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On One Inequality for Two Hyperbolic Transformations

Research Article

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Abstract: We made the numerical experiment to compare two functions $\tau(t,s,x)$ and $t_1(t,s,x)$. The computed values for $\tau(t,s,x)$ are positive and for $t_1(t,s,x)$ are negative. Further we proved analytically this conjecture. Then it is proved that $\tau(t,s,x) > t_1(t,s,x)$.

Keywords: Teichmller space, Fuchsian group, Maskit's hyperbolic transformations.

1. Introduction

The geometrical characteristics associated with the fractionally linear transformations are used in article [2] to establish relations between Thurston's metric on Teichmüller space and the isomorphisms between Fuchsian groups. In the work [3] a pair of hyperbolic transformations (g,h) are divided into three classes by using the geometrical characteristics associated with these transformations.

Let $k_1 = k(g)$, $k_2 = k(h)$ are two factors of hyperbolic transformation g and h, respectively; $\tau = (r(g), r(h), a(h), a(g))$ be a cross-ratio of attracting and repelling fixed points of transformations g and h. Let $f(k) = \sqrt{k} + \frac{1}{\sqrt{k}}$ for $k > 0$. Let

$$t_1 = \frac{2 - f(\frac{k_1}{k_2})}{f(k_1 \cdot k_2) - f(\frac{k_1}{k_2})} \text{ and } t_2 = \frac{-2 - f(\frac{k_1}{k_2})}{f(k_1 \cdot k_2) - f(\frac{k_1}{k_2})}. \tag{1}$$

Then

$$(g,h) \in \mathcal{H} \Leftrightarrow \tau \geq t_1, \tag{2}$$

$$(g,h) \in \mathcal{P} \Leftrightarrow \tau \geq t_2, \quad (g,h) \in \varepsilon \Leftrightarrow t_2 \leq \tau \leq t_1.$$

In this paper, we investigate the question: a pair of hyperbolic elements A and B which introduced by Maskit [1] for the parameterization of the Teichmller space of Riemann surfaces of topological type (1, 1) belong to which of these classes ?. The hyperbolic transformations A and B are in the upper half $H^2 = \{z : Im\ z > 0\}$ given by the matrices:

$$A = \begin{pmatrix} tx & x \\ t & x \end{pmatrix}, \quad B = \begin{pmatrix} 0 & x \\ -s & x(1+s) \end{pmatrix},$$

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where $x > 1$, $t \geq s > 0$. Fuchsian group G, generated by the transformations A and B, determines the Riemann surface $S = H^2/G$, is a torus with one boundary component. When $t = s$ we have a torus with one puncture, and if $t > s$ we have a torus with one deleted disc. The values $k_1 = k(A)$, $k_2 = k(B)$, $\tau = (r(A), r(B), a(h), a(B)), a(A))$ and t_1 can be explicitly expressed in the parameters x, t, s. The calculations done by using Maple programm. To find τ we have:

```
>e:=proc(x,t,s)#Description of the procedure
>a:=((t*x-x-sqrt((x^2)*((t-1)^2)+4*t*x))/(2*t));
>b:=((t*x-x+sqrt((x^2)*((t-1)^2)+4*t*x))/(2*t));
>c:=((x*(1+s)+sqrt((x^2)*((1+s)^2)-4*x*s))/(2*s));
>d:=((x*(1+s)-sqrt((x^2)*((1+s)^2)-4*x*s))/(2*s));
>e:=(((c-b)*(a-d))/((c-d)*(a-b)));
>end;
Warning, ‘a’is implicitly declared local to procedure ‘e’
Warning, ‘b’is implicitly declared local to procedure ‘e’
Warning, ‘c’is implicitly declared local to procedure ‘e’
Warning, ‘d’is implicitly declared local to procedure ‘e’
Warning, ‘e’is implicitly declared local to procedure ‘e’
e:=proc(x, t, s)
local a, b, c, d, e;
a:=1/2×(t×x-x-sqrt(x^2×(t-1)^2+4×t×x))/t;
b:=1/2×(t×x-x+sqrt(x^2×(t-1)^2+4×t×x))/t;
c:=1/2×(x×(1+s)+sqrt(x^2×(t+1)^2-4×x×s))/s;
d:=1/2×(x×(1+s)-sqrt(x^2×(t+1)^2-4×x×s))/s;
e:=(c-b)×(a-d)/((c-d)×(a-b))
end proc
>e(2.,2,1);#Calling procedures 0.3418861168
>e(4.,12,4);#Calling procedures 0.01749178422
>e(15.,36,13);# Calling procedures 0.0006288957839
>e(378.,673,444);#Calling procedures 0.36813700310^-7
>e(39845.,45378,33);#Calling procedures 0.217404980110^-7
>e(9632.,1234,1233(;#Calling procedures 0.272910507310^-9
>>e(2.,2,0.002);#Calling procedures 0.7227119246
```

