frac{\beta}{x}}} \left[1 - e^{-e^{\alpha x - \frac{\beta}{x}}}\right]^{a-1}}{1 - \left[1 - e^{-e^{\alpha x - \frac{\beta}{x}}}\right]^a}. \tag{6}$$

Figure 1-3, show the CDF, PDF and hazard of various KW-FWE distributions for different values of parameters.

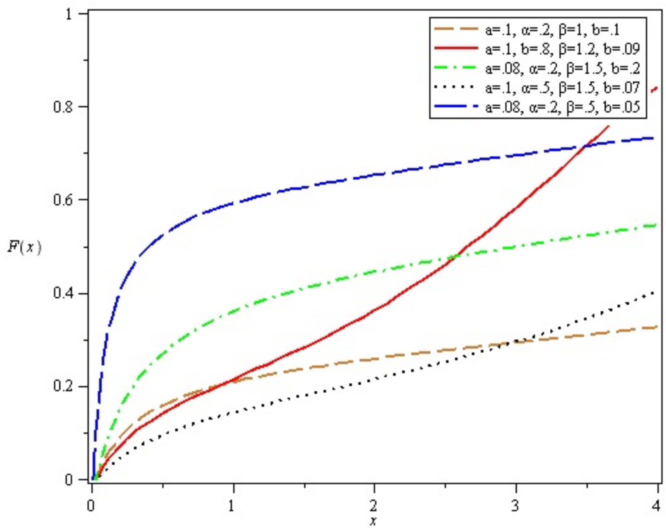

Figure 1: The CDF of various KW-FME distributions for some values of the parameters.

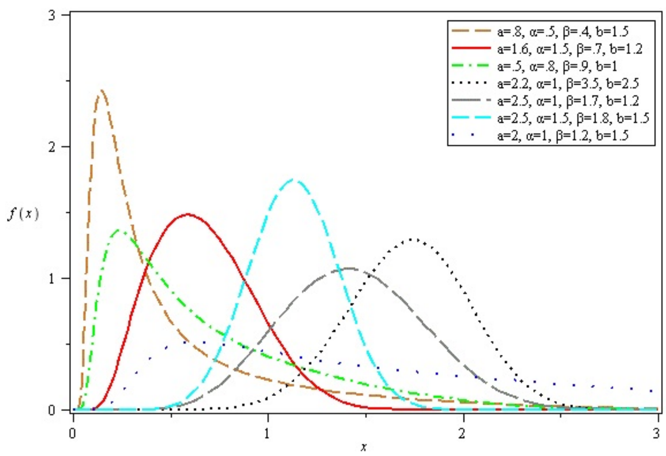

Figure 2: The pdf of various KW-FWE distributions for different values of parameters.

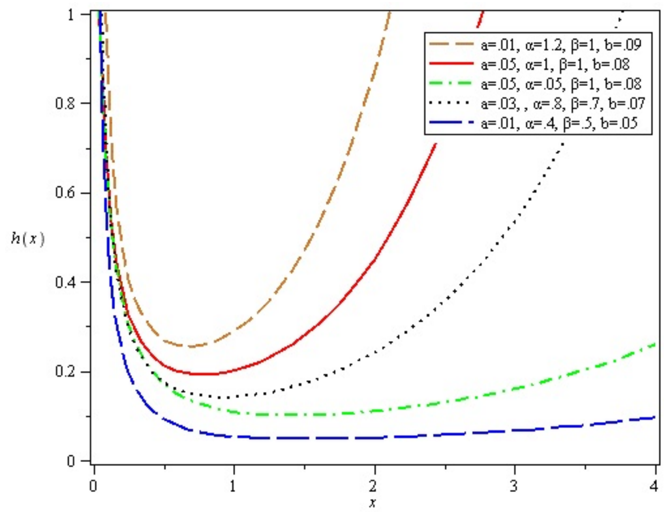

Figure 3: The hazard of various KW-FWE distributions for different values of parameters.

From Figures 1-3, the KW-FWE is a unimodal distribution and has increasing, decreasing and constant hazard rate function.

3. Statistical Properties

In this section we study and discuss some statistical properties for the Kumaraswamy flexible Weibull extension (KW-FWE), specially the quantile function, the median, the mode, the moments and the moment generating function.

3.1. Quantile, Median and Mode

The quantile of the KW-FWE distribution is obtained by solving the following equation, with respect to x_q

$$P\big(X \leq x_q\big) = q, \; 0 < q < 1. \tag{7}$$

Then we have

$$1 - \left[1 - \left[1 - e^{-e^{\alpha x_q - \frac{\beta}{x_q}}}\right]^a\right]^b = q. \tag{8}$$

By solving the above equation, we obtain x_q as follow

$$x_q = \frac{1}{2\alpha}\left\{\ln\left[-\ln\left[1-\left(1-(1-q)^{\frac{1}{b}}\right)^{\frac{1}{a}}\right]\right] \pm \sqrt{\left[\ln\left[-\ln\left[1-\left(1-(1-q)^{\frac{1}{b}}\right)^{\frac{1}{a}}\right]\right]\right]^2 + 4\alpha\beta}\right\}. \tag{9}$$

