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# Relationship Among Bivariate Ageing Classes

**Research Article**

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**Abstract:** In this paper, we introduce some new class of bivariate life distributions and we derive the relationship among them.

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**Keywords:** Ageing classes, Bivariate ageing classes, bivariate life distributions.



## 1. Introduction

Bivariate notions of aging and their related classes of life distributions defined by aging properties play a central role in survival analysis, reliability theory, maintenance models and many other actuarial science, engineering, economics, biometry and applied probability areas. They are also useful in obtaining fundamental inequalities of estimates. In the last four decades, remarkable studies have been done on the different aspects of univariate life distributions. Recently, studies have been attracted to establish bivariate life distributions.

## 2. Definitions and Some Related Concepts

In reliability theory, ageing life is usually characterized by a nonnegative random variable $x \geq 0$ with cumulative distribution function (cdf) $F(\cdot)$ and survival function $\overline{F}(\cdot) = 1 - F(\cdot)$. For any random variable $X$, let

$$X_t \cong [X - t|X > t], \quad t \in \{x : F(x) < 1\}$$

denote a random variable whose distribution is the same as the conditional distribution of $X - t$ given that $X > t$ When $X$ is the lifetime of a device, $X_t$ can be regarded as the residual lifetime of the device at time $t$, given that the device has survived upto time $t$. Its survival function is

$$\overline{F}_t(x) = \frac{\overline{F}(t+x)}{\overline{F}(t)}, \quad \overline{F}(t) > 0$$

where $F(x)$ is the survival function of $X$

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**Remark 2.1.** *If $F(\cdot)$ is an exponential distribution then $\overline{F}_t(x) = \overline{F}(x)$*

**Definition 2.2.** *A bivariate random variable $(X, Y)$ or its distribution $\overline{F}(t, s)$ is said to have Bivariate Increasing Failure Rate (BIFR), if*

$$\frac{\overline{F}(x+t, y+s)}{\overline{F}(t, s)}$$

*is decreasing in $x, y \geq 0$.*

**Definition 2.3.** *A bivariate random variable $(X, Y)$ or its distribution $\overline{F}(t, s)$ is said to have Bivariate Increasing Failure Rate Average (BIFRA), if*

$$\begin{aligned} &(i) \quad \frac{1}{\sqrt{u^2+v^2}} \int_0^u \int_0^v r(x, y)\; dy\, dx \text{ is increasing in } v > 0 \\ \text{and} \quad &(ii) \quad \frac{1}{\sqrt{u^2+v^2}} \int_0^u \int_0^v r(x, y)\; dy\; dx \text{ is increasing in } u > 0, \end{aligned}$$

*where $r(\cdot, \cdot) = \dfrac{f(\cdot, \cdot)}{\overline{F}(\cdot, \cdot)}$ denote the failure rate of $F$.*

**Definition 2.4.** *A bivariate random variable $(X, Y)$ or its distribution $\overline{F}(t, s)$ is said to have Bivariate New Better than Used (BNBU), if*

$$\overline{F}(x+t, y+s) \leq \overline{F}(x, y) \cdot \overline{F}(t, s)$$

*for $x, y, t, s \geq 0$.*

**Definition 2.5.** *A bivariate random variable $(X, Y)$ or its distribution $\overline{F}(t, s)$ is said to have Bivariate New Better than Used in Average (BNBUA), if*

$$\int_0^y \int_0^x \overline{F}(u+t, v+s)\; dv\, du \leq \overline{F}(t, s) \int_0^y \int_0^x \overline{F}(u, v)\; dv\, du$$

*for all $x, y > 0$.*

**Definition 2.6.** *A bivariate random variable $(X, Y)$ or its distribution $\overline{F}(t, s)$ is said to have Bivariate New Better than Used in Expectation (BNBUE), if*

$$\int_0^\infty \int_0^\infty \overline{F}(x+t, y+s)\; dt\, ds \leq \overline{F}(x, y) \int_0^\infty \int_0^\infty \overline{F}(t, s)\; dt\, ds,$$

