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M-Projective Curvature Tensor of a Semi-symmetric Metric Connection in a Kenmotsu Manifold

Research Article

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Abstract: In the present paper we consider a semi-symmetric metric connection in a Kenmotsu manifold. We deduce the relation between the Riemannian connection and the semi-symmetric metric connection on a Kenmotsu manifold.We investigate the curvature tensor and the Ricci tensor of a Kenmotsu manifold with respect to the semi-symmetric metric connection. We study M-projective curvature tensor with respect to the semi-symmetric metric connection satisfying certain curvature conditions.

MSC: 53C15, 53C25, 53D10

Keywords: Kenmotsu manifold, M-projective curvature tensor, η-Einstein manifold, Semi-symmetric metric connection.

1. Introduction

The notion of Kenmotsu manifolds was defined and studied by Kenmotsu [13] in 1972. They set up one of the three classes of almost contact metric manifolds M whose automorphism group attains the maximum dimension [17]. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c.

(i) If $c > 0$, M is a homogeneous Sasakian manifold of constant φ-sectional curvature. (ii) If $c = 0$, M is global Riemannian product of a line or a circle with a Kahler manifold of constant holomorphic sectional curvature. (iii) If $c < 0$, M is a warped product space $R \times_f C^n$. Kenmotsu [13] characterized the differential geometric properties of manifolds of class. (iv) The structure so obtained is now known as Kenmotsu structure. A Kenmotsu structure is not Sasakian.

In 1924, Friedman and Schouten [9] introduced the notion of semi-symmetric linear connection on a differentiable manifold. Then in 1932, Hayden [10] introduced the idea of metric connection with a torsion on a Riemannain manifold. In 1970, A systematic study of semi-symmetric metric connection on a Riemannian manifold has been given by Yano and later studied by K. Amur and S.S.Pujar [1], M. Prvanovic [15], U.C. De and S.C. Biswas [7], A. Sharfuddin and S.I. Hussain [16], T.Q. Binh [2], F. O Zengin and S.A. Uysal and S.A. Demirbag [24], S.K. Chaubey and R.H. Ojha ([5, 6]), H.B. Yilmaz [21] and others.

Let M be an n-dimensional Riemannian manifold of class C^∞ endowed with the Riemannian metric g and ∇ be the Levi-Civita connection on (M^n, g). A linear connection $\tilde{\nabla}$ defined on (M^n, g) is said to be semi-symmetric [9] if its torsion tensor T is of the form

$$T(X, Y) = \eta(Y)X - \eta(X)Y \tag{1}$$

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where η is a 1-form and ξ is a vector field given by

$$\eta(X) = g(X, \xi) \tag{2}$$

for all vector fields $X \in \chi(M^n), \chi(M^n)$ is the set of all differentiable vector fields on M^n. A semi-symmetric connection $\tilde{\nabla}$ is called a semi-symmetric metric connection [10] if it further satisfies

$$\tilde{\nabla} g = 0 \tag{3}$$

A relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ on (M^n, g) has been obtained by K.Yano [18] which is given by

$$\check{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi \tag{4}$$

we also have

$$(\check{\nabla}_X \eta)Y = (\nabla_X \eta)Y - \eta(Y)\eta(X) + \eta(\xi)g(X, Y) \tag{5}$$

Further, a relation between the curvature tensor R of the semi-symmetric metric connection $\tilde{\nabla}$ and the curvature tensor K of the Levi-Civita connection ∇ is given by

$$R(X, Y)Z = K(X, Y)Z + \alpha(X, Z)Y - \alpha(Y, Z)X + g(X, Z)Y - g(Y, Z)X \tag{6}$$

where α is a tensor field of type $(0, 2)$ and Q is a tensor field of type $(1, 1)$ which is given by

$$\alpha(Y, Z) = g(QY, Z) = (\nabla_Y \eta)Z - \eta(Y)\eta(Z) + \left(\frac{1}{2}\right)\eta(\xi)g(Y, Z) \tag{7}$$

from (6) and (7), we obtain

$$'R(X, Y, Z, W) =' K(X, Y, Z, W) + \alpha(X, Z)g(Y, W) - \alpha(Y, Z)g(X, W) + g(X, Z)\alpha(Y, W) - g(Y, Z)\alpha(X, W) \tag{8}$$

