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# PPF Dependent Fixed Point Theorem for $(\alpha - \psi)-$contractive Mappings in Banach Spaces

**Research Article**

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**Abstract:** We prove the existence of the PPF dependent fixed point theorems in the Razumikhin class for a pair of mappings satisfying $(\alpha - \psi)-$ contractive conditions in Banach spaces where the domains and ranges of the mappings are not the same. We also prove the existence of the PPF dependent common fixed point theorems and the PPF dependent coincidence points theorems. Our results extend and generalize the results of [2–4] and references therein.

**MSC:** 47H09, 47H10, 54E50.

**Keywords:** Razumikhin class, PPF dependent point, fixed point, Banach spaces.



## 1. Introduction and Preliminaries

In 1997, Bernfeld et al. [1] introduced the concept of PPF dependent fixed point or the fixed point with PPF dependence which is a one type of fixed point for mappings that have different domains and ranges. They also proved the existence of PPF dependent fixed point theorems in the Razumikhin class for Banach type contraction mappings. The PPF dependent fixed point theorems are useful for proving the solutions of nonlinear functional differential and integral equations which may depend upon the past history, present data and future consideration. Afterward, a number of papers appeared in which PPF dependent fixed point theorems have been discussed (see [2–4] and references therein).

Throughout this paper, $E$ denotes a Banach space with the norm $\|.\|_E$ $I$ denotes a closed intervel $[a, b]$ in $\mathbb{R}$, and $E_0 = C(I, E)$ denotes the set of all continuous $E-$valued functions on $I$ equipped with the superemum norm $\|.\|_{E_0}$ defined by

$$\|.\|_{E_0} = \sup_{t \in I} \|\phi(t)\|_E$$

for $\phi \in E_0$.

For a fixed element $c \in I$, the Razumikhin or minimal class of functions in $E_0$ is defined by

$$\mathcal{R}_c = \{\phi \in E_0 : \|.\|_E\}.$$

It is easy to see that the constant function in one of the mapping in $\mathcal{R}_c$. The class $\mathcal{R}_c$ is said to be algebraically closed with respect to difference if $\phi - \xi \in \mathcal{R}_c$ whenever $\phi, \xi \in \mathcal{R}_c$. Also we say the class $\mathcal{R}_c$ is said to be topologically closed and if it is closed with respect to the topology on $E_0$ generated by the norm $\|.\|_{E_0}$.

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**Definition 1.1.** *A point $\phi \in E_0$ is said to be a PPF dependent fixed point or a fixed point with PPF dependence of the non-self mapping $T : E_0 \to E$ if $T\phi = \phi(c)$ for some $c \in I$.*

**Definition 1.2.** *Let $X$ be a nonempty set, $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that $T$ is an $\alpha-$admissible mapping if it satisfies the following condition :*

$$for \;\; x, y \in X \;\; for \;\; which \;\; \alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1.$$

**Example 1.3.** *Let $X = [1, \infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Tx = x^2$ for all $x \in X$ and*

$$\alpha(x, y) = \begin{cases} 2, & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

*Then $T$ is $\alpha-$admissible.*

**Example 1.4.** *Let $X = [1, \infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Tx = \log x$ for all $x \in X$ and*

$$\alpha(x, y) = \begin{cases} e^{x-y}, & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

*Then $T$ is $\alpha-$admissible.*

**Example 1.5.** *Let $X = [1, \infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by*

$$Tx = \begin{cases} \ln x, & \text{if } x \geq 1 \\ \frac{x}{2} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

*and*

$$\alpha(x, y) = \begin{cases} 2, & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

*Then $T$ is $\alpha-$admissible.*

## 2. Main Results

**Definition 2.1.** *Let $c \in I$ and $T : E_0 \to E$, $\alpha : E \times E \to [0, \infty)$. We say that $T$ is an $\alpha_c-$admissible mapping if for $\phi, \xi \in E_0$,*

$$\alpha(\phi(c), \xi(c)) \geq 1 \implies \alpha(T\phi, T\xi) \geq 1.$$

**Example 2.2.** *Let $E = \mathbb{R}$ be real Banach spaces with usual norms and $I = [0, 1]$. Define $T : E_0 \to E$ and $\alpha : E \times E \to [0, \infty)$ by $T\phi = \phi(1)$ for all $\phi \in E_0$ and*

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

*Then $T$ is $\alpha_1-$admissible*

**Definition 2.3.** *Let $c \in I$ and $T : E_0 \to E$, $\alpha, \eta : E \times E \to [0, \infty)$. We say that $T$ is an $\alpha_c-$admissible mapping with respect to $\eta_c$ if for $\phi, \xi \in E_0$,*

$$\alpha(\phi(c), \xi(c)) \geq \eta(\phi(c), \xi(c)) \implies \alpha(T\phi, T\xi) \geq \eta(T\phi, T\xi).$$

Note that if we take $\eta(\phi(c), \xi(c)) = 1$, then we say $T$ is an $\alpha_c-$admissible mapping. Also, if we take $\alpha(\phi(c), \xi(c)) = 1$, then we say $T$ is an $\eta_c-$subadmissible mapping.

**Example 2.4.** *Let $E = \mathbb{R}$ be real Banach spaces with usual norms and $I = [0, 1]$. Define $T : E_0 \to E$ and $\alpha : E \times E \to [0, \infty)$ by $T\phi = \frac{1}{2}\phi(1)$ for all $\phi \in E_0$ and $\alpha, \eta : E \times E \to [0, \infty)$ by*

$$\alpha(x, y) = \begin{cases} x^4 + y^8 + 1, & \text{if } x \geq y \\ \frac{1}{3} & \text{otherwise} \end{cases}$$

*$\eta(x, y) = x^4 + \frac{1}{2}$. Then $T$ is $\alpha_1-$admissible with respect to $\eta_1$. In fact $\alpha(\phi(1), \xi(1)) \geq \eta(\phi(1), \xi(1))$, then $\phi(1) \geq \xi(1)$ and so, $\frac{1}{2}\phi(1) \geq \frac{1}{2}\xi(1)$. That is, $T\phi \geq T\xi$ which implies that $\alpha(T\phi, T\xi) \geq \eta(T\phi, T\xi)$.*

Khan et al. [6] introduced the notion of an altering distance function, which is a control function that alters distance between two points in an metric space.

**Definition 2.5.** *A function $\psi' : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if and only if*

*(i) $\psi'$ is continuous,*

*(ii) $\psi'$ is non-decreasing,*

*(iii) $\psi'(x) = 0 \iff x = 0$.*

Altering distance function have been generalized to a two variable function by Choudhury and Dutta [7] and to a three variable function by Choudhury [8] and was applied for obtaining fixed point results in metric spaces.

**Definition 2.6.** *Let $\Psi'_3$ be the set of all functions $\psi' : [0, +\infty)^3 \to [0, +\infty)$ is called a generalized altering distance function if and only if*

*(i) $\psi'$ is continuous,*

*(ii) $\psi'$ is non-decreasing in all the three variables,*

*(iii) $\psi'(x, y, z) = 0 \iff x = y = z = 0$.*

Rao et al. [5] introduced the generalized altering distance function in five variables as a generalization of three variables.

