International Journal of Mathematics And its Applications
Volume 4, Issue 1–A (2016), 63–68.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Generalization of Comparative Growth Properties of Entire Functions of Two Complex Variables Related to Relative Order

Research Article

Balram Prajapati[1] and Anupma Rastogi[1]*

1 Department of mathematics and Astronomy, University of Lucknow, Lucknow, Uttar Pradesh, India.

Abstract: In this paper we study some results related to comparative growth properties of entire functions of two complex variables on the basis of relative order and relative lower order of entire functions of two complex variables.

MSC: 32A15, 30D20.

Keywords: Entire and meromorphic function of two complex variable, relative order, composition, growth, asymptotic property, property (R).

1. Introduction, Definition and Notation

Let f be an entire function of two variables holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : |z_j| \leqslant r_j, j = 1, 2 \text{ for all } r_1 \geqslant 0, r_2 \geqslant 0\}$$

and

$$M_f(r_1, r_2) = \max\{|(z_1, z_2)| : |z_j| \leqslant r_j, j = 1, 2\}$$

by maximum principal and Hartogs theorem in [3], $M_f(r_1, r_2)$ is an increasing functions of r_1, r_2 see all standard definition and notation of the theory of entire functions in [3]. If f and g are two entire functions are said to be asymptotically equivalent if there exists k $(0 < k < \infty)$ such that

$$\frac{M_f(r_1, r_2)}{M_g(r_1, r_2)} \to k \text{ as } r_1, r_2 \to \infty$$

and in that case we can write $f \sim g$ clearly if $f \sim g$ then $g \sim f$. The following definition is well known:

Definition 1.1. *In [1, 3] the order ${}_{v_2}\rho_f$ and the lower order ${}_{v_2}\lambda_f$ of an entire function f of two variables are defined as*

$${}_{v_2}\rho_f = \lim_{r_1, r_2 \to \infty} \sup \frac{\log^{[2]} M_f(r_1, r_2)}{\log r_1 r_2}$$

* E-mail: anupmarastogi13121993@gmail.com

and

$$_{v_2}\lambda_f = \lim_{r_1,r_2\to\infty} \sup \frac{\log^{[2]} M_f(r_1,r_2)}{\log r_1 r_2}$$

If $_{v_2}\rho_f = {}_{v_2}\lambda_f$ *then function* f *is said to be of regular growth if* $_{v_2}\rho_f \neq {}_{v_2}\lambda_f$ *then function is irregular. In [2] Banerjee and Datta introduced the notion of relative order between two entire functions of two variables denoted by* $_{v_2}\rho_g(f)$ *as follows*

$$_{v_2}\rho_g(f) = \inf\{\mu : M_f(r_1,r_2) < M_g(r_1^{\mu}, r_2^{\mu}); r_1 \geqslant R(\mu), r_2 \geqslant R(\mu)\}$$

$$= \lim_{r_1,r_2\to\infty} \sup \frac{\log M_g^{-1} M_f(r_1,r_2)}{\log r_1 r_2}$$

similarly the relative lower order $_{v_2}\lambda_g(f)$ *is defined as*

$$_{v_2}\lambda_g(f) = \lim_{r_1,r_2\to\infty} \inf \frac{\log M_g^{-1} M_f(r_1,r_2)}{\log r_1 r_2}$$

In case of a meromorphic function this generalization was due to Banerjee and Lahiri [8]. They introduced the notion of relative order $_{v_2}\rho_g(f)$ *of* f *wih respect to* g *where* f *is meromorphic as follows*

$$_{v_2}\rho_g(f) = \inf\{\mu : T_f(r_1,r_2) < T_g(r_1^{\mu}, r_2^{\mu});\ \textit{for all } r_1 \geqslant 0, r_2 \geqslant 0\}$$

$$= \lim_{r_1,r_2\to\infty} \sup \frac{\log T_g^{-1} T_f(r_1,r_2)}{\log r_1 r_2}$$

similarly the relative lower order $_{v_2}\lambda_g(f)$ *of* f *with respect to* g *is defined by*

$$_{v_2}\lambda_g(f) = \lim_{r_1,r_2\to\infty} \inf \frac{\log T_g^{-1} T_f(r_1,r_2)}{\log r_1 r_2}$$