*for $x, y, t, s \geq 0$.*

**Definition 2.7.** *A bivariate random variable $(X, Y)$ or its distribution $\overline{F}(t, s)$ is said to have Bivariate Harmonic New Better than Used in Expectation (BHNBUE), if*

$$\int_t^\infty \int_s^\infty \overline{F}(x, y)\; dy\, dx \leq \mu \exp\left[-\frac{t+s}{\mu}\right] \;;\quad t \geq 0,\; s \geq 0.$$

**Definition 2.8.** *A bivariate random variable $(X, Y)$ or its distribution $\overline{F}(t, s)$, having failure rate $r(x, y)$, is said to have Bivariate New Better than Used in Failure Rate (BNBUFR), if*

$$r(0, 0) \leq r(x, y),$$

*for all $x, y \geq 0$.*

**Remark 2.9.** *A bivariate random variable* $(X, Y)$ *or its distribution* $\overline{F}(t, s)$*, having failure rate* $r(x, y)$*, is said to have Bivariate New Better than Used in Failure Rate (BNBUFR), if*

$$\overline{F}(x+t, y+s) \leq \exp\left[-r(0,0)\sqrt{t^2+s^2}\right] \overline{F}(x, y),$$

*for all* $x, y, t, s \geq 0$*.*

**Definition 2.10.** *A bivariate random variable* $(X, Y)$ *or its distribution* $\overline{F}(t, s)$*, having failure rate* $r(x, y)$*, is said to have Bivariate New Better than Used in Failure Rate Average (BNBUFRA), if*

$$r(0,0) \leq \frac{1}{\sqrt{t^2+s^2}} \int_0^t \int_0^s r(x, y) dy\, dx\,; \text{ for } 0 \leq t < \infty \text{ and } 0 \leq s < \infty.$$

**Definition 2.11.** *A bivariate random variable* $(X, Y)$ *or its distribution* $\overline{F}(t, s)$ *is said to have Bivariate New Better than Used in Convex order (BNBUC), if*

$$\int_u^\infty \int_v^\infty \overline{F}(x+t, y+s)\, ds\, dt \leq \overline{F}(x, y) \int_u^\infty \int_v^\infty \overline{F}(t, s)\, ds\, dt,$$

*for* $x, y \geq 0$ *and* $u, v > 0$*.*

**Definition 2.12.** *A bivariate random variable* $(X, Y)$ *or its distribution* $\overline{F}(t, s)$ *is said to have Bivariate Decreasing Mean Residual Life (BDMRL), if*

$$\mu(t, s) \geq \mu(x, y)\,; \quad 0 \leq t \leq x, \text{ and } 0 \leq s \leq y,$$

*where*

$$\mu(t, s) = \int_0^\infty \int_0^\infty \frac{\overline{F}(x+t, y+s)}{\overline{F}(x, y)}\, dx\, dy,$$

*for* $x, y, t, s \geq 0$*.*

**Definition 2.13.** *A bivariate random variable* $(X, Y)$ *or its distribution* $\overline{F}(t, s)$ *is said to have Bivariate New Better than Used of Third Order (BNBU (3)), if*

$$\int_0^\infty \int_0^\infty \int_0^x \int_0^y \frac{\overline{F}(t+u, s+v)}{\overline{F}(t, s)}\, dv\, du\, dy\, dx \leq \int_0^\infty \int_0^\infty \int_0^x \int_0^y \overline{F}(u, v)\, dv\, du\, dy\, dx$$

*for all* $x, y \geq 0$*.*

**Definition 2.14.** *A bivariate random variable* $(X, Y)$ *or its distribution* $\overline{F}(t, s)$ *is said to have Bivariate Decreasing Cumulative Conditional Survival (BDCCS), if*

$$\int_0^x \int_0^y \frac{\overline{F}(t_2+u, s_2+v)}{\overline{F}(t_2, s_2)}\, dv\, du \leq \int_0^x \int_0^y \frac{\overline{F}(t_1+u, s_1+v)}{\overline{F}(t_1, s_1)}\, dv\, du$$