Where

$$'R(X, Y, Z, W) = g(R(X, Y)Z, W),$$
$$'K(X, Y, Z, W) = g(K(X, Y)Z, W) \tag{9}$$

In an almost contact manifold M, the M-projective curvature tensor P with respect to semi-symmetric metric connection $\tilde{\nabla}$ is given by

$$P(X, Y)Z = R(X, Y)Z - \left(\frac{1}{4n}\right) S(Y, Z)X - S(X, Z) + g(Y, Z)QX - g(X, Z)QY \tag{10}$$

for $X, Y, Z \in \chi(M)$, where R,S and Q are the Riemannian curvature tensor, Ricci tensor and the Ricci operator with respect to the connection $\tilde{\nabla}$, respectively. From (10), it follows that

$$'P(X, Y, Z, W) = R(X, Y, Z, W) - \left(\frac{1}{4n}\right) [S(Y, Z)g(X, W) - S(X, Z)g(Y, W) + g(Y, Z)S(X, W) - g(X, Z)S(Y, W)] \tag{11}$$

And

$$'P(X, Y, Z, W) = g(P(X, Y)Z, W) \tag{12}$$

for all vector fields X, Y, Z on M. Where S is the Ricci tensor with respect to the Semi-symmetric metric connection.

In the present paper, we study M-projective curvature tensor on a Kenmotsu manifold with respect to the semi-symmetric metric connection. The organization of this paper is as follows:

First section contains basic concepts of semi-symmetric metric connection. In section 2, we give a brief account of the Kenmotsu manifolds and we also give curvature tensor and Ricci tensor of a Kenmotsu manifold with respect to the semi-symmetric metric connection. In section 3, we study the M-projectively flat Kenmotsu manifold with respect to the semi-symmetric metric connection and proved that it is Einstein manifold. Also, an example for M-projectively flat Kenmotsu manifold with respect to the semi-symmetric metric connection is given. Further, we shown quasi-M-projectively flat Kenmotsu manifold with respect to the semi-symmetric metric connection and we shown that the manifold is an η-Einstein manifold. In section 4, we investigate ξ-M-projectively flat Kenmotsu manifold with respect to the semi-symmetric metric connection. Section 4 is devoted to the studyof ϕ-M-projectively flat Kenmotsu manifold with respect to the semi-symmetric metric connection. In the last section 5, we investigate $P.S = 0$, in a Kenmotsu manifold with respect to the semi-symmetric metric connection.

2. Kenmotsu Manifolds

Let M be an $(2n+1-$dimensional almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ , a $1-$form η and a Riemannian metric g on M satisfying [3]

$$\varphi^2(X) = -X + \eta(X)\xi, \;\; g(X,\xi) = \eta(X) \tag{13}$$

$$\eta(\xi) = 1, \;\; \varphi(\xi) = 0, \;\; \eta(\varphi(X)) = 0 \tag{14}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{15}$$

for all vector fields X, Y on M. If an almost contact metric manifold satisfies

$$(\nabla_X \varphi)(Y) = g(\varphi X, Y)\xi - \eta(Y)\phi X \tag{16}$$

then M is called a Kenmotsu manifold [13]. From the above relations, it follows that

$$\nabla_X \xi = X - \eta(X)\xi \tag{17}$$

$$(\nabla_X \eta)(Y) = g(X,Y)\xi - \eta(X)\eta(Y) \tag{18}$$

Moreover the curvature tensor K and the Ricci tensor S of the Kenmotsu manifold with respect to the Levi-Civita connection satisfies

$$K(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{19}$$

$$K(\xi,Y)X = \eta(X)Y - g(X,Y)\xi, \tag{20}$$

$$K(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{21}$$

$$\tilde{S}(\varphi X, \varphi Y) = \tilde{S}(X,Y) + 2n\eta(X)\eta(Y), \tag{22}$$

$$S(X,\xi) = -2n\eta(X) \tag{23}$$

we state the following lemma which will be used in the next section:

Lemma 2.1 ([13])**.** *Let M be an $\eta-$Einstein Kenmotsu manifold of the form $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$. If $b =$constant (or,$a =$constant), then M is an Einstein manifold.*

3. M-projectively Flat and Quasi–M–Projectively Flat Kenmotsu Manifolds with Respect to the Semi-symmetric Metric Connection

Definition 3.1. *A Kenmotsu manifold is said to be $M-$projectively flat with respect to semi-symmetric metric connection if*

$$P(X,Y)Z = 0. \tag{24}$$

Definition 3.2. *A Kenmotsu manifold is said to be $M-$projectively flat with respect to semi-symmetric metric connection if*

$$g(P(X,Y)Z,\phi W) = 0. \tag{25}$$

Definition 3.3. *A Kenmotsu manifold is said to be an η-Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{26}$$

where a and b are smooth functions on the manifold.