**Definition 2.7.** *Let $\Psi'_5$ be the set of all functions $\psi' : [0, +\infty)^5 \to [0, +\infty)$ is called a generalized altering distance function if and only if*

*(i) $\psi'$ is continuous,*

*(ii) $\psi'$ is non-decreasing in all the five variables,*

*(iii) $\psi'(x, y, z, u, v) = 0 \iff x = y = z = u = v = 0$.*

Now we generalized the notion of altering distance function for five variables which is as follows,

**Definition 2.8.** *Let $\Psi_5$ denote the set of all functions $\psi:[0,+\infty)^5\to[0,+\infty)$ . Then $\psi$ is said to be a generalized altering distance function if and only if*

*(i)* $\psi$ *is continuous,*

*(ii)* $\psi(x,y,z,u,v)=0\Leftrightarrow x=y=z=u=v=0.$

*(iii) there exists* $k\in(0,1)$ *such that*

$$\psi(u,u,v,u+v,0)\le ku,$$

$$\psi(0,u,u,u,u)\le ku$$

*and*

$$\psi(u,0,0,u,u)\le ku.$$

**Example 2.9.** *Let* $\psi(x,y,z,u,v)=k\max\{x,y,z,u,v\}$ *for* $k\in(0,1)$, *then*

*(i)* $\psi$ *is continuous,*

*(ii)* $\psi(x,y,z,u,v)=0\Leftrightarrow x=y=z=u=v=0,$

*(iii)* $\psi(u,u,v,u+v,0)=\psi(0,u,u,u,u)=\psi(u,0,0,u,u)\le ku$ , $\forall u>0$ *and* $k\in(0,1)$.

*Therefore* $\psi\in\Psi_5$.

**Example 2.10.** *Let*

$$\psi(x,y,z,u,v)=k\max\{x,y,z,\frac{1}{2}(u+v)\}$$

*for* $k\in(0,1)$, *then*

*(i)* $\psi$ *is continuous,*

*(ii)* $\psi(x,y,z,u,v)=0\Leftrightarrow x=y=z=u=v=0,$

*(iii)* $\psi(u,u,v,u+v,0)=\psi(0,u,u,u,u)=\psi(u,0,0,u,u)\le ku$ , $\forall u>0$ *and* $k\in(0,1)$.

*Therefore* $\psi\in\Psi_5$.

**Example 2.11.** *Let*

$$\psi(x,y,z,u,v)=k_1x+k_2y+k_3z+k_4u+k_5v$$

*for* $k_i\in(0,1),i=1,2,3,4,5.$ *such that* $\Sigma_{i=1}^{5}k_i<1$, *then*

*(i)* $\psi$ *is continuous,*

*(ii)* $\psi(x,y,z,u,v)=0\Leftrightarrow x=y=z=u=v=0,$

*(iii)* $\psi(u,u,v,u+v,0)=\psi(0,u,u,u,u)=\psi(u,0,0,u,u)\le qu$ , $\forall u>0$ *and* $\Sigma_{i=1}^{5}k_i=q\in(0,1)$.

*Therefore* $\psi\in\Psi_5$.

Next we prove our main result of this section.

**Theorem 2.12.** *Let* $T:E_0\to E$ *and* $\alpha,\eta:E\times E\to[0,\infty)$ *be two mappings satisfying the following conditions:*

*(a) There exists $c \in I$ such that $\mathcal{R}_c$ is topological closed and algebraically closed with respect to difference.*

*(b) $T$ is $\alpha_c-$admissible with respect to $\eta_c$.*

*(c) For all $\phi, \xi \in E_0$, $\alpha(\phi(c), \xi(c)) \geq \eta(\phi(c), \xi(c))$ implies*

$$\|T\phi - T\xi\|_E \leq \psi(\|\phi - \xi\|_{E_0}, \|\phi - T\phi\|_E, \|\xi - T\xi\|_E, \|\phi - T\xi\|_E, \|\xi - T\phi\|_E)$$

*where $\psi \in \Psi_5$.*

*(d) If $\{\phi_n\}$ is a sequence in $E_0$ such that $\phi_n \to \phi$ as $n \to \infty$ and $\alpha(\phi_n(c), T\phi_n) \geq \eta(\phi_n(c), T\phi_n)$ for all $n \in \mathbb{N}$, then $\alpha(\phi(c), T\phi) \geq \eta(\phi(c), T\phi)$.*

*If there exists $\phi_0 \in \mathcal{R}_c$ such that*

$$\alpha(\phi_0(c), T\phi_0) \geq \eta(\phi_0(c), T\phi_0),$$

*then $T$ has a unique PPF dependent fixed point $\phi^*$ in $\mathcal{R}_c$ such that*

$$\alpha(\phi^*(c), T\phi^*) \geq \eta(\phi^*(c), T\phi^*).$$

*Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$ such that*

$$\alpha(\phi_0(c), T\phi_0) \geq \eta(\phi_0(c), T\phi_0),$$

*if a sequence $\{\phi_n\}$ of iterates of $T$ in $\mathcal{R}_c$ defined by*

$$T\phi_{n-1} = \phi_n(c) \tag{1}$$

*for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of $T$ in $\mathcal{R}_c$.*

*Proof.* Let $\phi_0$ be a point in $\mathcal{R}_c \subseteq E_0$ such that

$$\alpha(\phi_0(c), T\phi_0) \geq \eta(\phi_0(c), T\phi_0).$$

Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Choose $\phi_1 \in \mathcal{R}_c$ such that

$$x_1 = \phi_1(c).$$

Since $\phi_1 \in \mathcal{R}_c \subseteq E_0$ and by hypothesis, we get $T\phi_1 \in E$. This implies that there exists $x_2 \in E$ such that $T\phi_1 = x_2$. Thus, we can choose $\phi_2 \in \mathcal{R}_c$ such that

$$x_2 = \phi_2(c).$$

By continuing this process, by induction, we can construct the sequence $\{\phi_n\}$ in $\mathcal{R}_c \subseteq E_0$ such that

$$T\phi_{n-1} = \phi_n(c)$$

for all $n \in \mathbb{N}$. It follows from the fact that $\mathcal{R}_c$ is algebraically closed with respect to difference that

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \|\phi_{n-1}(c) - \phi_n(c)\|_E$$

for all $n \in \mathbb{N}$. Since $T$ is $\alpha_c\eta_c$- admissible and

$$\alpha(\phi_0(c), \phi_1(c)) = \alpha(\phi_0(c), T\phi_0) \geq \eta(\phi_0(c), T\phi_0) = \eta(\phi_0(c), \phi_1(c)),$$

we deduce that

$$\alpha(\phi_1(c), T\phi_1) = \alpha(T\phi_0, T\phi_1) \geq \eta(T\phi_0, T\phi_1) = \eta(\phi_1(c), T\phi_1)$$

. By continuing this process, we get

$$\alpha(\phi_{n-1}(c), T\phi_{n-1}) \geq \eta(\phi_{n-1}(c), T\phi_{n-1})$$

for all $n \in \mathbb{N}$. Next, we show that $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|\phi_n - \phi_{n+1}\|_{E_0} &= \|\phi_n(c) - \phi_{n+1}(c)\|_E = \|T\phi_{n-1} - T\phi_n\|_E \\
&\leq \psi(\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1} - T\phi_{n-1}\|_E, \|\phi_n - T\phi_n\|_E, \\
&\quad \|\phi_{n-1} - T\phi_n\|_E, \|\phi_n - T\phi_{n-1}\|_E) \\
&\leq \psi(\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}, \\
&\quad \|\phi_{n-1} - \phi_{n+1}\|_{E_0}, 0) \\
&\leq \psi(\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}, \\
&\quad \|\phi_{n-1} - \phi_n\|_{E_0} + \|\phi_n - \phi_{n+1}\|_{E_0}, 0) \\
&\leq k\|\phi_{n-1} - \phi_n\|_{E_0}.
\end{aligned}$$