*for* $0 \leq t_1 \leq t_2 \leq x$ *and* $0 \leq s_1 \leq s_2 \leq y$*.*

**Definition 2.15.** *A bivariate random variable* $(X, Y)$ *or its distribution* $\overline{F}(t, s)$ *is said to have Bivariate Increasing Failure Rate of Second Order (BIFR(2)), if and only if*

$$\int_0^x \int_0^y \frac{\overline{F}(u+t, v+s)}{\overline{F}(t, s)}\, du\, dv \geq \int_0^x \int_0^y \frac{\overline{F}(u+p, v+q)}{\overline{F}(p, q)}\, du\, dv$$

*for all* $x, y \geq 0,\ p \geq t,\ q \geq s$*.*

**Definition 2.16.** *A bivariate random variable* $(X, Y)$ *or its distribution* $\overline{F}(t, s)$ *is said to have Bivariate New Better than Used of Second Order [BNBU(2)], if and only if*

$$\int_0^x \int_0^y \overline{F}(u, v)\, du\, dv \geq \int_0^x \int_0^y \frac{\overline{F}(t+u, s+v)}{\overline{F}(t, s)}\, du\, dv$$

*for all* $x, y, t, s \geq 0$*.*

# 3. Relationship Among Positive Ageing Classes

In this section, we prove some properties of the newly defined positive bivariate ageing classes and we establish the implications on them.

**Theorem 3.1.** *Let $F$ be a BIFR distribution. Then $F$ is a BIFR(2) distribution.*

*Proof.* Since, $F$ be a BIFR distribution. Then we have

$$\frac{\overline{F}(u+t, v+s)}{\overline{F}(t,s)}$$

is decreasing in $x, y \geq 0$.

For, $t \leq p$ and $s \leq q$, we have

$$\frac{\overline{F}(u+t, v+s)}{\overline{F}(t,s)} \geq \frac{\overline{F}(u+p, v+q)}{\overline{F}(p,q)}.$$

On integrating both sides with respect to $u$ and then with respect to $v$, we obtain the desired inequality and this completes the proof. □

**Theorem 3.2.** *Let $F$ be a BIFR(2) distribution. Then $F$ is a BDMRL distribution.*

*Proof.* Since, $F$ be a BIFR(2) distribution. Then we have

$$\int_0^x \int_0^y \frac{\overline{F}(u+t, v+s)}{\overline{F}(t,s)}\, du\, dv \geq \int_0^x \int_0^y \frac{\overline{F}(u+p, v+q)}{\overline{F}(p,q)}\, du\, dv$$

for all $x, y \geq 0,\ p \geq t,\ q \geq s$. Taking limits as $x \to \infty$ and as $y \to \infty$, we obtain the desired inequality and this completes the proof. □

**Theorem 3.3.** *Let $F$ be a BNBU distribution. Then $F$ is a BNBU(2) distribution.*

*Proof.* Since $F$ be a BNBU distribution. Then we have

$$\overline{F}(x+t, y+s) \leq \overline{F}(x,y) \cdot \overline{F}(t,s)$$

for $x, y, t, s \geq 0$. Consider

$$\begin{aligned}\int_0^x \int_0^y \frac{\overline{F}(t+u, s+v)}{\overline{F}(t,s)}\, du\, dv &\leq \int_0^x \int_0^y \frac{\overline{F}(t,s)\cdot\overline{F}(u,v)}{\overline{F}(t,s)}\, du\, dv \\ &= \int_0^x \int_0^y \overline{F}(u,v)\, du\, dv\end{aligned}$$

This proves the theorem. □

**Theorem 3.4.** *Let $F$ be a BNBU(2) distribution. Then $F$ is a BNBUE distribution.*

*Proof.* Since, $F$ be a BNBU(2) distribution. Then we have

$$\int_0^x \int_0^y \frac{\overline{F}(t+u, s+v)}{\overline{F}(t,s)}\, du\, dv \leq \int_0^x \int_0^y \overline{F}(u,v)\, du\, dv$$

for all $x, y, t, s \geq 0$. This implies that

$$\frac{1}{\overline{F}(t,s)} \int_0^x \int_0^y \overline{F}(t+u, s+v)\, du\, dv \leq \int_0^x \int_0^y \overline{F}(u,v)\, du\, dv.$$