Using (7), (14) and (18) in (6), we obtain

$$R(X,Y)Z = K(X,Y)Z - 3g(Y,Z)X + 3g(X,Z)Y + 2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)Y + 2g(Y,Z)\eta(X)\xi - 2g(X,Z)\eta(Y) \tag{27}$$

using (9) in (27), we obtain

$$\begin{aligned} {}'R(X,Y,Z,W) ={}' K(X,Y,Z,W) &- 3g(Y,Z)g(X,W) + 3g(X,Z)g(Y,W) \\ &+ 2\eta(Y)\eta(Z)g(X,W) - 2\eta(X)\eta(Z)g(Y,W) \\ &+ 2g(Y,Z)\eta(X)\eta(W) - 2g(X,Z)\eta(Y)\eta(W) \end{aligned} \tag{28}$$

Contracting X in (27), we obtain

$$S(Y,Z) = S(Y,Z) - 2(3n-1)g(Y,Z) + 2(2n-1)\eta(Y)\eta(Z). \tag{29}$$

Substituting $Z = \xi$ in (29) and using (23), (13) and (14), we get

$$S(Y,\xi) = -4n\eta(Y) \tag{30}$$

Again contracting Y and Z in (29), we get

$$r = r - 2n(6n-1). \tag{31}$$

where r and $\tilde{r}$ are the scalar curvature with respect to the semi-symmetric metric connection and the Levi-Civita connection respectively.

Assume that M is M-projectively flat Kenmotsu manifold with respect to the connection $\tilde{\nabla}$. i.e., $P(X,Y)Z = 0$. Then from (10), we get

$$R(X,Y)Z = (\frac{1}{4n})S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \tag{32}$$

putting $Z = \xi$ in (26) and using (13) and (14), we get

$$R(X,Y)\xi = K(X,Y)\xi + \eta(X)Y - \eta(Y)X \tag{33}$$

using (19) in (33), we obtain

$$R(X,Y)\xi = 2\{\eta(X)Y - \eta(Y)X\} \tag{34}$$

putting $Z = \xi$ in (32) and taking inner product with W of (32) and using (34), we get

$$\{\eta(X)g(Y,W) - \eta(Y)g(X,W)\} = \left(\frac{1}{4n}\right)\{\eta(Y)S(X,W) - \eta(X)S(Y,W)\} \tag{35}$$

putting $Y = \xi$ in (35) and using (14) and (30), we get

$$S(X,W) = -\left(\frac{1}{4n}\right)g(X,W) \tag{36}$$

Hence (36) leads the following:

Theorem 3.4. *A M-projectively flat Kenmotsu manifold with respect to semi-symmetric metric connection is an Einstein manifold with respect to semi-symmetric metric connection.*

3.1. Example for M-projectively Flat Kenmotsu Manifold with Respect to Semi-symmetric Metric Connection

Let us consider a 5-dimensional manifold $M = \{(x_1, x_2, y_1, y_2, z) \in R^5 : z \neq 0\}$, where (x_1, x_2, y_1, y_2, z) are the standard coordinates in R^5. Let $e_1 = e^{-z}(\frac{\partial}{\partial x^1})$, $e_2 = e^{-z}(\frac{\partial}{\partial x^2})$, $e_3 = e^{-z}(\frac{\partial}{\partial y^1})$, $e_4 = e^{-z}(\frac{\partial}{\partial y^2})$, $e_5 = e^{-z}(\frac{\partial}{\partial z})$, which are linearly independent vector fields at each point of M. Define a Riemannian metric g on M as

$$g = e^{2z}(dx \otimes dx + dy \otimes dy) + \eta \otimes \eta$$

where η is the 1-form defined by $\eta(X) = g(X, e_5)$ for any vector X on M. Hence, $\{e_1, e_2, e_3, e_4, e_5\}$ is an orthonormal basis of M and ϕ be the tensor field of type $(1,1)$ defined as

$$\varphi\sum_{i=1}^{n}\left(X_i\frac{\partial}{\partial x_i} + Y_i\frac{\partial}{\partial y_i}\right) + Z\frac{\partial}{\partial z} = \sum_{i=1}^{n}\left(Y_i\frac{\partial}{\partial x_i} - X_i\frac{\partial}{\partial y_i}\right)$$