By repeating the above relation, we get

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq k^n\|\phi_0 - \phi_1\|_{E_0}$$

for all $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ with $m > n$, we obtain that

$$\begin{aligned}
\|\phi_n - \phi_m\|_{E_0} &\leq \|\phi_n - \phi_{n+1}\|_{E_0} + \|\phi_{n+1} - \phi_{n+2}\|_{E_0} + \\
&\quad + \cdots + \|\phi_{m-1} - \phi_m\|_{E_0} \\
&\leq (k^n + k^{n+1} + \cdots + k^{m-1})\|\phi_0 - \phi_1\|_{E_0} \\
&\leq \frac{k^n}{1-k}\|\phi_0 - \phi_1\|_{E_0}.
\end{aligned}$$

This implies that the sequence $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c \subseteq E_0$. By the completeness of $E_0$, we get that $\{\phi_n\}$ converges to a limit point $\phi^* \in E_0$, that is, $\lim_{n\to\infty} \phi_n = \phi^*$. Since $\mathcal{R}_c$ is topologically closed, we have $\phi^* \in \mathcal{R}_c$.

Now, we prove that $\phi^*$ is a PPF dependent fixed point of $T$. By (d) we have $\alpha(\phi^*(c), T\phi^*) \geq \eta(\phi^*(c), T\phi^*)$. From assumption (c), we get

$$\begin{aligned}
\|T\phi^* - \phi^*(c)\|_E &\leq \|T\phi^* - \phi_n(c)\|_E + \|\phi_n(c) - \phi^*(c)\|_E \\
&= \|T\phi^* - T\phi_{n-1}\|_E + \|\phi_n - \phi^*\|_{E_0} \\
&\leq \psi(\|\phi^* - \phi_{n-1}\|_{E_0}, \|\phi^* - T\phi^*\|_E, \|\phi_{n-1} - T\phi_{n-1}\|_E, \\
&\quad \|\phi^* - T\phi_{n-1}\|_E, \|\phi_{n-1} - T\phi^*\|_E) + \|\phi_n - \phi^*\|_{E_0}
\end{aligned}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the above inequality, we have

$$\|T\phi^* - \phi^*(c)\|_E \ \le \ k\|T\phi^* - \phi^*(c)\|_E$$

which contradiction

$$\|T\phi^* - \phi^*(c)\|_E \ = \ 0 \tag{2}$$

and so

$$T\phi^* = \phi^*(c).$$

This implies that $\phi^*$ is a PPF dependent fixed point of $T$ in $\mathcal{R}_c$. Finally, we prove the uniqueness of a PPF dependent fixed point of $T$ in $\mathcal{R}_c$. Let $\phi^*$ and $\xi^*$ be two PPF dependent fixed points of $T$ in $\mathcal{R}_c$ such that

$$\alpha(\phi^*(c), T\phi^*) \ge \eta(\phi^*(c), T\phi^*)$$

and

$$\alpha(\xi^*(c), T\xi^*) \ge \eta(\xi^*(c), T\xi^*).$$

Now we obtain that

$$\begin{aligned}
\|\phi^* - \xi^*\|_{E_0} &= \|\phi^*(c) - \xi^*(c)\|_E = \|T\phi^* - T\xi^*\|_E \\
&\le \psi(\|\phi^* - \xi^*\|_{E_0}, \|\phi^* - T\phi^*\|_E, \|\xi^* - T\xi^*\|_E, \\
&\quad \|\phi^* - T\xi^*\|_E, \|\xi^* - T\phi^*\|_E) \\
&\le k\|\phi^* - \xi^*\|_{E_0}.
\end{aligned}$$

Since $0 \le k < 1$, we get $\|\phi^* - \xi^*\|_{E_0} = 0$ and then $\phi^* = \xi^*$. Therefore, $T$ has a unique PPF dependent fixed point in $\mathcal{R}_c$. This completes the proof. □

**Theorem 2.13.** *Let $T : E_0 \to E$, $\alpha : E \times E \to [0, \infty)$ be two mappings satisfying the following conditions:*

*(a) There exists $c \in I$ such that $\mathcal{R}_c$ is topological closed and algebraically closed with respect to difference.*

*(b) $T$ is $\alpha_c-$admissible.*

*(c) For all $\phi, \xi \in E_0$, $\alpha(\phi(c), \xi(c)) \ge 1$ implies*

$$\|T\phi - T\xi\|_E \ \le \ \psi(\|\phi - \xi\|_{E_0}, \|\phi - T\phi\|_E, \|\xi - T\xi\|_E, \|\phi - T\xi\|_E, \|\xi - T\phi\|_E)$$

*where $\psi \in \Psi_5$.*

*(d) If $\{\phi_n\}$ is a sequence in $E_0$ such that $\phi_n \to \phi$ as $n \to \infty$ and*

$$\alpha(\phi_n(c), T\phi_n) \ge \eta(\phi_n(c), T\phi_n)$$

*for all $n \in \mathbb{N}$, then $\alpha(\phi(c), T\phi) \ge 1$.*

*If there exists* $\phi_0 \in \mathcal{R}_c$ *such that*

$$\alpha(\phi_0(c), T\phi_0) \geq 1,$$

*then* $T$ *has a unique PPF dependent fixed point* $\phi^*$ *in* $\mathcal{R}_c$ *such that*

$$\alpha(\phi^*(c), T\phi^*) \geq 1.$$

*Moreover, for a fixed* $\phi_0 \in \mathcal{R}_c$ *such that*

$$\alpha(\phi_0(c), T\phi_0) \geq 1,$$

*if a sequence* $\{\phi_n\}$ *of iterates of* $T$ *in* $\mathcal{R}_c$ *defined by*

$$T\phi_{n-1} = \phi_n(c) \tag{3}$$

*for all* $n \in \mathbb{N}$*, then* $\{\phi_n\}$ *converges to a PPF dependent fixed point of* $T$ *in* $\mathcal{R}_c$*.*

*Proof.* Let $\phi_0$ be a point in $\mathcal{R}_c \subseteq E_0$ such that

$$\alpha(\phi_0(c), T\phi_0) \geq 1.$$

Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Choose $\phi_1 \in \mathcal{R}_c$ such that

$$x_1 = \phi_1(c).$$

Since $\phi_1 \in \mathcal{R}_c \subseteq E_0$ and by hypothesis, we get $T\phi_1 \in E$. This implies that there exists $x_2 \in E$ such that $T\phi_1 = x_2$. Thus, we can choose $\phi_2 \in \mathcal{R}_c$ such that

$$x_2 = \phi_2(c).$$

By continuing this process, by induction, we can construct the sequence $\{\phi_n\}$ in $\mathcal{R}_c \subseteq E_0$ such that

$$T\phi_{n-1} = \phi_n(c)$$

for all $n \in \mathbb{N}$. It follows from the fact that $\mathcal{R}_c$ is algebraically closed with respect to difference that