Since the quantile x_q is positive, then we obtain x_q as follow

$$x_q = \frac{1}{2\alpha}\left\{\ln\left[-\ln\left[1-\left(1-(1-q)^{\frac{1}{b}}\right)^{\frac{1}{a}}\right]\right] + \sqrt{\left[\ln\left[-\ln\left[1-\left(1-(1-q)^{\frac{1}{b}}\right)^{\frac{1}{a}}\right]\right]\right]^2 + 4\alpha\beta}\right\}. \tag{10}$$

The median of KW-FWE can be obtained from equation (10) be setting $q = \frac{1}{2}$. That is, the median is obtained in the following form

$$Med = \frac{1}{2\alpha}\left\{\ln\left[-\ln\left[1-\left(1-\left(\frac{1}{2}\right)^{\frac{1}{b}}\right)^{\frac{1}{a}}\right]\right] + \sqrt{\left[\ln\left[-\ln\left[1-\left(1-\left(\frac{1}{2}\right)^{\frac{1}{b}}\right)^{\frac{1}{a}}\right]\right]\right]^2 + 4\alpha\beta}\right\}. \tag{11}$$

Also, the mode of the KW-FWE distribution can be obtained by deriving its pdf given in (4) with respect to x and equal it to zero. Thus the mode of the KW-FWE distribution can be obtained as a nonnegative solution of the following nonlinear equation

$$\frac{-2\beta}{x^3\left(\alpha+\frac{\beta}{x^2}\right)^2}+\left[1-e^{\alpha x-\frac{\beta}{x}}\right]+e^{\alpha x-\frac{\beta}{x}}e^{-e^{\alpha x-\frac{\beta}{x}}}\left[1-e^{-e^{\alpha x-\frac{\beta}{x}}}\right]^{a-1}\times \left[\frac{a-1}{\left[1-e^{-e^{\alpha x-\frac{\beta}{x}}}\right]^a}+\frac{a(b-1)}{1-\left[1-e^{-e^{\alpha x-\frac{\beta}{x}}}\right]^a}\right]=0. \tag{12}$$

It is not possible to get an explicit solution of the above equation in the general case and therefore numerical methods should be used such as bisection method or fixed-point method to solve it.

3.2. The Moments

Moments are necessary and very important in any statistical analysis, especially in the applications. It can be used to study the most important features and characteristics of the distribution (e.g., tendency,dispersion, skewness and kurtosis). The r^{th} moments of KW-FWE(Φ) is introduced by the following theorem.

Theorem 3.1. *The r^{th} moments of a random variable $X\sim KW-FWE(\Phi)$, where $\Phi=(a,\alpha,\beta,b)$ is given by*

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k+\ell}ab\beta^{\ell}(j+1)^k}{k!\ell!}\binom{b-1}{i}\binom{a(i+1)-1}{j}\times \left[\frac{(r-\ell)!}{\alpha^{r-\ell}(k+1)^{r-2\ell+1}}+\frac{\beta(r-\ell-2)!}{\alpha^{r-\ell-1}(k+1)^{r-2\ell-1}}\right].$$

Proof. The rth moment of the positive random variable X with probability density function f(x) is given by

$$\mu_r^{'}=\int_0^{\infty}x^r f(x)dx. \tag{13}$$

Substituting from (4) into (13), we obtain

$$\mu_r^{'}=ab\int_0^{\infty}x^r\left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}}e^{-e^{\alpha x-\frac{\beta}{x}}}\left[1-e^{-e^{\alpha x-\frac{\beta}{x}}}\right]^{a-1}\left[1-\left[1-e^{-e^{\alpha x-\frac{\beta}{x}}}\right]^a\right]^{b-1}dx. \tag{14}$$

Since $0<\left[1-[1-e^{-e^{\alpha x-\frac{\beta}{x}}}]^a\right]^{b-1}<1$, we obtain

$$\left[1-[1-e^{-e^{\alpha x-\frac{\beta}{x}}}]^a\right]^{b-1}=\sum_{i=0}^{\infty}\binom{b-1}{i}(-1)^i\left[1-e^{-e^{\alpha x-\frac{\beta}{x}}}\right]^{ai}. \tag{15}$$

Substituting from (15) into (14), we get

$$\mu_r^{'}=\sum_{i=0}^{\infty}\binom{b-1}{i}(-1)^i ab\int_0^{\infty}x^r\left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}}\ e^{-e^{\alpha x-\frac{\beta}{x}}}\left[1-e^{-e^{\alpha x-\frac{\beta}{x}}}\right]^{a(i+1)-1}dx. \tag{16}$$

Since $0<\left[1-e^{-e^{\alpha x-\frac{\beta}{x}}}\right]^{a(i+1)-1}<1$, we obtain

$$\left[1-e^{-e^{\alpha x-\frac{\beta}{x}}}\right]^{a(i+1)-1}=\sum_{j=0}^{\infty}\binom{a(i+1)-1}{j}(-1)^j e^{-je^{\alpha x-\frac{\beta}{x}}}. \tag{17}$$