Thus, we have $\varphi(e_1) = e_3$, $\varphi(e_2) = e_4$, $\varphi(e_3) = -e_1$, $\varphi(e_4) = -e_2$, $\varphi(e_5) = 0$. Then by applying linearity of φ and g, we have $\eta(e_5) = 1$, $\varphi^2 X = -X + \eta(X)e_5$, $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$, for any vector fields X, Y on M. Hence for $e_5 = \xi$, $M(\varphi, \xi, \eta, g)$ defines an almost contact metric manifold. The 1-form η is closed. In addition, we have

$$\varphi\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = g\left(\frac{\partial}{\partial x}, \varphi\frac{\partial}{\partial y}\right) = g\left(\frac{\partial}{\partial x}, -\frac{\partial}{\partial x}\right) = -e^{2z}$$

Thus, we obtain $\varphi = -e^{2z}dx \wedge dy$. Hence $d\varphi = -e^{2z}dz \wedge dx \wedge dy = 2\eta \wedge \varphi$. Therefore, $M(\varphi, \xi, \eta, g)$ is an almost Kenmostu manifold. It can be seen that $M(\varphi, \xi, \eta, g)$ is normal. So, it is a Kenmotsu manifold. Moreover, we get

$$[X_i, \xi] = X_i, \qquad [Y_i, \xi] = Y_i, \qquad [X_i, X_j] = 0,$$
$$[X_i, Y_i] = 0, \qquad [X_i, Y_j] = 0, \qquad [Y_i, Y_j] = 0, \quad 1 \leq i, j \geq 2$$

The Riemannian connection ∇ of the metric is given by

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y).$$

By using Koszul's Formula, we get $\nabla_{X_i}X_i = \xi$, $\nabla_{Y_i}Y_i = \xi$, $\nabla_{X_i}X_j = \nabla_{Y_i}Y_j = \nabla_{X_i}Y_i = \nabla_{X_i}Y_j = 0$, $\nabla_{X_i}\xi = X_i$, $\nabla_{Y_i}\xi = Y_i$, $1 \leq i,j \geq 2$. Therefore, the semi-symmetric metric connection on M is given by $\nabla_{X_i}X_i = 0$, $\nabla_{Y_i}Y_i = 0$, $\nabla_{X_i}X_j = \nabla_{Y_i}Y_j = \nabla_{X_i}Y_i = \nabla_{X_i}Y_j = 0$, $\nabla_{X_i}\xi = 2X_i$, $\nabla_{Y_i}\xi = 2Y_i, 1 \leq i,j \geq 2$.

So, it can be seen that $R = 0$. Thus, $M(\varphi,\xi,\eta,g)$ is a M-projectively flat Kenmotsu manifold with respect to semi-symmetric metric connection. From above theorem, $M(\varphi,\xi,\eta,g)$ is an Einstein manifold with respect to semi-symmetric metric connection. Next, Substituting $X = \varphi X$ and $Y = \varphi Y$ in (10) and using (12), we get

$$g(P(\varphi X,Y)Z,\varphi W) = {}'R(\varphi X,Y,Z,\varphi W) - \left(\frac{1}{4n}\right)[S(Y,Z)g(\varphi X,\varphi W) - S(\varphi X,Z)g(Y,\varphi W) + g(Y,Z)S(\varphi X,\varphi W) - g(\varphi X,Z)S(Y,\varphi W) \tag{37}$$

we begin with the following:

Lemma 3.5. *Let M be a $(2n+1)-$demensional Kenmotsu manifold. If M satisfies*

$$g(P(\varphi X,Y)Z,\varphi W) = 0, \quad X,Y,Z,W \in \chi(M), \tag{38}$$

then M is an $\eta-$Einstein manifold.