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \|\phi_{n-1}(c) - \phi_n(c)\|_E$$

for all $n \in \mathbb{N}$. Since $T$ is $\alpha_c\eta_c$- admissible and

$$\alpha(\phi_0(c), \phi_1(c)) = \alpha(\phi_0(c), T\phi_0) \geq 1$$

we deduce that

$$\alpha(\phi_1(c), T\phi_1) = \alpha(T\phi_0, T\phi_1) \geq 1$$

. By continuing this process, we get

$$\alpha(\phi_{n-1}(c), T\phi_{n-1}) \geq 1$$

for all $n \in \mathbb{N}$. Next, we show that $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|\phi_n - \phi_{n+1}\|_{E_0} &= \|\phi_n(c) - \phi_{n+1}(c)\|_E = \|T\phi_{n-1} - T\phi_n\|_E \\
&\leq \alpha(\phi_{n-1}(c), T\phi_{n-1})\alpha(\phi_n(c), T\phi_n)\|T\phi_{n-1} - T\phi_n\|_E \\
&\leq \psi(\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1} - T\phi_{n-1}\|_E, \|\phi_n - T\phi_n\|_E, \\
&\quad \|\phi_{n-1} - T\phi_n\|_E, \|\phi_n - T\phi_{n-1}\|_E) \\
&\leq \psi(\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}, \\
&\quad \|\phi_{n-1} - \phi_{n+1}\|_{E_0}, 0) \\
&\leq \psi(\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}, \\
&\quad \|\phi_{n-1} - \phi_n\|_{E_0} + \|\phi_n - \phi_{n+1}\|_{E_0}, 0) \\
&\leq k\|\phi_{n-1} - \phi_n\|_{E_0}.
\end{aligned}$$

By repeating the above relation, we get

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq k^n\|\phi_0 - \phi_1\|_{E_0}$$

for all $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ with $m > n$, we obtain that

$$\begin{aligned}
\|\phi_n - \phi_m\|_{E_0} &\leq \|\phi_n - \phi_{n+1}\|_{E_0} + \|\phi_{n+1} - \phi_{n+2}\|_{E_0} + \\
&\quad + \cdots + \|\phi_{m-1} - \phi_m\|_{E_0} \\
&\leq (k^n + k^{n+1} + \cdots + k^{m-1})\|\phi_0 - \phi_1\|_{E_0} \\
&\leq \frac{k^n}{1-k}\|\phi_0 - \phi_1\|_{E_0}.
\end{aligned}$$

This implies that the sequence $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c \subseteq E_0$. By the completeness of $E_0$, we get that $\{\phi_n\}$ converges to a limit point $\phi^* \in E_0$, that is, $\lim_{n\to\infty} \phi_n = \phi^*$. Since $\mathcal{R}_c$ is topologically closed, we have $\phi^* \in \mathcal{R}_c$. Now, we prove that $\phi^*$ is a PPF dependent fixed point of $T$. By (d) we have $\alpha(\phi^*(c), T\phi^*) \geq 1$. From assumption (c), we get

$$\begin{aligned}
\|T\phi^* - \phi^*(c)\|_E &\leq \|T\phi^* - \phi_n(c)\|_E + \|\phi_n(c) - \phi^*(c)\|_E \\
&= \|T\phi^* - T\phi_{n-1}\|_E + \|\phi_n - \phi^*\|_{E_0} \\
&\leq \alpha(\phi^*(c), T\phi^*)\alpha(\phi_{n-1}(c), T\phi_{n-1})\|T\phi^* - T\phi_{n-1}\|_E + \|\phi_n - \phi^*\|_{E_0} \\
&\leq \psi(\|\phi^* - \phi_{n-1}\|_{E_0}, \|\phi^* - T\phi^*\|_E, \|\phi_{n-1} - T\phi_{n-1}\|_E, \\
&\quad \|\phi^* - T\phi_{n-1}\|_E, \|\phi_{n-1} - T\phi^*\|_E) + \|\phi_n - \phi^*\|_{E_0}
\end{aligned}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the above inequality, we have

$$\|T\phi^* - \phi^*(c)\|_E \leq k\|T\phi^* - \phi^*(c)\|_E$$

which contradiction

$$\|T\phi^* - \phi^*(c)\|_E = 0 \tag{4}$$

and so

$$T\phi^* = \phi^*(c).$$

This implies that $\phi^*$ is a PPF dependent fixed point of $T$ in $\mathcal{R}_c$. Finally, we prove the uniqueness of a PPF dependent fixed point of $T$ in $\mathcal{R}_c$. Let $\phi^*$ and $\xi^*$ be two PPF dependent fixed points of $T$ in $\mathcal{R}_c$ such that $\alpha(\phi^*(c), T\phi^*) \geq 1$ and $\alpha(\xi^*(c), T\xi^*) \geq 1$. Now we obtain that

$$\begin{aligned}
\|\phi^* - \xi^*\|_{E_0} &= \|\phi^*(c) - \xi^*(c)\|_E \\
&= \|T\phi^* - T\xi^*\|_E \\
&\leq \alpha(\phi^*(c), T\phi^*)\alpha(\xi^*(c), T\xi^*)\|T\phi^* - T\xi^*\|_E \\
&\leq \psi(\|\phi^* - \xi^*\|_{E_0}, \|\phi^* - T\phi^*\|_E, \|\xi^* - T\xi^*\|_E, \\
&\quad \|\phi^* - T\xi^*\|_E, \|\xi^* - T\phi^*\|_E) \\
&\leq k\|\phi^* - \xi^*\|_{E_0}.
\end{aligned}$$

Since $0 \leq k < 1$, we get $\|\phi^* - \xi^*\|_{E_0} = 0$ and then $\phi^* = \xi^*$. Therefore, $T$ has a unique PPF dependent fixed point in $\mathcal{R}_c$. This completes the proof. □

**Theorem 2.14.** *Let $T : E_0 \to E$, $\alpha : E \times E \to [0, \infty)$ be two mappings satisfying the following conditions:*

*(a) There exists $c \in I$ such that $\mathcal{R}_c$ is topological closed and algebraically closed with respect to difference.*

*(b) $T$ is $\alpha_c-$admissible.*

*(c) For all $\phi, \xi \in E_0$,*

$$\begin{aligned}
(\|T\phi - T\xi\|_E + \epsilon)^{\alpha(\phi(c),T\phi)\alpha(\xi(c),T\xi)} &\leq \psi(\|\phi - \xi\|_{E_0}, \|\phi - T\phi\|_E, \|\xi - T\xi\|_E, \\
&\quad \|\phi - T\xi\|_E, \|\xi - T\phi\|_E) + \epsilon
\end{aligned}$$

*where $\epsilon \geq 1$.*

*(d) If $\{\phi_n\}$ is a sequence in $E_0$ such that $\phi_n \to \phi$ as $n \to \infty$ and*

$$\alpha(\phi_n(c), T\phi_n) \geq 1$$

*for all $n \in \mathbb{N}$, then*

$$\alpha(\phi(c), T\phi) \geq 1.$$

*If there exists $\phi_0 \in \mathcal{R}_c$ such that*

$$\alpha(\phi_0(c), T\phi_0) \geq 1,$$

*then $T$ has a unique PPF dependent fixed point $\phi^*$ in $\mathcal{R}_c$ such that*

$$\alpha(\phi^*(c), T\phi^*) \geq 1.$$

*Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$ such that*

$$\alpha(\phi_0(c), T\phi_0) \geq 1,$$

*if a sequence $\{\phi_n\}$ of iterates of $T$ in $\mathcal{R}_c$ defined by*

$$T\phi_{n-1} = \phi_n(c) \tag{5}$$

*for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of $T$ in $\mathcal{R}_c$.*