That is, $$\int_0^x \int_0^y \overline{F}(t+u, s+v)\, du\, dv \leq \overline{F}(t,s) \int_0^x \int_0^y \overline{F}(u,v)\, du\, dv.$$

As $x \to \infty$ and $y \to \infty$, we obtain the desired result. □

**Theorem 3.5.** *Let $F$ be a BIFR distribution. Then $F$ is a BNBU distribution.*

*Proof.* Since, $F$ be a BIFR distribution. Then we have

$$\frac{\overline{F}(u+t, v+s)}{\overline{F}(t,s)}$$

is decreasing in $x, y \geq 0$.

For, $t \leq p$ and $s \leq q$, we have

$$\frac{\overline{F}(u+t, v+s)}{\overline{F}(t,s)} \geq \frac{\overline{F}(u+p, v+q)}{\overline{F}(p,q)}.$$

Taking $t = 0 = s$, we have

$$\frac{\overline{F}(u,v)}{\overline{F}(0,0)} \geq \frac{\overline{F}(u+p, v+q)}{\overline{F}(p,q)}.$$

This implies that

$$\overline{F}(u,v) \cdot \overline{F}(p,q) \geq \overline{F}(u+p, v+q).$$

Hence, $F$ is BNBU distribution. □

**Theorem 3.6.** *Let $F$ be a BNBU distribution. Then $F$ is a BNBUC distribution.*

*Proof.* Since, $F$ be a BNBU distribution. Then, we have

$$\overline{F}(x+t, y+s) \leq \overline{F}(x,y)\ \overline{F}(t,s).$$

On integrating both sides with respect to $s$ and $t$ and taking limits for $s$ from $v$ to infinity and for $t$ from $u$ to infinity, we find that $F$ is BNBUC distribution. □

**Theorem 3.7.** *Let $F$ be a BNBUC distribution. Then $F$ is a BNBUE distribution.*

*Proof.* Since, $F$ be a BNBUC distribution. Then, we have

$$\int_u^\infty \int_v^\infty \frac{\overline{F}(t+x, s+y)}{\overline{F}(t,s)}\ ds\, dt \leq \int_u^\infty \int_v^\infty \overline{F}(t,s)\ ds\, dt$$

for $x, y, u, v \geq 0$. As $u$ tends to zero and $v$ tends to zero, it follows that $F$ is BNBUE. This proves the theorem. □

**Theorem 3.8.** *Let $F$ be a BNBU distribution. Then $F$ is a BNBUFR distribution.*

*Proof.* Since, $F$ be a BNBU distribution. Then, we have

$$\overline{F}(t+x, s+y) \leq \overline{F}(t,s) \cdot \overline{F}(x,y)$$

for $x, y, t, s \geq 0$. This implies that

$$\frac{\overline{F}(t+x, s+y)}{\overline{F}(t,s)} - 1 \leq \overline{F}(x,y) - 1$$

That is, $\dfrac{\overline{F}(t+x,s+y)-\overline{F}(t,s)}{\overline{F}(t,s)} \leq -F(x,y)$. Therefore,

$$\begin{aligned}
F(x,y) &\leq \frac{\overline{F}(t,s)-\overline{F}(t+x,s+y)}{\overline{F}(t,s)} \\
&= \frac{[(1-F(t,s)]-[1-F(t+x,s+y)]}{\overline{F}(t,s)} \\
&\leq \frac{F(t+x,s+y)-F(t,s)}{\overline{F}(t,s)} \\
\frac{F(x,y)}{xy} &\leq \frac{F(t+x,s+y)-F(t,s)}{xy\,\overline{F}(t,s)}
\end{aligned}$$