To find t_1 have:

```
>m:=proc(x,t,s)# Description of the procedure
>f:=(((x^2)*(t^2)+(x^2)+2*t*x+x*(t+1)*sqrt((x^2)*((t-1)^2)+4*t*x))/(2*t*x*(x-1)));
>g:=((2*(x^2)*((s+1)^2)+2*x*(s+1)*sqrt((x^2)*((s+1)^2)-4*x*s)-4*x*s)/(4*x*s));
>h:=((f)*g);
>i:=((f)/g);
>k:=((h+1)/sqrt(h));
>l:=((i+1)/sqrt(i));
>m:=((2-l)/(k-l));
>end;
```

```
Warning, 'f' is implicitly declared local to procedure 'm'
Warning, 'g' is implicitly declared local to procedure 'm'
Warning, 'h' is implicitly declared local to procedure 'm'
Warning, 'i' is implicitly declared local to procedure 'm'
Warning, 'k' is implicitly declared local to procedure 'm'
Warning, 'l' is implicitly declared local to procedure 'm'
Warning, 'm' is implicitly declared local to procedure 'm'
m:=proc(x, t, s)
local f, g, h, i, k, l, m;
f:=1/2×(x^2×t^2+x^2+2×t×x+x×(t+1)×sqrt(x^2×(t-1)^2+4×t×x))/(t×x×(x-1));
g:=1/4×(2×x^2×(s+1)^2+2×x×(s+1)×sqrt(x^2×(s+1)^2-4×x×s))/(x×s);
h:=f×g;
i:=f/g;
k:=(h+1)/sqrt(h);
l:=(i+1)/sqrt(i);
m:=(2-l)/(k-l)
end proc
>m(2.,2,1);# Calling procedures -0.001469702412
>m(4.,12,4);# Calling procedures -0.001430106612
>m(15.,36,13);# Calling procedures -0.009094777041
>m(378.,673,444);# Calling procedures -0.001302296832
>m(39845.,45378,33);# Calling procedures -0.00001480652040
>m(9632.,1234,1233);# Calling procedures -0,0007945098000
>m(2.,2,0.002);# Calling procedures -0,1438764752
```

Thus, the table of calculation results is as follows:

x	t	s	τ	t_1
2	2	1	0.341886168	-0.001469702
4	12	4	0.017491784	-0.001430107
15	36	13	0.000628896	-0.009094777
378	678	444	0.368137000	-0.001302297
39845	45378	33	$0.217404.10^{-7}$	-0.000014807
9632	1234	1233	$0.272910.10^{-9}$	-0.000794510
2	2	0.002	0.722711925	-0.143876475

The table shows that in all cases we have $\tau > t_1$, that's mean the pair (A, B) according to (2) belongs to $\mathcal{H}$. We shall prove that the same is true in the general case.

2. The Main Result

Theorem 2.1. *Any pair (A, B) of Maskit's hyperbolic transformations belong to the class $\mathcal{H}$.*

Proof. The axis of hyperbolic transformations A and B will intersect (see. [1]),

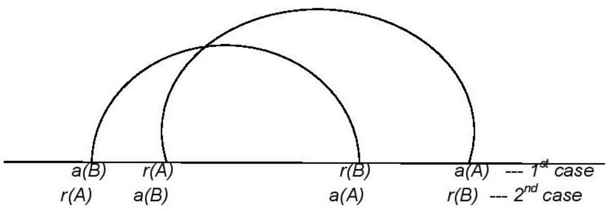
$$r(A) = \frac{x(t-1) - \sqrt{x^2(t-1)^2 + 4tx}}{2t}, \qquad a(A) = \frac{x(t-1) + \sqrt{x^2(t-1)^2 + 4tx}}{2t},$$
$$r(B) = \frac{x(s+1) - \sqrt{x^2(s+1)^2 - 4sx}}{2t}, \qquad a(B) = \frac{x(s+1) + \sqrt{x^2(s+1)^2 - 4sx}}{2t}.$$

It is possible to show by the geometric considerations. Axis transformations A and B are semicircle orthogonal to the real axis and ending at fixed points $r(A)$, $a(A)$ and $r(B)$, $a(B)$. Since, obviously, $r(A) < a(A)$ and $r(B) > a(B)$, then there are two cases:

All 4 numbers are real and in both cases, a fraction

$$\tau = (r(A), r(B), a(B), a(A)) = \frac{a(B) - r(A)}{a(B) - r(B)} : \frac{a(A) - r(A)}{a(A) - r(B)} > 0,$$

for arbitrary pairs (A, B).



On the other hand, the count gives

$$k_1 = \frac{x(t+1) + \sqrt{x^2(t-1)^2 + 4tx}}{x(t+1) - \sqrt{x^2(t-1)^2 + 4tx}} > 1,$$

Since $x(t+1) > \sqrt{x^2(t-1)^2 + 4tx}$ by the condition $x > 1$.

$$k_2 = \frac{x(s+1) + \sqrt{x^2(s+1)^2 - 4sx}}{x(s+1) - \sqrt{x^2(s+1)^2 - 4sx}} > 1.$$

Therefore, it's clear that from these inequalities and by (1), we obtain

$$t_1 = \frac{2 - \sqrt{\frac{k_1}{k_2}} - \sqrt{\frac{k_2}{k_1}}}{\sqrt{k_1 k_2} + \frac{1}{\sqrt{k_1 k_2}} - \sqrt{\frac{k_1}{k_2}} - \sqrt{\frac{k_2}{k_1}}} = \frac{2\sqrt{k_1 k_2} - k_1 - k_2}{k_1 k_2 + 1 - k_1 - k_2} = \frac{-(\sqrt{k_1} - \sqrt{k_2})^2}{(k_1 - 1)(k_2 - 1)} < 0.$$

Consequently, $\tau > 0 > t_1$ and according to (2), we conclude that $(A, B) \in \mathcal{H}$. □

References

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