*Proof.* Let $\phi_0$ be a point in $\mathcal{R}_c \subseteq E_0$ such that

$$\alpha(\phi_0(c), T\phi_0) \geq 1.$$

Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Choose $\phi_1 \in \mathcal{R}_c$ such that

$$x_1 = \phi_1(c).$$

Since $\phi_1 \in \mathcal{R}_c \subseteq E_0$ and by hypothesis, we get $T\phi_1 \in E$. This implies that there exists $x_2 \in E$ such that $T\phi_1 = x_2$. Thus, we can choose $\phi_2 \in \mathcal{R}_c$ such that

$$x_2 = \phi_2(c).$$

By continuing this process, by induction, we can construct the sequence $\{\phi_n\}$ in $\mathcal{R}_c \subseteq E_0$ such that

$$T\phi_{n-1} = \phi_n(c)$$

for all $n \in \mathbb{N}$. It follows from the fact that $\mathcal{R}_c$ is algebraically closed with respect to difference that

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \|\phi_{n-1}(c) - \phi_n(c)\|_E$$

for all $n \in \mathbb{N}$. Since $T$ is $\alpha_c$- admissible and

$$\alpha(\phi_0(c), \phi_1(c)) = \alpha(\phi_0(c), T\phi_0) \geq 1,$$

we deduce that

$$\alpha(\phi_1(c), T\phi_1) = \alpha(T\phi_0, T\phi_1) \geq 1.$$

By continuing this process, we get

$$\alpha(\phi_{n-1}(c), T\phi_{n-1}) \geq 1$$

for all $n \in \mathbb{N}$. Next, we show that $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|\phi_n - \phi_{n+1}\|_{E_0} + \epsilon &= \|\phi_n(c) - \phi_{n+1}(c)\|_E + \epsilon = \|T\phi_{n-1} - T\phi_n\|_E + \epsilon \\
&\leq (\|T\phi_{n-1} - T\phi_n\|_E + \epsilon)^{\alpha(\phi_{n-1}(c), T\phi_{n-1})\alpha(\phi_n(c), T\phi_n)} \\
&\leq \psi(\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1} - T\phi_{n-1}\|_E, \|\phi_n - T\phi_n\|_E, \\
&\quad \|\phi_{n-1} - T\phi_n\|_E, \|\phi_n - T\phi_{n-1}\|_E) + \epsilon \\
&\leq \psi(\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}, \\
&\quad \|\phi_{n-1} - \phi_{n+1}\|_{E_0}, 0) + \epsilon \\
&\leq k\|\phi_{n-1} - \phi_n\|_{E_0} + \epsilon.
\end{aligned}$$

This implies that

$$\|\phi_n-\phi_{n+1}\|_{E_0}\leq k\|\phi_{n-1}-\phi_n\|_{E_0}$$

for all $n\in\mathbb{N}$. By repeating the above relation, we get

$$\|\phi_n-\phi_{n+1}\|_{E_0} \;\leq\; k^n\|\phi_0-\phi_1\|_{E_0}$$

for all $n\in\mathbb{N}$. For $m,n\in\mathbb{N}$ with $m>n$, we obtain that

$$\begin{aligned}\|\phi_n-\phi_m\|_{E_0} &\leq \|\phi_n-\phi_{n+1}\|_{E_0}+\|\phi_{n+1}-\phi_{n+2}\|_{E_0}+\\ &\quad +\cdots+\|\phi_{m-1}-\phi_m\|_{E_0}\\ &\leq (k^n+k^{n+1}+\cdots+k^{m-1})\|\phi_0-\phi_1\|_{E_0}\\ &\leq \frac{k^n}{1-k}\|\phi_0-\phi_1\|_{E_0}.\end{aligned}$$

This implies that the sequence $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c\subseteq E_0$. By the completeness of $E_0$, we get that $\{\phi_n\}$ converges to a limit point $\phi^*\in E_0$, that is, $\lim_{n\to\infty}\phi_n=\phi^*$. Since $\mathcal{R}_c$ is topologically closed, we have $\phi^*\in\mathcal{R}_c$. Now, we prove that $\phi^*$ is a PPF dependent fixed point of $T$. By (d) we have $\alpha(\phi^*(c),T\phi^*)\geq 1$. From assumption (c), we get

$$\begin{aligned}\|T\phi^*-\phi^*(c)\|_E+\epsilon &\leq \|T\phi^*-\phi_n(c)\|_E+\|\phi_n(c)-\phi^*(c)\|_E+\epsilon\\ &= \|T\phi^*-T\phi_{n-1}\|_E+\|\phi_n-\phi^*\|_{E_0}+\epsilon\\ &\leq (\|T\phi^*-T\phi_{n-1}\|_E+\epsilon)^{\alpha(\phi^*(c),T\phi^*)\alpha(\phi_{n-1}(c),T\phi_{n-1})}+\|\phi_n-\phi^*\|_{E_0}\\ &\leq \psi(\|\phi^*-\phi_{n-1}\|_{E_0},\|\phi^*-T\phi^*\|_E,\|\phi_{n-1}-\phi_n\|_{E_0},\\ &\quad \|\phi^*-T\phi_n\|_E,\|\phi_n-T\phi^*\|_E)+\|\phi_n-\phi^*\|_{E_0}+\epsilon\end{aligned}$$

for all $n\in\mathbb{N}$. Taking the limit as $n\to\infty$ in the above inequality, we have

$$\|T\phi^*-\phi^*(c)\|_E+\epsilon\leq k\|T\phi^*-\phi^*(c)\|_E+\epsilon \tag{6}$$

which contradiction, and so

$$\|T\phi^*-\phi^*(c)\|_E=0$$

and so

$$T\phi^*=\phi^*(c).$$

This implies that $\phi^*$ is a PPF dependent fixed point of $T$ in $\mathcal{R}_c$. Finally, we prove the uniqueness of a PPF dependent fixed point of $T$ in $\mathcal{R}_c$. Let $\phi^*$ and $\xi^*$ be two PPF dependent fixed points of $T$ in $\mathcal{R}_c$ such that $\alpha(\phi^*(c),T\phi^*)\geq 1$ and $\alpha(\xi^*(c),T\xi^*)\geq 1$. Now we obtain that

$$\begin{aligned}\|\phi^*-\xi^*\|_{E_0}+\epsilon &= \|\phi^*(c)-\xi^*(c)\|_E+\epsilon=\|T\phi^*-T\xi^*\|_E+\epsilon\\ &\leq (\|T\phi^*-T\xi^*\|_E+\epsilon)^{\alpha(\phi^*(c),T\phi^*)\alpha(\xi^*(c),T\xi^*)}\\ &\leq \psi(\|\phi^*-\xi^*\|_{E_0},\|\phi^*-T\phi^*\|_E,\|\xi^*-T\xi^*\|_E,\\ &\quad \|\phi^*-T\xi^*\|_E,\|\xi^*-T\phi^*\|_E)+\epsilon\\ &\leq k\|\phi^*-\xi^*\|_{E_0}+\epsilon.\end{aligned}$$

Since $0 \leq k < 1$, we get $\|\phi^* - \xi^*\|_{E_0} = 0$ and then $\phi^* = \xi^*$. Therefore, $T$ has a unique PPF dependent fixed point in $\mathcal{R}_c$. This completes the proof. □