Taking limit as $(x,y)\longrightarrow(0,0)$, we have

$$f(0,0) \leq \frac{f(t,s)}{\overline{F}(t,s)}.$$

Since, $\overline{F}(0,0)=1$, the above inequality becomes

$$\frac{f(0,0)}{\overline{F}(0,0)} \leq \frac{f(t,s)}{\overline{F}(t,s)}.$$

This shows that $r(0,0)\leq r(t,s)$. Hence $F$ is BNBUFR. This proves the theorem. ☐

**Theorem 3.9.** *Let $F$ be a BNBUFR distribution. Then $F$ is a BNBUFRA distribution.*

*Proof.* Since, $F$ be a BNBUFR distribution. Then, we have

$$\begin{aligned}
r(0,0) &\leq r(t,s) \\
&\leq \frac{1}{\sqrt{t^2+s^2}}\int_0^t\int_0^s r(x,y)dy\,dx\,; \quad \text{for } 0\leq t<\infty \text{ and } 0\leq s<\infty.
\end{aligned}$$

It follows that, $F$ is BNBUFRA. This proves the theorem. ☐

**Theorem 3.10.** *Let $F$ be a BNBUE distribution. Then $F$ is a BHNBUE distribution.*

*Proof.* Since, $F$ be a BNBUE distribution. Then, we have

$$\begin{aligned}
\int_0^\infty\int_0^\infty \overline{F}(x+t,y+s)\,ds\,dt &\leq \overline{F}(x,y)\int_0^\infty\int_0^\infty \overline{F}(t,s)\,ds\,dt \\
\int_0^\infty\int_0^\infty \frac{\overline{F}(x+t,y+s)}{\overline{F}(t,s)}\,ds\,dt &\leq \int_0^\infty\int_0^\infty \overline{F}(t,s)\,ds\,dt
\end{aligned}$$

Hence, $\mu(t,s)\leq\mu$, where $\mu=\displaystyle\int_0^\infty\int_0^\infty \overline{F}(t,s)\,ds\,dt$ denotes the mean of the bivariate distribution $F(x,y)$ and for all $x,y,u,v\geq 0$. This implies that

$$\begin{aligned}
\frac{1}{\mu(t,s)} &\geq \frac{1}{\mu} \\
\mu^{-1}(t,s) &\geq \frac{1}{\mu} \\
\int_0^\infty\int_0^\infty \mu^{-1}(x,y)\,dy\,dx &\geq \frac{ts}{\mu} \\
\frac{1}{\frac{1}{ts}\int_0^\infty\int_0^\infty \mu^{-1}(x,y)\,dy\,dx} &\leq \mu.
\end{aligned}$$

It follows that $F$ is BHNBUE. This proves the theorem. ☐

**Theorem 3.11.** *Let $F$ be a BIFR distribution. Then $F$ is a BIFRA distribution.*

*Proof.* $F$ is BIFR if $-\log \overline{F}(x,y)$ is convex. Also, $F$ is BIFRA if $\log \overline{F}(x,y)$ is star shaped. That is,

$$-\log \overline{F}(\lambda x, \lambda y) \leq -\lambda \log \overline{F}(x,y), \quad 0 \leq \lambda \leq 1 \ \ and \ \ x \geq 0.$$

Since, a convex function is star-shaped. It follows that, $F$ is BIFRA. This proves the theorem. □

**Theorem 3.12.** *Let $F$ be a BIFRA distribution. Then $F$ is a BNBU distribution.*

*Proof.* $F$ is BIFRA if $-\dfrac{\log \overline{F}(t,s)}{\sqrt{t^2+s^2}}$ is increasing in $t, s \geq 0$. That is, $\log \left[\overline{F}(t,s)\right]^{\dfrac{1}{\sqrt{t^2+s^2}}}$ is decreasing in $t, s \geq 0$. This implies that, $\left[\overline{F}(t,s)\right]^{\dfrac{1}{\sqrt{t^2+s^2}}}$. Hence, if $t > u$ and $s > v$, then