Substituting from (17) into (16), we get

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \tbinom{b-1}{i}\tbinom{a(i+1)-1}{j}(-1)^{i+j}ab\int_0^{\infty} x^r\left(\alpha+\tfrac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}}\; e^{-(j+1)e^{\alpha x-\frac{\beta}{x}}}dx.$$

Using series expansion of $e^{-(j+1)e^{\alpha x-\frac{\beta}{x}}}$, we obtain

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \tbinom{b-1}{i}\tbinom{a(i+1)-1}{j}\frac{(-1)^{i+j+k}ab(j+1)^k}{k!}\int_0^{\infty} x^r\left(\alpha+\tfrac{\beta}{x^2}\right)e^{(k+1)\alpha x}e^{\frac{-(k+1)\beta}{x}}dx.$$

Using series expansion of $e^{\frac{-\beta(k+1)}{x}}$, we obtain

$$\begin{aligned}\mu_r^{'} &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k+\ell}ab\beta^{\ell}(j+1)^k(k+1)^{\ell}}{k!\ell!}\binom{b-1}{i}\binom{a(i+1)-1}{j}\times\\ &\qquad\left[\alpha\int_0^{\infty}x^{r-\ell}e^{\alpha(k+1)x}dx+\beta\int_0^{\infty}x^{r-\ell-2}e^{\alpha(k+1)x}dx\right].\end{aligned}$$

By using the definition of gamma function in the form

$$\Gamma(z) = x^z\int_0^{\infty}e^{tx}t^{z-1}dt,\;\; z,x>0.$$

Finally, we obtain the rth moment of KW-FWE in the form

$$\begin{aligned}\mu_r^{'} &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k+\ell}ab\beta^{\ell}(j+1)^k}{k!\ell!}\binom{b-1}{i}\binom{a(b-i)-1}{j}\times\\ &\qquad\left[\frac{(r-\ell)!}{\alpha^{r-\ell}(k+1)^{r-2\ell+1}}+\frac{\beta(r-\ell-2)!}{\alpha^{r-\ell-1}(k+1)^{r-2\ell-1}}\right].\end{aligned}\tag{18}$$

This completes the proof. □

3.3. Moment Generating Function

In this subsection we derive the moment generating function of KW-FWE distribution as infinite series expansion.

Theorem 3.2. *The moment generating function $M_X(t)$ of a random variable $X \sim KW-FWE(\Phi)$, where $\Phi = (a,\alpha,\beta,,b)$ is given by*

$$\begin{aligned}M_X(t) &= \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k+\ell}ab\beta^{L}(j+1)^k t^r}{r!k!L!}\binom{b-1}{i}\binom{a(b-i)-1}{j}\times\\ &\qquad\left[\frac{(r-\ell)!}{\alpha^{r-\ell}(k+1)^{r-2\ell+1}}+\frac{\beta(r-\ell-2)!}{\alpha^{r-\ell-1}(k+1)^{r-2\ell-1}}\right].\end{aligned}$$

Proof. The moment generating function $M_X(t)$ is defined by

$$M_X(t) = \int_0^{\infty}e^{tx}f(x)dx.$$

Using series expansion of e^{tx}, we obtain

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty}x^r f(x)dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r^{'}\tag{19}$$

Substituting from (18) into (19), we obtain

$$\begin{aligned}M_X(t) &= \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k+\ell}ab\beta^{\ell}(j+1)^k t^r}{r!k!\ell!}\binom{b-1}{i}\binom{a(b-i)-1}{j}\times\\ &\qquad\left[\frac{(r-\ell)!}{\alpha^{r-\ell}(k+1)^{r-2\ell+1}}+\frac{\beta(r-\ell-2)!}{\alpha^{r-\ell-1}(k+1)^{r-2\ell-1}}\right].\end{aligned}$$

This completes the proof. □

4. Order Statistics

Let $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ denote the order statistics obtained from a random sample $X_1, X_2, \cdots, X_n$ which taken from a continuous population with cumulative distribution function $F(x, \Phi)$ and probability density function $f(x, \Phi)$, then the probability density function of $X_{r:n}$ is given by

$$f_{r:n}(x, \Phi) = \frac{1}{B(r, n-r+1)} [F(x, \Phi)]^{r-1} [1 - F(x, \Phi)]^{n-r} f(x, \Phi), \tag{20}$$

where $f(x, \Phi)$ and $F(x, \Phi)$ are the probability density function and the cumulative distribution function of KW-FWE(Φ) distribution given by (3) and (4) respectively and $B(.,.)$ is the beta function, also we define first order statistics $X_{1:n} = \min(X_1, X_2, \cdots, X_n)$, and the last order statistics as $X_{n:n} = \max(X_1, X_2, \cdots, X_n)$. Since $0 < F(x, \Phi) < 1$ for $x > 0$, we can use the binomial expansion of $[1 - F(x, \Phi)]^{n-r}$ given as follows

$$[1 - F(x, \Phi)]^{n-r} = \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i [F(x, \Phi)]^i . \tag{21}$$

Substituting from (21) into (20), we obtain

$$f_{r:n}(x, \Phi) = \frac{f(x; \Phi)}{B(r, n-r+1)} \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i [F(x, \Phi)]^{i+r-1} . \tag{22}$$

Substituting from (3) and (4) into (22), we obtain the probability density function for $X_{r:n}$.