The p-th relative order $_{v_2}\rho_g^{[p]}(f)$ *and p-th relative lower order*$_{v_2}\lambda_g^{[p]}(f)$ *of* f *with respect to* g *where* f *is meromorphic defined as*

$$_{v_2}\rho_g^{[p]}(f) = \lim_{r_1,r_2\to\infty} \sup \frac{\log^{[p]} T_g^{-1} T_f(r_1,r_2)}{\log r_1 r_2}$$

and

$$_{v_2}\lambda_g^{[p]}(f) = \lim_{r_1,r_2\to\infty} \inf \frac{\log^{[p]} T_g^{-1} T_f(r_1,r_2)}{\log r_1 r_2}$$

where $\log^{[p]} z = \log(\log^{[p-1]} z)$ *for* $p = 1, 2, 3, ...$ *and* $\log^{[0]} z = z$*. In this paper we study some results related to comparative growth properties of entire functions of two variables on the basis of relative order and relative lower order of entire function of two variables.*

2. Lemma

Lemma 2.1. *If* g_1 *and* g_2 *be two entire functions of two variable with property (R) such that* $g_1 \sim g_2$*.If* f *be meromorphic then*

$$_{v_2}\rho_{g_1}^{[p]}(f) = {}_{v_2}\rho_{g_2}^{[p]}(f).$$

Lemma 2.2 ([7])**.** *If* f*,* g *be two meromorphic of two variable function* f *and* g *is of regular growth.Then*

$$_{v_2}\rho_g^{[p]}(f) = \frac{_{v_2}\rho_f^{[p]}}{_{v_2}\rho_g^{[p]}}.$$

3. Theorem

In this section we prove main results of this paper.

Theorem 3.1. *Let f be meromorphic and g, h be entire functions of two variable with non-zero finite orders.Then*

$$\lim_{r_1,r_2\to\infty}\inf\frac{\log^{[p]} T_g^{-1}T_f(r_1,r_2)}{\log^{[q]} T_h^{-1}T_f(r_1,r_2)}\leqslant\frac{{}_{v_2}\rho_g^{[p]}(f)}{{}_{v_2}\rho_h^{[q]}(f)}\leqslant\lim_{r_1,r_2\to\infty}\sup\frac{\log^{[p]} T_g^{-1}T_f(r_1,r_2)}{\log^{[q]} T_h^{-1}T_f(r_1,r_2)}.$$

Proof. From the definition of relative order we get for arbitrary $\epsilon(>0)$ and for all sufficiently large values of r_1, r_2

$$\log^{[p]} T_g^{-1}T_f(r_1,r_2)<({}_{v_2}\rho_g^{[p]}(f)+\epsilon)\log r_1r_2 \tag{1}$$

also for a sequence of values of r_1, r_2 tending to infinity we get,

$$\log^{[p]} T_g^{-1}T_f(r_1,r_2)>({}_{v_2}\rho_g^{[p]}(f)-\epsilon)\log r_1r_2 \tag{2}$$

again for arbitrary $\epsilon(>0)$ and for all sufficiently large values of r_1, r_2 we get

$$\log^{[q]} T_h^{-1}T_f(r_1,r_2)<({}_{v_2}\rho_h^{[q]}(f)+\epsilon)\log r_1r_2 \tag{3}$$

and for a sequence of values of r_1, r_2 tending to infinity we obtain that

$$\log^{[q]} T_h^{-1}T_f(r_1,r_2)>({}_{v_2}\rho_h^{[q]}(f)-\epsilon)\log r_1r_2 \tag{4}$$

now,from equation (1) and (4) we get for a sequence of values of r_1, r_2 tending to infinity,

$$\frac{\log^{[p]} T_g^{-1}T_f(r_1,r_2)}{\log^{[q]} T_h^{-1}T_f(r_1,r_2)}<\frac{{}_{v_2}\rho_g^{[p]}(f)+\epsilon}{{}_{v_2}\rho_h^{[q]}(f)-\epsilon}$$