$$\begin{aligned}
\left[\overline{F}(t+u, s+v)\right]^{\frac{1}{\sqrt{t^2+u^2+s^2+v^2}}} &\leq \left[\overline{F}(t,s)\right]^{\frac{1}{\sqrt{t^2+s^2}}} \\
&\leq \left[\overline{F}(t,s)\right]^{\frac{1}{\sqrt{t^2+s^2}}} \\
&\leq \left[\overline{F}(t,s)\right]^{\frac{\sqrt{t^2+u^2+s^2+v^2}}{\sqrt{t^2+s^2}}} \\
&\leq \left[\overline{F}(t,s)\right]^{\frac{1+\sqrt{u^2+v^2}}{\sqrt{t^2+s^2}}} \\
&= \overline{F}(t,s) \cdot \left[\overline{F}(t,s)\right]^{\frac{\sqrt{u^2+v^2}}{\sqrt{t^2+s^2}}} \\
&\leq \overline{F}(u,v) \cdot \left[\overline{F}(u,v)\right]^{\frac{\sqrt{u^2+v^2}}{\sqrt{t^2+s^2}}} \\
&\leq \overline{F}(u,v) \cdot \overline{F}(t,s)
\end{aligned}$$

It follows that $F$ is BNBU. This proves the theorem. □

**Theorem 3.13.** *Let $F$ be a BDMRL distribution. Then $F$ is a BNBUE distribution.*

*Proof.* $F$ is BDMRL if $\mu(t,s) \geq \mu(x,y): \quad 0 \leq t \leq x, \ 0 \leq s \leq y$, where

$$\mu(t,s) = \int_0^\infty \int_0^\infty \frac{\overline{F}(x+t, y+s)}{\overline{F}(x,y)} \, dx \, dy,$$

for $x, y, t, s \geq 0$. We have to prove,

$$\int_0^\infty \int_0^\infty \overline{F}(x+t, y+s) \, ds \, dt \leq \overline{F}(x,y) \int_0^\infty \int_0^\infty \overline{F}(t,s) \, ds \, dt$$

for $x, y \geq 0$.

Now consider, for all $0 \leq x \leq t$ and $0 \leq y \leq s$,

$$\begin{aligned}
\int_0^\infty \int_0^\infty \overline{F}(x+t, y+s) \, ds \, dt &= \overline{F}(x,y) \cdot \int_0^\infty \int_0^\infty \frac{\overline{F}(x+t, y+s)}{\overline{F}(x,y)} \, ds \, dt \\
&= \overline{F}(x,y) \cdot \mu(t,s) \\
&\leq \overline{F}(x,y) \cdot \mu(x,y) \\
&= \overline{F}(x,y) \cdot \int_0^\infty \int_0^\infty \frac{\overline{F}(u+x, v+y)}{\overline{F}(u,v)} \, dv \, du \\
&\leq \overline{F}(x,y) \cdot \int_0^\infty \int_0^\infty \overline{F}(x,y) \, dx \, dy
\end{aligned}$$

As $u \longrightarrow 0$ and $v \longrightarrow 0$, the theorem is proved. □

**Theorem 3.14.** *Let $F$ be a BNBU(2) distribution. Then $F$ is a BNBU(3) distribution.*

*Proof.* Assume that $F$ is BNBU(2). That is,

$$\int_0^x \int_0^y \frac{\overline{F}(t+u,s+v)}{\overline{F}(t,s)}\,dv\,du \le \int_0^x \int_0^y \overline{F}(u,v)\,dv\,du, \quad \textit{for all } x,y,t,s \ge 0.$$

Integrating twice both sides from 0 to $\infty$ with respect to $y$ and $x$, the result follows. □

**Theorem 3.15.** *Let $F$ be a BNBUA distribution. Then $F$ is a BNBUE distribution.*

*Proof.* Assume that $F$ is BNBUA. That is,

$$\int_0^y \int_0^x \overline{F}(u+t,v+s)\,dv\,du \le \overline{F}(t,s)\int_0^y \int_0^x \overline{F}(u,v)\,dv\,du \quad \text{for } x,y>0.$$