5. Estimation and inference

In this section we discussed the estimation of the KW-FWE parameters by using the method of maximum likelihood based on a complete sample.

5.1. Maximum Likelihood Estimators

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from KW-FWE$(a, \alpha, \beta, b\ )$ with observed values $x_1, x_2, \cdots, x_n$, then the likelihood function can be written as

$$L = \prod_{i=1}^{n} f(x_i, a, b, \alpha, \beta). \tag{23}$$

Substituting from (4) into (23), we get

$$L = \prod_{i=1}^{n} ab \left(\alpha + \frac{\beta}{x_i^2}\right) e^{\alpha x_i - \frac{\beta}{x_i}} e^{-e^{\alpha x_i - \frac{\beta}{x_i}}} \left[1 - e^{-e^{\alpha x_i - \frac{\beta}{x_i}}}\right]^{a-1} \left[1 - \left[1 - e^{-e^{\alpha x - \frac{\beta}{x}}}\right]^{a}\right]^{b-1} .$$

The log-likelihood function can be written as

$$\begin{aligned} \mathcal{L} \ = \ & n \ln(a) + n \ln(b) + \sum_{i=1}^{n} \ln \left[\alpha + \frac{\beta}{x_i^2}\right] + \sum_{i=1}^{n} \left[\alpha x_i - \frac{\beta}{x_i}\right] - \sum_{i=1}^{n} \left[e^{\alpha x_i - \frac{\beta}{x_i}}\right] \\ & + (a-1) \sum_{i=1}^{n} \ln \left[1 - e^{-e^{\alpha x_i - \frac{\beta}{x_i}}}\right] + (b-1) \sum_{i=1}^{n} \ln \left[1 - \left[1 - e^{-e^{\alpha x_i - \frac{\beta}{x_i}}}\right]^{a}\right] \end{aligned} \tag{24}$$

The maximum likelihood estimates of the parameters are obtained by differentiating the log-likelihood function $\mathcal{L}$ with respect to the parameters b, a, α, β and setting the result to zero

$$\frac{\partial \mathcal{L}}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \ln(1 - D_i^a) = 0, \tag{25}$$

$$\frac{\partial \mathcal{L}}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \ln(D_i) - (b-1) \sum_{i=1} n \frac{D_i^a \ln(D_i)}{1 - D_i^a} = 0, \tag{26}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \alpha} &= \sum_{i=1}^{n} \frac{x_i^2}{\alpha x_i^2 + \beta} + \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i e^{\alpha x_i - \frac{\beta}{x_i}} + (a-1) \sum_{i=1}^{n} \frac{x_i e^{\alpha x_i - \frac{\beta}{x_i}}}{R_i} \\ &\quad -a(b-1) \sum_{i=1}^{n} \frac{x_i e^{\alpha x_i - \frac{\beta}{x_i}} (1 - D_i) D_i^{a-1}}{1 - D_i^a} = 0, \end{aligned} \tag{27}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \beta} &= \sum_{i=1}^{n} \frac{1}{\alpha x_i^2 + \beta} - \sum_{i=1}^{n} \frac{1}{x_i} + \sum_{i=1}^{n} \frac{1}{x_i} e^{\alpha x_i - \frac{\beta}{x_i}} - (a-1) \sum_{i=1}^{n} \frac{e^{\alpha x_i - \frac{\beta}{x_i}}}{x_i R_i} \\ &\quad +a(b-1) \sum_{i=1}^{n} \frac{e^{\alpha x_i - \frac{\beta}{x_i}} (1 - D_i) D_i^{a-1}}{x_i (1 - D_i^a)} = 0, \end{aligned} \tag{28}$$

where the nonlinear functions D_i and R_i are given by

$$D_i = 1 - e^{-e^{\alpha x_i - \frac{\beta}{x_i}}}, \qquad R_i = e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1.$$

From equation (25), we obtain the maximum likelihood estimate of b in a closed form as follow

$$\hat{b} = \frac{-n}{\sum_{i=1}^{n} \ln(1 - D_i^a)}.$$

Substituting from (25) into (26), (27) and (28), we get the MLEs of a, α, β by solving the following system of non-linear equations

$$\frac{n}{\hat{a}} + \sum_{i=1}^{n} \ln(D_i) - (\hat{b} - 1) \sum_{i=1}^{n} \frac{D_i^{\hat{a}} \ln(D_i)}{1 - D_i^{\hat{a}}} = 0,$$

$$\sum_{i=1}^{n} \frac{x_i^2}{\hat{\alpha} x_i^2 + \hat{\beta}} + \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i e^{\hat{\alpha} x_i - \frac{\hat{\beta}}{x_i}} + (\hat{a} - 1) \sum_{i=1}^{n} \frac{x_i e^{\hat{\alpha} x_i - \frac{\hat{\beta}}{x_i}}}{R_i} - \hat{a}(\hat{b} - 1) \times \sum_{i=1}^{n} \frac{x_i e^{\hat{\alpha} x_i - \frac{\hat{\beta}}{x_i}} (1 - D_i) D_i^{\hat{a}-1}}{1 - D_i^{\hat{a}}} = 0,$$