as $\epsilon(>0)$ is arbitrary it follows that

$$\lim_{r_1,r_2\to\infty}\inf\frac{\log^{[p]} T_g^{-1}T_f(r_1,r_2)}{\log^{[q]} T_h^{-1}T_f(r_1,r_2)}\leqslant\frac{{}_{v_2}\rho_g^{[p]}(f)}{{}_{v_2}\rho_h^{[q]}(f)} \tag{5}$$

also from (2) and (3) for sequence r_1, r_2 tending to infinity

$$\frac{\log^{[p]} T_g^{-1}T_f(r_1,r_2)}{\log^{[q]} T_h^{-1}T_f(r_1,r_2)}>\frac{{}_{v_2}\rho_g^{[p]}(f)-\epsilon}{{}_{v_2}\rho_h^{[q]}(f)+\epsilon}$$

as $\epsilon(>0)$ is arbitrary it follows that

$$\lim_{r_1,r_2\to\infty}\sup\frac{\log^{[p]} T_g^{-1}T_f(r_1,r_2)}{\log^{[q]} T_h^{-1}T_f(r_1,r_2)}\geqslant\frac{{}_{v_2}\rho_g^{[p]}(f)}{{}_{v_2}\rho_h^{[q]}(f)} \tag{6}$$

thus from (5) and (6) we get completes the proof. □

Corollary 3.2. *If g and h are of regular growths then using Lemma 2.2 we get from theorem 1 that*

$$\lim_{r_1,r_2\to\infty}\inf\frac{\log^{[p]} T_g^{-1}T_f(r_1,r_2)}{\log^{[q]} T_h^{-1}T_f(r_1,r_2)}\leqslant\frac{{}_{v_2}\rho_h^{[p]}}{{}_{v_2}\rho_g^{[q]}}\leqslant\lim_{r_1,r_2\to\infty}\sup\frac{\log^{[p]} T_g^{-1}T_f(r_1,r_2)}{\log^{[q]} T_h^{-1}T_f(r_1,r_2)}$$

Corollary 3.3. *If g and h are of regular growths and $g \sim h$ then using Lemma 2.1 we get from Theorem 3.1 that*

$$\lim_{r_1,r_2\to\infty} \inf \frac{\log^{[p]} T_g^{-1} T_f(r_1,r_2)}{\log^{[q]} T_h^{-1} T_f(r_1,r_2)} \leqslant 1 \leqslant \lim_{r_1,r_2\to\infty} \sup \frac{\log^{[p]} T_g^{-1} T_f(r_1,r_2)}{\log^{[q]} T_h^{-1} T_f(r_1,r_2)}$$

converse of Corollary 3.2 is not always true we see in example.

Example 3.4. *Let $g(z_1,z_2) = \exp z_1z_2$, $h(z) = \exp 2z_1z_2$ so that $M_g(r_1,r_2) = \exp r_1r_2$ and $M_h(r_1,r_2) = \exp 2r_1r_2$. Now*

$$\frac{M_g(r_1,r_2)}{M_h(r_1,r_2)} \to 0 \text{ as } r_1,r_2 \to \infty$$

and so $g_1 \sim g_2$ also

$$T_g(r_1,r_2) = \frac{r_1r_2}{\pi} \text{ and } T_h(r_1,r_2) = \frac{r_1r_2}{\pi}$$

and therefore

$$T_g^{-1}(r_1,r_2) = \pi r_1r_2 \text{ and } T_h^{-1}(r_1,r_2) = \frac{\pi r_1r_2}{2}$$

but

$$\lim_{r_1,r_2\to\infty} \frac{\log^{[p]} T_g^{-1} T_f(r_1,r_2)}{\log^{[q]} T_h^{-1} T_f(r_1,r_2)} = \lim_{r_1,r_2\to\infty} \frac{\log^{[p]} \pi T_f(r_1,r_2)}{\log^{[q]} \frac{\pi}{2} T_f(r_1,r_2)} = 1$$