Proof. Using (38) in (37), we have

$${}'R(\varphi X,Y,Z,\varphi W) = \left(\frac{1}{4n}\right)[S(Y,Z)g(\varphi X,\varphi W) - S(\varphi X,Z)g(Y,\varphi W) + g(Y,Z)S(\varphi X,\varphi W) - g(\varphi X,Z)S(Y,\varphi W) \tag{39}$$

Again using (27) and (28) in (39), we get

$$\begin{aligned} {}'K(\varphi X,Y,Z,\varphi W) = &\left(\frac{1}{n}\right)g(Y,Z)g(\varphi X,\varphi W) - \left(\frac{1}{n}\right)g(\varphi X,Z)g(Y,\varphi W) \\ &- \left(\frac{(2n+1)}{2n}\right)\eta(Y)\eta(Z)g(\varphi X,\varphi W) - \left(\frac{1}{4n}\right)[S(Y,Z)g(\varphi X,\varphi W) \\ &- S(\varphi X,Z)g(Y,\varphi W) + g(Y,Z)S(\varphi X,\varphi W) - g(\varphi X,Z)S(Y,\varphi W)] \end{aligned} \tag{40}$$

Let $\{e_1, e_2, e_3, \ldots, e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in M, then $\{\varphi e_1, \varphi e_2, \varphi e_3, \ldots, \varphi e_{2n}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (40) and summing over $= 1, \ldots, 2n$, we get

$$\begin{aligned} \sum_{i=1}^{2n} {}'K(\varphi e_i,Y,Z,\varphi e_i) = &\frac{1}{n}\sum_{i=1}^{2n} g(Y,Z)g(\varphi e_i,\varphi e_i) - \frac{1}{n}\sum_{i=1}^{2n} g(\varphi e_i,Z)g(Y,\varphi e_i) \\ &- \left(\frac{(2n+1)}{2n}\right)\sum_{i=1}^{2n}\eta(Y)\eta(Z)g(\varphi e_i,\varphi e_i) - \left(\frac{1}{4n}\right)\sum_{i=1}^{2n}[S(Y,Z)g(\varphi e_i,\varphi e_i) \\ &- S(\varphi e_i,Z)g(Y,\varphi e_i) + g(Y,Z)S(\varphi e_i,\varphi e_i) - g(\varphi e_i,Z)S(Y,\varphi e_i)] \end{aligned} \tag{41}$$

from (41), we get

$$S(Y,Z) = (\frac{(10-n+r)}{2(n+1)})g(Y,Z) - (\frac{2n+1}{2(n+1)})\eta(Y)\eta(Z) \tag{42}$$

Therefore, $S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z)$, where

$$a = \left(\frac{(10-n+\tilde{r})}{2(n+1)}\right), \quad b = \left(\frac{2n+1}{2(n+1)}\right)\eta(Y)\eta(Z)$$

This result shows that the manifold is an $\eta-$Einstein manifold. This proves the lemma. ∎

In view of Lemma 3.5, we can state the following theorem:

Theorem 3.6. *If a Kenmotsu manifold is quasi-M-projectively flat with respect to the semi-symmetric metric connection, then the manifold is an η-Einstein manifold.*

Since a and b are both constant, by Lemma 2.1, we get following:

Corollary 3.7. *If a Kenmotsu manifold is quasi-M-projectively flat with respect to the semi-symmetric metric connection, then the manifold is an Einstein manifold.*

4. ξ-M-Projectively Flat and φ-M-Projectively Flat Kenmotsu Manifolds With Respect to Semi-Symmetric Metric Connection

Let W^* be the Weyl conformal curvature tensor of a $(2n+1)-$dimensional manifold M. Since at each point $p \in M$ the tangent space $\chi_p(M)$ can be decomposed into the direct sum $\chi_p(M) = \varphi(\chi_p(M)) \oplus L(\xi_p)$, where $L(\xi_p)$ is an $1-$dimensional linear subspace of $\chi_p(M)$ generated by ξ_p. Then we have a map

$$W^* : \chi_p(M) \times \chi_p(M) \to \varphi(\chi_p(M)) \oplus L(\xi_p),$$

Let us consider the following particular cases:

(1) $W^* : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \to L(\xi_p)$, i.e., the projection of the image of W^* in $\varphi(\chi_p(M))$ is zero.