$$\sum_{i=1}^{n} \frac{1}{\hat{\alpha} x_i^2 + \hat{\beta}} - \sum_{i=1}^{n} \frac{1}{x_i} + \sum_{i=1}^{n} \frac{1}{x_i} e^{\hat{\alpha} x_i - \frac{\hat{\beta}}{x_i}} - (\hat{a} - 1) \sum_{i=1}^{n} \frac{e^{\hat{\alpha} x_i - \frac{\hat{\beta}}{x_i}}}{x_i R_i} + \hat{a}(\hat{b} - 1) \times \sum_{i=1}^{n} \frac{e^{\hat{\alpha} x_i - \frac{\hat{\beta}}{x_i}} (1 - D_i) D_i^{\hat{a}-1}}{x_i (1 - D_i^{\hat{a}})} = 0.$$

There is no closed form solution to these equations, so statistical software or numerical technique must be applied.

5.2. Asymptotic Confidence Bounds

In this subsection, we derive the asymptotic confidence intervals of the unknown parameters a, α, β, b when $a, \alpha, \beta > 0$ and $b > 0$. The simplest large sample approach is to assume that the MLEs(a, α, β, b) are approximately multivariate normal with mean (a, α, β, b) and covariance matrix I_0^{-1} where I_0^{-1} the inverse of the observed information matrix which is defined by

$$
\begin{aligned}
I_0^{-1} &= -\begin{pmatrix} \frac{\partial^2\mathcal{L}}{\partial a^2} & \frac{\partial^2\mathcal{L}}{\partial a\partial\alpha} & \frac{\partial^2\mathcal{L}}{\partial a\partial\beta} & \frac{\partial^2\mathcal{L}}{\partial a\partial b} \\ \frac{\partial^2\mathcal{L}}{\partial\alpha\partial a} & \frac{\partial^2\mathcal{L}}{\partial\alpha^2} & \frac{\partial^2\mathcal{L}}{\partial\alpha\partial\beta} & \frac{\partial^2\mathcal{L}}{\partial\alpha\partial b} \\ \frac{\partial^2\mathcal{L}}{\partial\beta\partial a} & \frac{\partial^2\mathcal{L}}{\partial\beta\partial\alpha} & \frac{\partial^2\mathcal{L}}{\partial\beta^2} & \frac{\partial^2\mathcal{L}}{\partial\beta\partial b} \\ \frac{\partial^2\mathcal{L}}{\partial b\partial a} & \frac{\partial^2\mathcal{L}}{\partial b\partial\alpha} & \frac{\partial^2\mathcal{L}}{\partial b\partial\beta} & \frac{\partial^2\mathcal{L}}{\partial b^2} \end{pmatrix}^{-1}. \\
&= \begin{pmatrix} var(\hat{a}) & cov(\hat{a},\hat{\alpha}) & cov(\hat{a},\hat{\beta}) & cov(\hat{a},\hat{b}) \\ cov(\hat{\alpha},\hat{a}) & var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) & cov(\hat{\alpha},\hat{b}) \\ cov(\hat{\beta},\hat{a}) & cov(\hat{\beta},\hat{\alpha}) & var(\hat{\beta}) & cov(\hat{\beta},\hat{b}) \\ cov(\hat{b},\hat{a}) & cov(\hat{b},\hat{\alpha}) & cov(\hat{b},\hat{\beta}) & var(\hat{b}) \end{pmatrix}. \qquad (29)
\end{aligned}
$$