**Theorem 2.15.** *Let $T : E_0 \to E$, $\alpha : E \times E \to [0, \infty)$ be two mappings satisfying the following conditions:*

*(a) There exists $c \in I$ such that $\mathcal{R}_c$ is topological closed and algebraically closed with respect to difference.*

*(b) $T$ is $\alpha_c-$admissible.*

*(c) For all $\phi, \xi \in E_0$,*

$$(\alpha(\phi(c), T\phi)\alpha(\xi(c), T\xi) - 1 + \epsilon')^{\|T\phi - T\xi\|_E} \leq \epsilon^{\psi(\|\phi-\xi\|_{E_0}, \|\phi-T\phi\|_E, \|\xi-T\xi\|_E, \|\phi-T\xi\|_E, \|\xi-T\phi\|_E)} \tag{7}$$

*where $1 < \epsilon \leq \epsilon'$.*

*(d) If $\{\phi_n\}$ is a sequence in $E_0$ such that $\phi_n \to \phi$ as $n \to \infty$ and*

$$\alpha(\phi_n(c), T\phi_n) \geq 1$$

*for all $n \in \mathbb{N}$, then*

$$\alpha(\phi(c), T\phi) \geq 1.$$

*If there exists $\phi_0 \in \mathcal{R}_c$ such that*

$$\alpha(\phi_0(c), T\phi_0) \geq 1,$$

*then $T$ has a unique PPF dependent fixed point $\phi^*$ in $\mathcal{R}_c$ such that*

$$\alpha(\phi^*(c), T\phi^*) \geq 1.$$

*Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$ such that*

$$\alpha(\phi_0(c), T\phi_0) \geq 1,$$

*if a sequence $\{\phi_n\}$ of iterates of $T$ in $\mathcal{R}_c$ defined by*

$$T\phi_{n-1} = \phi_n(c) \tag{8}$$

*for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of $T$ in $\mathcal{R}_c$.*

*Proof.* Let $\phi_0$ be a point in $\mathcal{R}_c \subseteq E_0$ such that

$$\alpha(\phi_0(c), T\phi_0) \geq 1.$$

Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Choose $\phi_1 \in \mathcal{R}_c$ such that

$$x_1 = \phi_1(c).$$

Since $\phi_1 \in \mathcal{R}_c \subseteq E_0$ and by hypothesis, we get $T\phi_1 \in E$. This implies that there exists $x_2 \in E$ such that $T\phi_1 = x_2$. Thus, we can choose $\phi_2 \in \mathcal{R}_c$ such that

$$x_2 = \phi_2(c).$$

By continuing this process, by induction, we can construct the sequence $\{\phi_n\}$ in $\mathcal{R}_c \subseteq E_0$ such that

$$T\phi_{n-1} = \phi_n(c)$$

for all $n \in \mathbb{N}$. It follows from the fact that $\mathcal{R}_c$ is algebraically closed with respect to difference that

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \|\phi_{n-1}(c) - \phi_n(c)\|_E$$

for all $n \in \mathbb{N}$. Since $T$ is $\alpha_c$- admissible and

$$\alpha(\phi_0(c), \phi_1(c)) = \alpha(\phi_0(c), T\phi_0) \geq 1,$$

we deduce that

$$\alpha(\phi_1(c), T\phi_1) = \alpha(T\phi_0, T\phi_1) \geq 1.$$

By continuing this process, we get

$$\alpha(\phi_{n-1}(c), T\phi_{n-1}) \geq 1$$

for all $n \in \mathbb{N}$. Next, we show that $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\epsilon^{\|\phi_n - \phi_{n+1}\|_{E_0}} &= \epsilon^{\|\phi_n(c) - \phi_{n+1}(c)\|_E} = \epsilon^{\|T\phi_{n-1} - T\phi_n\|_E} \\
&\leq (\alpha(\phi_{n-1}(c), T\phi_{n-1})\alpha(\phi_n(c), T\phi_n) - 1 + \epsilon')^{\|T\phi_{n-1} - T\phi_n\|_E} \\
&\leq (\alpha(\phi_{n-1}(c), T\phi_{n-1})\alpha(\phi_n(c), T\phi_n) - 1 + \epsilon')^{M(\phi_{n-1}, \phi_n)} \\
&\leq \epsilon^{k\|\phi_{n-1} - \phi_n\|_{E_0}}
\end{aligned}$$

where

$$M(\phi_{n-1}, \phi_n) = \psi(\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1} - T\phi_{n-1}\|_E, \|\phi_n - T\phi_n\|_E, \|\phi_{n-1} - T\phi_n\|_E, \|\phi_n - T\phi_{n-1}\|_E).$$

This implies that

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq k\|\phi_{n-1} - \phi_n\|_{E_0}$$

for all $n \in \mathbb{N}$. By repeating the above relation, we get

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq k^n\|\phi_0 - \phi_1\|_{E_0}$$

for all $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ with $m > n$, we obtain that

$$\begin{aligned}
\|\phi_n - \phi_m\|_{E_0} &\leq \|\phi_n - \phi_{n+1}\|_{E_0} + \|\phi_{n+1} - \phi_{n+2}\|_{E_0} + \cdots + \|\phi_{m-1} - \phi_m\|_{E_0} \\
&\leq (k^n + k^{n+1} + \cdots + k^{m-1})\|\phi_0 - \phi_1\|_{E_0} \\
&\leq \frac{k^n}{1-k}\|\phi_0 - \phi_1\|_{E_0}.
\end{aligned}$$

This implies that the sequence $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c \subseteq E_0$. By the completeness of $E_0$, we get that $\{\phi_n\}$ converges to a limit point $\phi^* \in E_0$, that is, $\lim_{n\to\infty} \phi_n = \phi^*$. Since $\mathcal{R}_c$ is topologically closed, we have $\phi^* \in \mathcal{R}_c$. Now, we prove that $\phi^*$ is a PPF dependent fixed point of $T$. By (d) we have $\alpha(\phi^*(c), T\phi^*) \geq 1$. From assumption (c), we get

$$\begin{aligned}
\epsilon^{\|T\phi^*-\phi^*(c)\|_E} &\leq \epsilon^{\|T\phi^*-\phi_n(c)\|_E+\|\phi_n(c)-\phi^*(c)\|_E} = \epsilon^{\|T\phi^*-T\phi_{n-1}\|_E+\|\phi_n-\phi^*\|_{E_0}}\\
&\leq \epsilon^{\|T\phi^*-T\phi_{n-1}\|_E}\epsilon^{\|\phi_n-\phi^*\|_{E_0}}\\
&\leq (\alpha(\phi^*(c), T\phi^*)\alpha(\phi_{n-1}(c), T\phi_{n-1}) - 1 + \epsilon')^{\|T\phi^*-T\phi_{n-1}\|_E}\epsilon^{\|\phi_n-\phi^*\|_{E_0}}\\
&\leq (\alpha(\phi^*(c), T\phi^*)\alpha(\phi_{n-1}(c), T\phi_{n-1}) - 1 + \epsilon')^{M(\phi^*,\phi_{n-1})}\epsilon^{\|\phi_n-\phi^*\|_{E_0}}\\
&\leq \epsilon^{k\|\phi^*-\phi_{n-1}\|_{E_0}}\epsilon^{\|\phi_n-\phi^*\|_{E_0}}\\
&\leq \epsilon^{k\|\phi^*-\phi_{n-1}\|_{E_0}+\|\phi_n-\phi^*\|_{E_0}}
\end{aligned}$$