Theorem 3.5. *Let f, h be meromorphic and g be entire functions of two variable with non zero finite order.Then*

$$\lim_{r_1,r_2\to\infty} \inf \frac{\log^{[p]} T_g^{-1} T_f(r_1,r_2)}{\log^{[q]} T_g^{-1} T_h(r_1,r_2)} \leqslant \frac{{}_{v_2}\rho_g^{[p]}(f)}{{}_{v_2}\rho_g^{[q]}(h)} \leqslant \lim_{r_1,r_2\to\infty} \sup \frac{\log^{[p]} T_g^{-1} T_f(r_1,r_2)}{\log^{[q]} T_g^{-1} T_h(r_1,r_2)}$$

Proof. From the definition of relative order we get for arbitrary $\epsilon(>0)$ and for all sufficiently large values of r_1, r_2 that

$$\log^{[q]} T_g^{-1} T_h(r_1,r_2) < ({}_{v_2}\rho_g^{[q]}(h) + \epsilon) \log r_1r_2 \tag{7}$$

and for a sequence of values of r_1, r_2 tending to infinity we obtain that

$$\log^{[q]} T_g^{-1} T_h(r_1,r_2) > ({}_{v_2}\rho_g^{[q]}(h) - \epsilon) \log r_1r_2 \tag{8}$$

now from (1) and (8) we get for a sequence of values of r_1, r_2 tending to infinity that

$$\frac{\log^{[p]} T_g^{-1} T_f(r_1,r_2)}{\log^{[q]} T_g^{-1} T_h(r_1,r_2)} < \frac{{}_{v_2}\rho_g^{[p]}(f) + \epsilon}{{}_{v_2}\rho_g^{[q]}(h) - \epsilon}$$

as $\epsilon(>0)$ is arbitrary it follows that

$$\lim_{r_1,r_2\to\infty} \inf \frac{\log^{[p]} T_g^{-1} T_f(r_1,r_2)}{\log^{[q]} T_g^{-1} T_h(r_1,r_2)} \leqslant \frac{{}_{v_2}\rho_g^{[p]}(f)}{{}_{v_2}\rho_g^{[q]}(h)} \tag{9}$$

also from (2) and (7) we get for a sequence of values of r_1, r_2 tending to infinity,

$$\frac{\log^{[p]} T_g^{-1} T_f(r_1,r_2)}{\log^{[q]} T_g^{-1} T_h(r_1,r_2)} > \frac{{}_{v_2}\rho_g^{[p]}(f) - \epsilon}{{}_{v_2}\rho_g^{[q]}(h) + \epsilon}$$

as $\epsilon(>0)$ is arbitrary it follows that

$$\lim_{r_1,r_2\to\infty} \sup \frac{\log^{[p]} T_g^{-1} T_f(r_1,r_2)}{\log^{[q]} T_g^{-1} T_h(r_1,r_2)} \geqslant \frac{{}_{v_2}\rho_g^{[p]}(f)}{{}_{v_2}\rho_g^{[q]}(h)} \tag{10}$$

from (9) and (10) we obtain we obtain Theorem 3.5 this completes the proof. □

Corollary 3.6. *If g and h of regular growths then using Lemma 2.2 we get Theorem 3.5 that*

$$\lim_{r_1,r_2\to\infty}\inf\frac{\log^{[p]}T_g^{-1}T_f(r_1,r_2)}{\log^{[q]}T_g^{-1}T_h(r_1,r_2)}\leqslant\frac{{}_{v_2}\rho_f^{[p]}}{{}_{v_2}\rho_h^{[q]}}\leqslant\lim_{r_1,r_2\to\infty}\sup\frac{\log^{[p]}T_g^{-1}T_f(r_1,r_2)}{\log^{[q]}T_g^{-1}T_h(r_1,r_2)}.$$

Theorem 3.7. *Let f, h be meromorphic and g, k be entire functions of two variable with non-zero finite orders then*

$$\lim_{r_1,r_2\to\infty}\inf\frac{\log^{[p]}T_g^{-1}T_f(r_1,r_2)}{\log^{[q]}T_k^{-1}T_h(r_1,r_2)}\leqslant\frac{{}_{v_2}\rho_g^{[p]}(f)}{{}_{v_2}\rho_k^{[q]}(h)}\leqslant\lim_{r_1,r_2\to\infty}\sup\frac{\log^{[p]}T_g^{-1}T_f(r_1,r_2)}{\log^{[q]}T_k^{-1}T_h(r_1,r_2)}.$$