The second partial derivatives included in I_0^{-1} are given as follows

$$
\begin{aligned}
\frac{\partial^2\mathcal{L}}{\partial b^2} &= \frac{-n}{b^2}, \qquad \frac{\partial^2\mathcal{L}}{\partial b\partial a} = -\sum_{i=1}^{n}\frac{D_i^a\ln(D_i)}{1-D_i^a}, \\
\frac{\partial^2\mathcal{L}}{\partial b\partial\alpha} &= -a\sum_{i=1}^{n}\frac{x_i e^{\alpha x_i-\frac{\beta}{x_i}}(1-D_i)D_i^{a-1}}{1-D_i^a}, \\
\frac{\partial^2\mathcal{L}}{\partial b\partial\beta} &= a\sum_{i=1}^{n}\frac{e^{\alpha x_i-\frac{\beta}{x_i}}(1-D_i)D_i^{a-1}}{x_i(1-D_i^a)}, \\
\frac{\partial^2\mathcal{L}}{\partial a^2} &= \frac{-n}{a^2}-(b-1)\sum_{i=1}^{n}\frac{D_i^a\left[\ln(D_i)\right]^2}{(1-D_i^a)^2}, \\
\frac{\partial^2\mathcal{L}}{\partial a\partial\alpha} &= \sum_{i=1}^{n}\frac{x_i e^{\alpha x_i-\frac{\beta}{x_i}}}{R_i}-(b-1)\sum_{i=1}^{n}\frac{x_i e^{\alpha x_i-\frac{\beta}{x_i}}(1-D_i)D_i^{a-1}L_i}{(1-D_i^a)^2}, \\
\frac{\partial^2\mathcal{L}}{\partial a\partial\beta} &= -\sum_{i=1}^{n}\frac{e^{\alpha x_i-\frac{\beta}{x_i}}}{x_i R_i}+(b-1)\sum_{i=1}^{n}\frac{e^{\alpha x_i-\frac{\beta}{x_i}}(1-D_i)D_i^{a-1}L_i}{x_i(1-D_i^a)^2}, \\
\frac{\partial^2\mathcal{L}}{\partial\alpha^2} &= -\sum_{i=1}^{n}\frac{x_i^4}{(\alpha x_i^2+\beta)^2}-\sum_{i=1}^{n}x_i^2 e^{\alpha x_i-\frac{\beta}{x_i}}+(a-1)\sum_{i=1}^{n}\frac{x_i^2 e^{\alpha x_i-\frac{\beta}{x_i}}K_i}{R_i^2} \\
&\quad -a(b-1)\sum_{i=1}^{n}\frac{x_i^2 e^{\alpha x_i-\frac{\beta}{x_i}}(1-D_i)D_i^{a-2}}{(1-D_i^a)^2}\left[D_iN_i+e^{\alpha x_i-\frac{\beta}{x_i}}(1-D_i)W_i\right], \\
\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\beta} &= -\sum_{i=1}^{n}\frac{x_i^2}{(\alpha x_i^2+\beta)^2}+\sum_{i=1}^{n}e^{\alpha x_i-\frac{\beta}{x_i}}-(a-1)\sum_{i=1}^{n}\frac{e^{\alpha x_i-\frac{\beta}{x_i}}K_i}{R_i^2} \\
&\quad +a(b-1)\sum_{i=1}^{n}\frac{e^{\alpha x_i-\frac{\beta}{x_i}}(1-D_i)D_i^{a-2}}{(1-D_i^a)^2}\left[D_iN_i+e^{\alpha x_i-\frac{\beta}{x_i}}(1-D_i)W_i\right],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\mathcal{L}}{\partial\beta^2} &= -\sum_{i=1}^{n}\frac{1}{(\alpha x_i^2+\beta)^2}-\sum_{i=1}^{n}\frac{1}{x_i^2}e^{\alpha x_i-\frac{\beta}{x_i}}+(a-1)\sum_{i=1}^{n}\frac{e^{\alpha x_i-\frac{\beta}{x_i}}K_i}{x_i^2R_i^2} \\
&\quad +a(b-1)\sum_{i=1}^{n}\frac{e^{\alpha x_i-\frac{\beta}{x_i}}(1-D_i)D_i^{a-2}}{x_i^2(1-D_i^a)^2}\left[D_iN_i+e^{\alpha x_i-\frac{\beta}{x_i}}(1-D_i)W_i\right],
\end{aligned}
$$

where the nonlinear functions D_i, R_i, Li, K_i, N_i and W_i are given by

$$
\begin{aligned}
D_i &= 1-e^{-e^{\alpha x_i-\frac{\beta}{x_i}}}, \qquad R_i = e^{e^{\alpha x_i-\frac{\beta}{x_i}}}-1, \\
L_i &= a\ln\left[1-e^{-e^{\alpha x_i-\frac{\beta}{x_i}}}\right]-\left[1-e^{-e^{\alpha x_i-\frac{\beta}{x_i}}}\right]^a+1, \quad K_i = e^{e^{\alpha x_i-\frac{\beta}{x_i}}}\left[1-e^{\alpha x_i-\frac{\beta}{x_i}}\right]-1 \\
N_i &= \left[1-e^{\alpha x_i-\frac{\beta}{x_i}}\right]\left[1-\left[1-e^{-e^{\alpha x_i-\frac{\beta}{x_i}}}\right]^a\right], \quad W_i = \left[1-e^{-e^{\alpha x_i-\frac{\beta}{x_i}}}\right]^a+a-1.
\end{aligned}
$$

The above approach is used to derive the $(1-\delta)100\%$ confidence intervals for the parameters a, α, β and b as in the following forms

$$\hat{a} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{a})}, \quad \hat{\alpha} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{\alpha})}, \quad \hat{\beta} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{\beta})}, \quad \hat{b} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{b})},$$

where is $Z_{\frac{\delta}{2}}$ the upper $(\frac{\delta}{2})$th percentile of the standard normal distribution.

6. Data Analysis

Now we use a real data set to show that the KW-FWE distribution can be a better model, comparing with many known distributions such as flexible Weibull extension distribution (FWED), Weibull distribution (WD), linear failure rate distribution (LFRD), generalized exponential distribution (GED), exponentiated Weibull distribution (EWD) and exponentiated flexible Weibull distribution(EFWD). Consider the data have been obtained from Aarset [1], and widely reported in many literatures. It represents the lifetimes of 50 devices, and also, possess a bathtub-shaped failure rate property.

Table 1: The data from Aarset [1].