(2) $W^* : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \to \varphi(\chi_p(M))$, i.e., the projection of the image of W^* in $L(\xi_p)$ is zero.

$$W^*(X,Y)\xi = 0 \tag{43}$$

(3) $W^* : \varphi(\chi_p(M)) \times \varphi(\chi_p(M)) \times \varphi(\chi_p(M)) \to L(\xi_p)$, i.e., when W^* is restricted to $\varphi(\chi_p(M)) \times \varphi(\chi_p(M)) \times \varphi(\chi_p(M))$, the projection of the image of W^* in $\varphi(\chi_p(M))$ is zero. This condition is equivalent to

$$\varphi^2 W^*(\varphi X, \varphi Y)\varphi Z = 0 \tag{44}$$

Here the cases $1, 2$ and 3 are conformally symmetric, $\xi-$conformally flat and $\varphi-$conformally flat respectively. The cases (1) and (2) were considered in [4] and [22] respectively. the case (3) was considered in [23] for the case M is a K-contact manifold. Analogous to the definition of $\xi-$conformally flat and $\varphi-$conformally flat, we give the following defintions:

Definition 4.1. *A Kenmotsu manifold with respect to the semi-symmetric metric connection is said to be ξ-M-projectively flat if*

$$P(X,Y)\xi = 0 \tag{45}$$

Definition 4.2. *A Kenmotsu manifold is said to be φ-M-projectively flat with respect to the semi-symmetric metric connection if*

$$g(P(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0 \tag{46}$$

where $X, Y, Z, W \in \chi(M)$.

Putting $Z = \xi$ in (26) and using (13) and (14), we get

$$R(X,Y)\xi = K(X,Y)\xi + \eta(X)Y - \eta(Y)X \tag{47}$$

using (19) in (47), we get

$$R(X,Y)\xi = 2K(X,Y)\xi \tag{48}$$

Putting $Z = \xi$ in (10), we have

$$P(X,Y)\xi = R(X,Y)\xi - (\frac{1}{4n})[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY \tag{49}$$

Using (29) and (48) in (49), we get

$$P(X,Y)\xi = 0 \tag{50}$$

This leads the following:

Theorem 4.3. *If a Kenmotsu manifold admits a semi-symmetric metric connection, then the Kenmotsu manifold is ξ-M-projectively flat with respect to the semi-symmetric metric connection.*

Putting $Y = \varphi Y$ and $Z = \varphi Z$ in (37), we obtain

$$\begin{aligned} g(P(\varphi X,\varphi Y)\varphi Z,\varphi W) = g(R(\varphi X,\varphi Y)\varphi Z,\varphi W) - \left(\frac{1}{4n}\right)[S(\varphi Y,\varphi Z)g(\varphi X,\varphi W) - S(\varphi X,\varphi Z)g(\varphi Y,\varphi W) \\ + g(\varphi Y,\varphi Z)S(\varphi X,\varphi W) - g(\varphi X,\varphi Z)S(\varphi Y,\varphi W) \end{aligned} \tag{51}$$

Using (13), (14), (26) and (28) in (51), we get

$$\begin{aligned} g(P(\varphi X,\varphi Y)\varphi Z,\varphi W) = g(K(\varphi X,\varphi Y)\varphi Z,\varphi W) - \left(\frac{1}{n}\right)g(\varphi Y,\varphi Z)g(\varphi X,\varphi W) + \left(\frac{1}{n}\right)g(\varphi X,\varphi Z)g(\varphi Y,\varphi W) \\ - \left(\frac{1}{4n}\right)[S(\varphi Y,\varphi Z)g(\varphi X,\varphi W) - S(\varphi X,\varphi Z)g(Y,\varphi W) + g(\varphi Y,\varphi Z)S(\varphi X,\varphi W) \\ - g(\varphi X,\varphi Z)S(\varphi Y,\varphi W)] \end{aligned} \tag{52}$$

Using (46) in (52), we get

$$\begin{aligned} g(K(\varphi X,\varphi Y)\varphi Z,\varphi W) = (\frac{1}{n})g(\varphi Y,\varphi Z)g(\varphi X,\varphi W) + (\frac{1}{n})g(\varphi X,\varphi Z)g(\varphi Y,\varphi W) \\ - \left(\frac{1}{4n}\right)[S(\varphi Y,\varphi Z)g(\varphi X,\varphi W) - S(\varphi X,\varphi Z)g(Y,\varphi W) \\ + g(\varphi Y,\varphi Z)S(\varphi X,\varphi W) - g(\varphi X,\varphi Z)S(\varphi Y,\varphi W)] \end{aligned} \tag{53}$$