Proof. From the definition of relative order we get for arbitrary $\epsilon(>0)$ and for all sufficiently large values of r_1, r_2 that

$$\log^{[q]}T_k^{-1}T_h(r_1,r_2)<({}_{v_2}\rho_k^{[q]}(h)+\epsilon)\log r_1r_2 \tag{11}$$

also for a sequence of values of r_1, r_2 tending to infinity,

$$\log^{[q]}T_k^{-1}T_h(r_1,r_2)>({}_{v_2}\rho_k^{[q]}(h)-\epsilon)\log r_1r_2 \tag{12}$$

from equation (1) and (12) we get for a sequence of values of $r_1, r_2 \to \infty$

$$\frac{\log^{[p]}T_g^{-1}T_f(r_1,r_2)}{\log^{[q]}T_k^{-1}T_h(r_1,r_2)}<\frac{{}_{v_2}\rho_g^{[p]}(f)+\epsilon}{{}_{v_2}\rho_k^{[q]}(h)-\epsilon}$$

as $\epsilon(>0)$ is arbitrary it follows that

$$\lim_{r_1,r_2\to\infty}\inf\frac{\log^{[p]}T_g^{-1}T_f(r_1,r_2)}{\log^{[q]}T_k^{-1}T_h(r_1,r_2)}\leqslant\frac{{}_{v_2}\rho_g^{[p]}(f)}{{}_{v_2}\rho_k^{[q]}(h)} \tag{13}$$

now from (2) and (11) we get for a sequence of values of r_1, r_2 tending to infinity that

$$\frac{\log^{[p]}T_g^{-1}T_f(r_1,r_2)}{\log^{[q]}T_k^{-1}T_h(r_1,r_2)}>\frac{{}_{v_2}\rho_g^{[p]}(f)-\epsilon}{{}_{v_2}\rho_k^{[q]}(h)+\epsilon}$$

as $\epsilon(>0)$ is arbitrary then

$$\lim_{r_1,r_2\to\infty}\inf\frac{\log^{[p]}T_g^{-1}T_f(r_1,r_2)}{\log^{[q]}T_k^{-1}T_h(r_1,r_2)}\geqslant\frac{{}_{v_2}\rho_g^{[p]}(f)}{{}_{v_2}\rho_k^{[q]}(h)} \tag{14}$$

from (13) and (14) we obtain Theorem 3.7 this is the complete proof . □

Acknowledgement

The authors are thankful to the referee for valuable suggestions towards the improvement of the paper.

References

[1] A.K.Agarwal, *On the properties of entire function of two complex variables*, Canadian Journal of Mathematics, 20(1968), 51-57.

[2] D.Banerjee and R.K.Dutta, *Relative order of entire functions of two complex variables*, International J. of math.Sci. and Engg. Appls.(IJMSEA), 1(1)(2007), 141-154.

[3] A.B.Fuks, *Theory of Analytic functions of several complex variables Moscow*, (1963).

[4] L.Bernal, *Orden relation decremiento de functiones enteras*, collectanea mathematica, 39(1988), 209-229.

[5] S.K.Datta and T.Biswas, *Relative order of composite entire functions and some related growth properties*, Bull. Cal. Math. Soc., 102(2010), 259-266.

[6] S.K.Datta and A.Biswas, *Estimation of relative order of entire and meromorphic functions in terms of slowly changing functions*, Int. J. Contemp. Math. Sciences, 6(2011), 1175-1186.

[7] S.K.Datta and A.Biswas, *A note on relative order of entire and meromorphic functions*, International J. Math. Sci. and Engg. Appls., 6(2012), 413-421.

[8] B.K.Lahiri and D.Banerjee, *Relative order of entire and meromorphic functions*, Proc. Nat. Acad. Sci., 69(1999), 339-354.