where

$$M(\phi^*, \phi_{n-1}) = \psi(\|\phi^* - \phi_{n-1}\|_{E_0}, \|\phi^* - T\phi^*\|_E, \|\phi_{n-1} - T\phi_{n-1}\|_{E_0}, \|\phi^* - T\phi_{n-1}\|_{E_0}, \|\phi_{n-1} - T\phi^*\|_{E_0})$$

for all $n \in \mathbb{N}$. Since the exponential function is a real function, we can take the limit as $n \to \infty$ in the above inequality, we have

$$\|T\phi^* - \phi^*(c)\|_E = 0$$

and so

$$T\phi^* = \phi^*(c).$$

This implies that $\phi^*$ is a PPF dependent fixed point of $T$ in $\mathcal{R}_c$. Finally, we prove the uniqueness of a PPF dependent fixed point of $T$ in $\mathcal{R}_c$. Let $\phi^*$ and $\xi^*$ be two PPF dependent fixed points of $T$ in $\mathcal{R}_c$ such that $\alpha(\phi^*(c), T\phi^*) \geq 1$ and $\alpha(\xi^*(c), T\xi^*) \geq 1$. Now we obtain that

$$\begin{aligned}
\epsilon^{\|\phi^*-\xi^*\|_{E_0}} &= \epsilon^{\|\phi^*(c)-\xi^*(c)\|_E} = \epsilon^{\|T\phi^*-T\xi^*\|_E}\\
&\leq (\alpha(\phi^*(c), T\phi^*)\alpha(\xi^*(c), T\xi^*) - 1 + \epsilon')^{\|T\phi^*-T\xi^*\|_E}\\
&\leq (\alpha(\phi^*(c), T\phi^*)\alpha(\xi^*(c), T\xi^*) - 1 + \epsilon')^{M(\phi^*,\xi^*)}\\
&\leq \epsilon^{k\|\phi^*-\xi^*\|_{E_0}}.
\end{aligned}$$

$$M(\phi^*, \xi^*) = \psi(\|\phi^* - \xi^*\|_{E_0}, \|\phi^* - T\phi^*\|_E, \|\xi^* - T\xi^*\|_E, \|\phi^* - T\xi^*\|_E, \|\xi^* - T\phi^*\|_E)$$

Since $0 \leq k < 1$, we get $\|\phi^* - \xi^*\|_{E_0} = 0$ and then $\phi^* = \xi^*$. Therefore, $T$ has a unique PPF dependent fixed point in $\mathcal{R}_c$. This completes the proof. □

**Remark 2.16.** *If the Razumikhin class $\mathcal{R}_c$ is not topological closed, then the limit of the sequence $\{\phi_n\}$ in Theorem 2.12, Theorem 2.14 and Theorem 2.15 may be outside of $\mathcal{R}_c$, which may not be unique.*

# 3. PPF Dependent Coincidence Point Theorems

In this section, we discuss some relation between PPF dependent fixed point results and PPF dependent coincidence point results. First, we give the concept of PPF dependent coincidence point.

**Definition 3.1.** *Let $S : E_0 \to E_0$ and $T : E_0 \to E$, $\alpha : E \times E \to [0, \infty)$. We say that $(S, T)$ is an $\alpha_c-$ admissible pair if for $\phi, \xi \in E_0$,*

$$\alpha((S\phi)(c), (S\xi)(c)) \geq 1 \text{ implies } \alpha(T\phi, T\xi) \geq 1.$$

**Remark 3.2.** *It easy to see that if $(S, T)$ is an $\alpha_c-$admissible pair and $S$ is an identity mapping, then $T$ is also an $\alpha_c-$admissible mapping.*

Now, we indicate that Theorem 2.12 can be utilized to derive a PPF dependent coincidence point theorem.

**Theorem 3.3.** *Let $S : E_0 \to E_0$ and $T : E_0 \to E$, $\alpha : E \times E \to [0, \infty)$ be three mappings satisfying the following conditions:*

*(a) There exists $c \in I$ such that $\mathcal{R}_c$ is topological closed and algebraically closed with respect to difference.*

*(b) $(S, T)$ is $\alpha_c-$admissible.*

*(c) For all $\phi, \xi \in E_0$,*

$$\begin{aligned}\alpha((S\phi)(c), T\phi)\alpha((S\xi(c)), T\xi)\|T\phi - T\xi\|_E \ \leq\ & \psi(\|S\phi - S\xi\|_{E_0}, \|S\phi - T\phi\|_E, \\ & \|S\xi - T\xi\|_E, \|S\phi - T\xi\|_E, \|S\xi - T\phi\|_E)\end{aligned}$$

*where $\psi \in \Psi_5$.*

*(d) If $\{S\phi_n\}$ is a sequence in $E_0$ such that $S\phi_n \to S\phi$ as $n \to \infty$ and*

$$\alpha((S\phi_n)(c), T\phi_n) \geq 1$$

*for all $n \in \mathbb{N}$, then $\alpha((S\phi)(c), T\phi) \geq 1$.*

*(e) $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$.*

*If there exists $\phi_0 \in \mathcal{R}_c$ such that*

$$\alpha(\phi_0(c), T\phi_0) \geq 1,$$

*then $S$ and $T$ have a PPF dependent coincidence point $\omega$ in $\mathcal{R}_c$ such that*

$$\alpha((S\omega)(c), T\omega) \geq 1.$$

*Proof.* Consider the mapping $S : E_0 \to E_0$. We obtain that there exists $F_0 \subseteq E_0$ such that $S(T_0) = S(E_0)$ and $S|F_0$ is one-to-one. Since $T(F_0) \subseteq T(E_0) \subseteq E$, we can define a mapping $\mathcal{A} : S(F_0) \to E$ by

$$\mathcal{A}(S\phi) = T\phi \tag{9}$$

for all $\phi \in F_0$. Since $S|F_0$ is one-to-one, then $\mathcal{A}$ is well defined. From (9) and condition (c), we have

$$\begin{aligned}&\alpha((S\phi)(c), \mathcal{A}(S\phi))\alpha((S\xi(c)), \mathcal{A}(S\xi))\|\mathcal{A}(S\phi) - \mathcal{A}(S\xi)\|_E \\ \leq\ &\psi(\|S\phi - S\xi\|, \|S\phi - T\phi\|, \|S\xi - T\xi\|, \|S\phi - T\xi\|, \|S\xi - T\phi\|)\end{aligned}$$

for all $S\phi, S\xi \in S(E_0)$. This shows that $\mathcal{A}$ satisfies condition (c) of Theorem 8.

Now, we use Theorem-8 we a mapping $\mathcal{A}$, then there exists a unique PPF dependent fixed point $\varphi \in S(F_0)$ of $\mathcal{A}$, that is $\mathcal{A}\varphi = \varphi(c)$ and

$$\alpha(\varphi(c), \mathcal{A}\varphi) \geq 1.$$

Since $\varphi \in S(F_0)$, we can find $\omega \in F_0$ such that $\varphi = S\omega$. Therefore, we get

$$T\omega = \mathcal{A}(S\omega) = \mathcal{A}\varphi = \varphi(c) = (S\omega)(c)$$

and

$$\alpha((S\omega)(c), T\omega) = \alpha(\varphi(c), \mathcal{A}\varphi) \geq 1.$$

This implies that $\omega$ is a PPF dependent coincident point of $T$ and $S$. This completes the proof. □

Similarly, we can apply Theorem 2.14 and Theorem 2.15 to the Theorem 3.4 and Theorem 3.5. Then, in order to avoid repetition, then proof is omitted.