Let $\{e_1, e_2, e_3, \ldots, e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in M, then $\{\varphi e_1, \varphi e_2, \varphi e_3, \ldots, \varphi e_{2n}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (53) and summing over $= 1, \ldots, 2n$, we get

$$\begin{aligned} \sum_{i=1}^{2n} g(K(\varphi e_i,\varphi Y)\varphi Z,\varphi e_i) = \frac{1}{n}\sum_{i=1}^{2n} g(\varphi Y,\varphi Z)g(\varphi e_i,\varphi e_i) + \frac{1}{n}\sum_{i=1}^{2n} g(\varphi e_i,\varphi Z)g(\varphi Y,\varphi e_i) \\ - \frac{1}{4n}\sum_{i=1}^{2n} S(\varphi Y,\varphi Z)g(\varphi e_i,\varphi e_i) + S(\varphi e_i,\varphi Z)g(Y,\varphi e_i) \\ - g(\varphi Y,\varphi Z)S(\varphi e_i,\varphi e_i) - g(\varphi e_i,\varphi Z)S(\varphi Y,\varphi e_i)] \end{aligned} \tag{54}$$

From (54), we get

$$S(\varphi Y, \varphi Z) = \frac{10n - 4 + r}{2(n+1)} g(\varphi Y, \varphi Z) \tag{55}$$

Using (15) and (22) in (55) we get

$$S(Y, Z) = \frac{10n - 4 + r}{2(n+1)} g(Y, Z) - \frac{4n^2 + 14n - 4 + r}{2(n+1)} \eta(Y)\eta(Z). \tag{56}$$

Therefore, $S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z)$. Where

$$a = \frac{10n - 4 + r}{2(n+1)}, \quad b = -\frac{4n^2 + 14n - 4 + r}{2(n+1)}$$

This leads the following:

Theorem 4.4. *If a Kenmotsu manifold is φ-M-projectively flat with respect to the semi-symmetric metric connection, then the manifold is an $\eta-$Einstein manifold.*

Since a and b are both constant, by Lemma 2.1 we get following:

Corollary 4.5. *If a Kenmotsu manifold is φ-M-projectively flat with respect to the semi-symmetric metric connection, then the manifold is an Einstein manifold.*

5. Kenmotsu Manifold with Respect to the Semi-Symmetric Metric Connection Satisfying $P.S = 0$

In this Section we consider Kenmotsu Manifold with respect to the semi-symmetric metric connection M^{2n+1} satisfying condition

$$(P(U, Y).S)(Z, X) = 0$$

Then we have

$$S(P(U, Y)Z, X) + S(Z, P(U, Y)X) = 0 \tag{57}$$

Putting $U = \xi$ in (57), we get

$$S(P(\xi, Y)Z, X) + S(Z, P(\xi, Y)X) = 0 \tag{58}$$

Putting $X = \xi$ and using (28) and (29) in (10), we obtain

$$P(\xi, Y)Z = R(\xi, Y)Z - \left(\frac{1}{4n}\right)[S(Y, Z)\xi - 2(5n-1)g(Y, Z)\xi + 2(2n+1)\eta(Y)\eta(Z)\xi \tag{59}$$

Again putting $X = \xi$ in (26) and using (20), we get

$$R(\xi, Y)Z = 2[\eta(Z)Y - g(Y, Z)\xi] \tag{60}$$

Using (28), (29), (59) and (60) in (58), we obtain

$$S(Y, Z) = (\frac{2(11n+7)}{3})g(Y, Z) + (\frac{4(1-2n)}{3})\eta(Y)\eta(Z) \tag{61}$$

Therefore, $S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z)$. Where

$$a = \left(\frac{2(11n+7)}{3}\right), b = \left(\frac{4(1-2n)}{3}\right)$$

This leads the following:

Theorem 5.1. *If a Kenmotsu manifold with respect to the semi-symmetric metric connection satisfying $P.S = 0$, then the manifold is an $\eta-$Einstein manifold.*

Since a and b are both constant, by Lemma 2.1, we get following:

Corollary 5.2. *If a Kenmotsu manifold with respect to the semi-symmetric metric connection satisfying $P.S = 0$, then the manifold is an Einstein manifold.*

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