0.1	0.2	1	1	1	1	1	2	3	6	7	11	12	18	18	18	18
18	21	32	36	40	45	46	47	50	55	60	63	63	67	67	67	67
72	75	79	82	82	83	84	84	84	85	85	85	85	85	86	86	

The MLEs of the unknown parameters a, α, β, b and the corresponding Kolmogorov–Smirnov(K–S) test statistic with its corresponding p-value for the seven distributions are given in Table 2. Also, the values of the negative of the log-likelihood functions (-$\mathcal{L}$), AIC (Akaike Information Criterion), the statistics AICC (Akaike Information Criterion with correction), BIC (Bayesian Information Criterion) and HQIC (Hannan-Quinn information criterion) are given in Table 3 for the seven distributions in order to verify which distribution fits better to these data.

Table 2: the MLES of the parameters, the K–S values and p-values.

	MLE of the parameters						
The model	$\hat{a}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{b}$	K–S	P-value(K-S)
FW(α, β)	–	0.0122	0.7002	–	–	0.4386	4.29×10^{-9}
W(α, β)	–	0.0220	0.949	–	–	0.2397	0.0052
LFR(a, b)	0.014	–	–	–	2.4×10^{-4}	0.1955	0.0370
GE(α, β)	–	0.0212	0.9012	–	–	0.1940	0.0514
EW(α, β, θ)	–	91.023	4.6900	0.146	–	0.1841	0.0590
EFW(α, β, θ)	–	0.0147	0.1330	4.220	–	0.1433	0.2617
KW-FW(a, α, β, b)	4.648	0.0130	0.1270	–	1.324	0.1196	0.4534

Table 3: -$\mathcal{L}$, AIC, AICC, BIC and HQIC for devices data.

The model	-$\mathcal{L}$	AIC	AICC	BIC	HQAIC
FW(α, β)	250.81	505.620	505.88	509.448	507.076
W(α, β)	241.002	486.004	486.26	489.828	487.460
LFR(a, b)	238.064	480.128	480.383	483.952	481.584
GE(α, β)	240.3855	484.7710	485.0264	488.5951	486.227
EW(α, β, θ)	235.926	477.852	478.37	483.588	480.036
EFW(α, β, θ)	226.989	459.979	460.65	465.715	462.162
KW-FW(a, α, β, b)	226.674	461.348	462.236	468.996	464.260

Substituting the MLEs of the unknown parameters into (29), we get estimation of the variance covariance matrix as the following:

$$I_0^{-1} = \begin{bmatrix} 6.6761 & -0.16724 & -0.154609 & 4.370682 \\ -0.16724 & 3.651 \times 10^{-5} & 4.002 \times 10^{-4} & -0.100224 \\ -0.154609 & 4.002 \times 10^{-4} & 1.7455 \times 10^{-3} & -0.103647 \\ 4.370682 & -0.0100224 & -0.103647 & 2.666949 \end{bmatrix}.$$

The approximate 95% two sided confidence intervals of the unknown parameters a, α, β and b are given respectively as $[0, 9.712]$, $\left[1.16 \times 10^{-3}, 0.0248\right]$, $[0.0451, 0.2089]$, $[0, 4.524]$.

Based on Tables 2 and 3, it is shown that KW-FWE$(a, \alpha, \beta, b\ )$ model provide better fit to the data rather than other distributions which we compared with. To show that the likelihood equation have unique solution, we plot the profiles of the log-likelihood function of a, α, β and b in Figures 4-5.

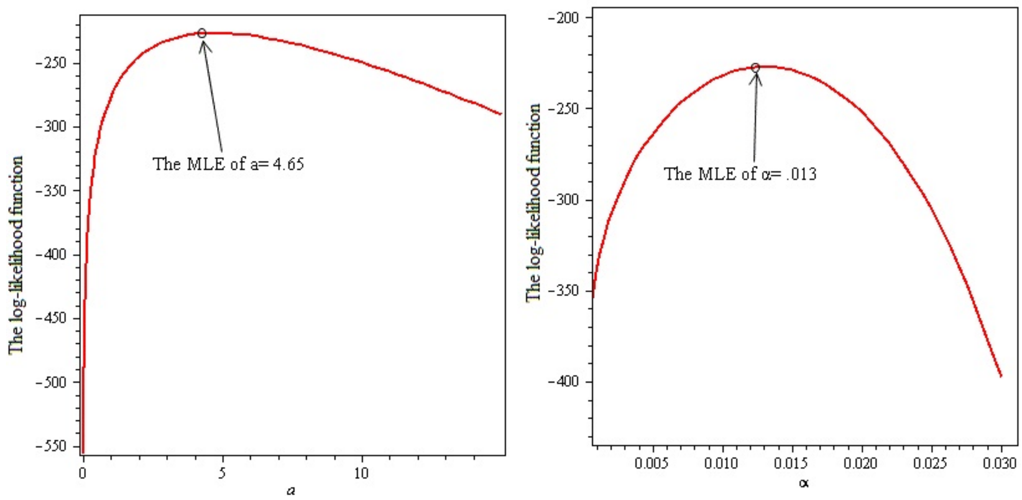

Figure 4: The profile of the log-likelihood function of a, α.