**Theorem 3.4.** *Let $S : E_0 \to E_0$ and $T : E_0 \to E$, $\alpha : E \times E \to [0, \infty)$ be three mappings satisfying the following conditions:*

*(a) There exists $c \in I$ such that $\mathcal{R}_c$ is topological closed and algebraically closed with respect to difference.*

*(b) $(S, T)$ is $\alpha_c-$admissible.*

*(c) For all $\phi, \xi \in E_0$,*

$$\begin{aligned}&(\|T\phi - T\xi\|_E + \epsilon)^{\alpha((S\phi)(c),T\phi)\alpha((S\xi(c)),T\xi)} \\ \leq\ &\psi(\|S\phi - S\xi\|_{E_0}, \|S\phi - T\phi\|_E, \|S\xi - T\xi\|_E, \|S\phi - T\xi\|_E, \|S\xi - T\phi\|_E) + \epsilon\end{aligned}$$

*where $\psi \in \Psi_5$ and $\epsilon \geq 1$.*

*(d) If $\{S\phi_n\}$ is a sequence in $E_0$ such that $S\phi_n \to S\phi$ as $n \to \infty$ and*

$$\alpha((S\phi_n)(c), T\phi_n) \geq 1$$

*for all $n \in \mathbb{N}$, then*

$$\alpha((S\phi)(c), T\phi) \geq 1.$$

*(e) $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$.*

*If there exists $\phi_0 \in \mathcal{R}_c$ such that*

$$\alpha(\phi_0(c), T\phi_0) \geq 1,$$

*then $S$ and $T$ have a PPF dependent coincidence point $\omega$ in $\mathcal{R}_c$ such that*

$$\alpha((S\omega)(c), T\omega) \geq 1.$$

**Theorem 3.5.** *Let $S : E_0 \to E_0$ and $T : E_0 \to E$, $\alpha : E \times E \to [0, \infty)$ be three mappings satisfying the following conditions:*

*(a) There exists $c \in I$ such that $\mathcal{R}_c$ is topological closed and algebraically closed with respect to difference.*

*(b) $(S, T)$ is $\alpha_c-$admissible.*

*(c) For all $\phi, \xi \in E_0$,*

$$(\alpha((S\phi)(c), T\phi)\alpha((S\xi(c)), T\xi) - 1 + \epsilon')^{\|T\phi - T\xi\|_E} \leq \epsilon^{M(\phi,\xi)}$$

*where*

$$M(\phi, \xi) = \psi(\|S\phi - S\xi\|_{E_0}, \|S\phi - T\phi\|_E, \|S\xi - T\xi\|_E, \|S\phi - T\xi\|_E, \|S\xi - T\phi\|_E),$$

*$\psi \in \Psi_5$ and $1 < \epsilon \leq \epsilon'$.*

*(d) If $\{S\phi_n\}$ is a sequence in $E_0$ such that $S\phi_n \to S\phi$ as $n \to \infty$ and*

$$\alpha((S\phi_n)(c), T\phi_n) \geq 1$$

*for all $n \in \mathbb{N}$, then $\alpha((S\phi)(c), T\phi) \geq 1$.*

*(e) $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$.*

*If there exists $\phi_0 \in \mathcal{R}_c$ such that*

$$\alpha(\phi_0(c), T\phi_0) \geq 1,$$

*then $S$ and $T$ have a PPF dependent coincidence point $\omega$ in $\mathcal{R}_c$ such that*

$$\alpha((S\omega)(c), T\omega) \geq 1.$$

# 4. Some Results in Banach Spaces Endowed with a Graph

Let $(E, d)$ be a metric space where $d(x, y) = \|x - y\|_E$ for all $x, y \in E$ and $\Delta$ denotes the diagonal of the Cartesian product of $X \times X$. Consider a directed graph $G$ such that the set $V(G)$ of its vertices coincides with $X$, and the set $E(G)$ of its edges contains all loops; that is $\Delta \subseteq E(G)$. We assume that $G$ has no parallel edges, so we can identify $G$ with the pair $(V(G), E(G))$. Moreover, we may treat $G$ as a weighted graph by assigning to each edge the distance between its vertices. If $x$ and $y$ are vertices in a graph $G$ , then a path in $G$ from $x$ to $y$ of length $N (N \in \mathbb{N})$ is a sequence $\{x_i\}_{i=0}^N$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \ldots, N$. A graph $G$ is connected if there is a path between any two vertices. $G$ is weakly connected if $\tilde{G}$ is connected.

**Theorem 4.1.** *Let $T : E_0 \to E$ and $E$ endowed with a graph $G$. Suppose that the following assertions holds true:*

*(a) there exists $c \in I$ such that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference;*

*(b) if $(\phi(c), \xi(c)) \in E(G)$ then $(T\phi, T\xi) \in E(G)$;*

*(c) assume that $(\phi(c), \xi(c)) \in E(G)$ implies*

$$\begin{aligned} &\alpha(\phi(c), T\phi)\alpha(\xi(c), T\xi)\|T\phi - T\xi\|_E \\ \leq\ &\psi(\|\phi - \xi\|_{E_0}, \|\phi - T\phi\|_E, \|\xi - T\xi\|_E, \|\phi - T\xi\|_E, \|\xi - T\phi\|_E) \end{aligned}$$

*where $\psi \in \Psi_5$.*

*(d) If $\{\phi_n\}$ is a sequence in $E_0$ such that $\phi_n \to \phi$ as $n \to \infty$ and*

$$(\phi_n(c), \phi_{n+1}(c)) \in E(G)$$

*for all $n \in \mathbb{N}$, then*

$$(\phi_n(c), \phi(c)) \in E(G)$$

*for all $n \in \mathbb{N}$;*

*(e) there exists $\phi_0 \in \mathcal{R}_c$ such that $(\phi_0(c), T\phi_0) \in E(G)$.*

*Then, $T$ has a PPF dependent fixed point.*

*Proof.* Define $\alpha : E \times E \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E \\ 0 & \text{otherwise} \end{cases}$$

First we prove that $T$ is an $\alpha_c-$admissible non-self mapping. Assume that

$$\alpha(\phi(c), \xi(c)) \geq 1.$$

Then, we have

$$(\phi(c), \xi(c)) \in E(G).$$

From (b), we have

$$(T\phi, T\xi) \in E(G),$$

that is

$$\alpha(T\phi, T\xi) \geq 1.$$

Thus $T$ is an $\alpha_c-$admissible. From (e) there exists $\phi_0 \in \mathcal{R}_c$ such that

$$\alpha(\phi_0(c), T\phi_0) \geq 1.$$

Let $\{\phi_n\}$ be a sequence in $E_0$ such that $\phi_n \to \phi$ as $n \to \infty$ and

$$(\phi_n(c), \phi_{n+1}(c)) \in E(G)$$

for all $n \in \mathbb{N}$. Then

$$\alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$$

for all $n \in \mathbb{N}$. Thus from (d) we get

$$(\phi_n(c), \phi) \in E(G)$$

for all $n \in \mathbb{N}$ that is

$$\alpha(\phi_n(c), \phi) \geq 1$$

for all $n \in \mathbb{N}$.

Therefore all condition of Theorem 2.13 hold true and $T$ has a PPF dependent fixed point. □

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