Letting $x$ tend to infinity and $y$ tend to infinity,we have

$$\int_0^\infty \int_0^\infty \overline{F}(u+t,v+s)\,dv\,du \le \overline{F}(t,s)\int_0^\infty \int_0^\infty \overline{F}(u,v)\,dv\,du,$$

so that $F$ is BNBUE distribution. □

**Theorem 3.16.** *Let $F$ be a BDCCS distribution. Then $F$ is a BNBUA distribution.*

*Proof.* Assume that $F$ is BDCCS. Then,

$$\int_0^x \int_0^y \frac{\overline{F}(t_2+u,s_2+v)}{\overline{F}(t_2,s_2)}\,dv\,du \le \int_0^x \int_0^y \frac{\overline{F}(t_1+u,s_1+v)}{\overline{F}(t_1,s_1)}\,dv\,du,$$

for $0 \le t_1 \le t_2 \le x;\quad 0 \le s_1 \le s_2 \le y$. Letting $t_1$ tend to 0 and $s_1$ tend to 0, we obtain the desired result. □

**Theorem 3.17.** *Let $F$ be a BNBU distribution. Then $F$ is a BNBUA distribution.*

*Proof.* Since, $F$ be a BNBU distribution. Then, we have

$$\overline{F}(u+t,v+s) \le \overline{F}(u,v)\cdot\overline{F}(t,s)$$

for $x,y,t,s \ge 0$. Integrating twice both sides from 0 to $x$ and 0 to $y$, we obtain

$$\int_0^y \int_0^x \overline{F}(u+t,v+s)\,dv\,du \le \overline{F}(t,s)\int_0^y \int_0^x \overline{F}(u,v)\,dv\,du \quad \text{for } x,y>0.$$

Hence, $F$ is a BNBUA distribution. □

**Theorem 3.18.** *If $(X,Y)=Z$ is a bivariate NBUA, then $Z$ is bivariate NBUE.*

*Proof.* Since $Z=(X,Y)$ is bivariate NBUA,

$$\int_0^y \int_0^x \overline{F}(u+t,v+s)\,du\,dv \le \overline{F}(t,s)\int_0^y \int_0^x \overline{F}(u,v)\,du\,dv.$$

Letting $x$ to tend to infinity and $y$ to tend to infinity, we have

$$\int_0^\infty \int_0^\infty \overline{F}(u+t,v+s)\,du\,dv \le \overline{F}(t,s)\int_0^\infty \int_0^\infty \overline{F}(u,v)\,du\,dv,$$

so that $F$ is bivariate NBUE. □

**Theorem 3.19.** *If $(X, Y) = Z$ is a bivariate DCCS, then $Z$ is bivariate NBUA.*

*Proof.* Since $Z = (X, Y)$ is bivariate DCCS, we have for $0 \leq t_1 \leq t_2 \leq x$ and $0 \leq s_1 \leq s_2 \leq y$

$$\int_0^x \int_0^y \frac{\overline{F}(t_2 + u, s_2 + v)}{\overline{F}(t_2, s_2)}\, dv\, du \leq \int_0^x \int_0^y \frac{\overline{F}(t_1 + u, s_1 + v)}{\overline{F}(t_1, s_1)}\, dv\, du$$

Letting $t_1$ to tend to 0, we have

$$\int_0^x \int_0^y \frac{\overline{F}(t_2 + u, s_2 + v)}{\overline{F}(t_2, s_2)}\, dv\, du \leq \int_0^x \int_0^y \overline{F}(u, v)\, dv\, du$$

Equivalently,

$$\int_0^x \int_0^y \overline{F}(t + u, s + v)\, dv\, du \leq \overline{F}(t, s) \int_0^x \int_0^y \overline{F}(u, v)\, dv\, du,$$

so that $Z$ is BNBUA. This completes the theorem. □

**Theorem 3.20.** *If $(X, Y) = Z$ is a bivariate NBU, then $Z$ is bivariate NBUA.*

*Proof.* The result at once follows from the definition. □

# 4. Conclusion

In this paper, we have introduced and studied the relationship among the bivariate ageing classes. Implications among some of the bivariate ageing classes are also proved.

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