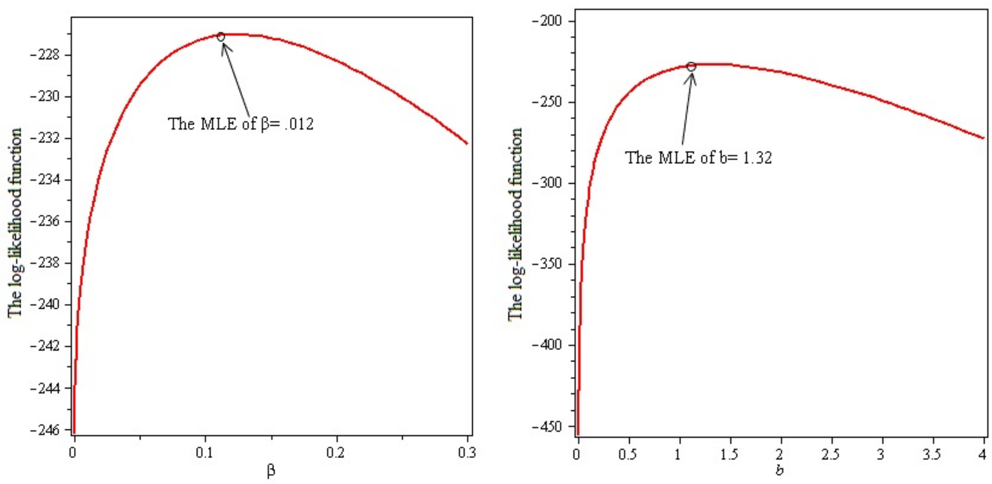

Figure 5: The profile of the log-likelihood function of β, b.

The estimated survival functions for FWE, W, LFR, GE, EW, EFWE and KW-FWE distributions and the empirical survival for Aarset data are given in figure 6.

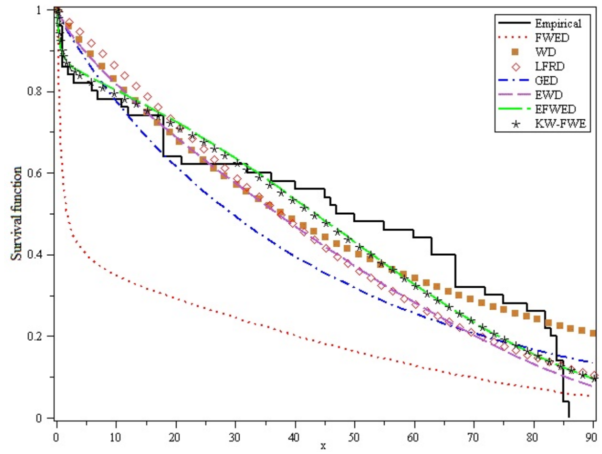

Figure 6: Empirical survival function and fitted survival functions.

Figure 7 and 8, give the form of the probability density functions and the hazard functions for the FWE, W, LFR, GE, EW, EFWE and KW-FWE distributions which are used to fit the data after replacing the unknown parameters included in each distribution by their MLE.

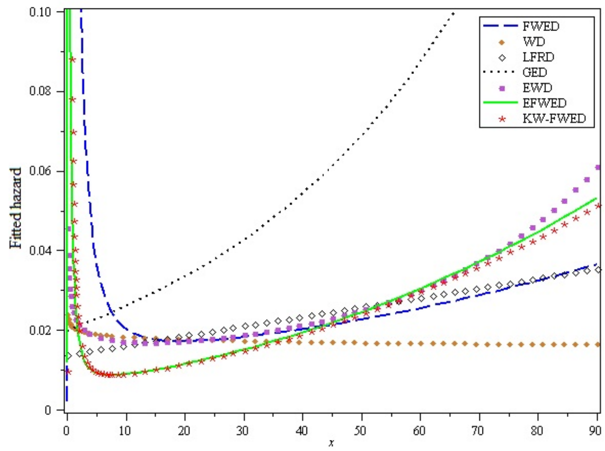

Figure 7: The Fitted hazard functions for the data.

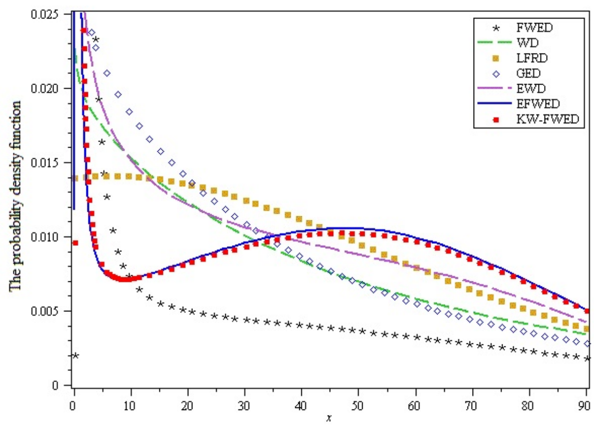

Figure 8: The Fitted probability density functions for the data.

7. Conclusion

In this paper, a new four parameters continuous distribution which generalizes the flexible Weibull extension distribution we called it the Kumaraswamy flexible-Weibull extension(KW-FWE) distribution. Several mathematical and statistical properties have been derived and discussed. We derive expansions for the moments and the moment generating function. The estimation of parameters is approached by the method of maximum likelihood, also the fisher information matrix is derived. An application of the KW-FWE distribution to real data show that the new distribution can be used quite effectively to provide better fits than other distributions we compared with.

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