M-Projective Curvature Tensor of a Semi-symmetric Metric Connection in a Kenmotsu Manifold

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Abstract: In the present paper we consider a semi-symmetric metric connection in a Kenmotsu manifold. We deduce the relation between the Riemannian connection and the semi-symmetric metric connection on a Kenmotsu manifold. We investigate the curvature tensor and the Ricci tensor of a Kenmotsu manifold with respect to the semi-symmetric metric connection. We study M-projective curvature tensor with respect to the semi-symmetric metric connection satisfying certain curvature conditions.

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1. Introduction

The notion of Kenmotsu manifolds was defined and studied by Kenmotsu [13] in 1972. They set up one of the three classes of almost contact metric manifolds M whose automorphism group attains the maximum dimension [17]. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c.

(i) If c > 0, M is a homogeneous Sasakian manifold of constant ϕ-sectional curvature. (ii) If c = 0, M is global Riemannian product of a line or a circle with a Kahler manifold of constant holomorphic sectional curvature. (iii) If c < 0, M is a warped product space $\mathbb{R} \times f C^n$. Kenmotsu [13] characterized the differential geometric properties of manifolds of class. (iv) The structure so obtained is now known as Kenmotsu structure. A Kenmotsu structure is not Sasakian.


Let M be an n-dimensional Riemannian manifold of class $C^\infty$ endowed with the Riemannian metric g and $\nabla$ be the Levi-Civita connection on $(M^n, g)$. A linear connection $\nabla$ defined on $(M^n, g)$ is said to be semi-symmetric [9] if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y$$

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where \( \eta \) is a 1-form and \( \xi \) is a vector field given by

\[
\eta(X) = g(X, \xi)
\]  

(2)

for all vector fields \( X \in \chi(M^n) \), \( \chi(M^n) \) is the set of all differentiable vector fields on \( M^n \). A semi-symmetric connection \( \tilde{\nabla} \) is called a semi-symmetric metric connection [10] if it further satisfies

\[
\tilde{\nabla} g = 0
\]  

(3)

A relation between the semi-symmetric metric connection \( \tilde{\nabla} \) and the Levi-Civita connection \( \nabla \) on \( (M^n, g) \) has been obtained by K.Yano [18] which is given by

\[
\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi
\]  

(4)

we also have

\[
(\tilde{\nabla}_X \eta)Y = (\nabla_X \eta)Y - \eta(Y)\eta(X) + \eta(\xi)g(X, Y)
\]  

(5)

Further, a relation between the curvature tensor \( R \) of the semi-symmetric metric connection \( \tilde{\nabla} \) and the curvature tensor \( K \) of the Levi-Civita connection \( \nabla \) is given by

\[
R(X, Y)Z = K(X, Y)Z + \alpha(X, Z)Y - \alpha(Y, Z)X + g(X, Z)Y - g(Y, Z)X
\]  

(6)

where \( \alpha \) is a tensor field of type \((0, 2)\) and \( Q \) is a tensor field of type \((1, 1)\) which is given by

\[
\alpha(Y, Z) = g(QY, Z) = (\nabla_Y \eta)Z - \eta(Y)\eta(Z) + \left( \frac{1}{2} \right) \eta(\xi)g(Y, Z)
\]  

(7)

from (6) and (7), we obtain

\[
' R(X, Y, Z, W) = ' K(X, Y, Z, W) + \alpha(X, Z)g(Y, W) - \alpha(Y, Z)g(X, W) + g(X, Z)\alpha(Y, W) - g(Y, Z)\alpha(X, W)
\]  

(8)

Where

\[
'R(X, Y, Z, W) = g(R(X, Y)Z, W),
\]

\[
'K(X, Y, Z, W) = g(K(X, Y)Z, W)
\]  

(9)

In an almost contact manifold \( M \), the M-projective curvature tensor \( P \) with respect to semi-symmetric metric connection \( \tilde{\nabla} \) is given by

\[
P(X, Y)Z = R(X, Y)Z - \left( \frac{1}{4n} \right) S(Y, Z)X - S(X, Z) + g(Y, Z)QX - g(X, Z)QY
\]  

(10)

for \( X, Y, Z \in \chi(M) \), where \( R, S \) and \( Q \) are the Riemannian curvature tensor, Ricci tensor and the Ricci operator with respect to the connection \( \nabla \), respectively. From (10), it follows that

\[
' P(X, Y, Z, W) = R(X, Y, Z, W) - \left( \frac{1}{4n} \right) [S(Y, Z)g(X, W) - S(X, Z)g(Y, W) + g(Y, Z)S(X, W) - g(X, Z)S(Y, W)]
\]  

(11)

And

\[
' P(X, Y, Z, W) = g(P(X, Y)Z, W)
\]  

(12)
for all vector fields $X, Y, Z$ on $M$. Where $S$ is the Ricci tensor with respect to the Semi-symmetric metric connection.

In the present paper, we study $M$-projective curvature tensor on a Kenmotsu manifold with respect to the semi-symmetric metric connection. The organization of this paper is as follows:

First section contains basic concepts of semi-symmetric metric connection. In section 2, we give a brief account of the Kenmotsu manifolds and we also give curvature tensor and Ricci tensor of a Kenmotsu manifold with respect to the semi-symmetric metric connection. In section 3, we study the $M$-projectively flat Kenmotsu manifold with respect to the semi-symmetric metric connection and proved that it is Einstein manifold. Also, an example for $M$-projectively flat Kenmotsu manifold with respect to the semi-symmetric metric connection is given. Further, we shown quasi-$M$-projectively flat Kenmotsu manifold with respect to the semi-symmetric metric connection. In the last section 5, we investigate $P.S = 0$, in a Kenmotsu manifold with respect to the semi-symmetric metric connection.

2. Kenmotsu Manifolds

Let $M$ be an $(2n + 1)$-dimensional almost contact metric manifold with an almost contact metric structure $(\phi, \xi, \eta, g)$ consisting of a $(1, 1)$ tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$ on $M$ satisfying [3]

\[
\varphi^2(X) = -X + \eta(X)\xi, \quad g(X, \xi) = \eta(X) \quad (13)
\]

\[
\eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta(\varphi(X)) = 0 \quad (14)
\]

\[
g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (15)
\]

for all vector fields $X, Y$ on $M$. If an almost contact metric manifold satisfies

\[
(\nabla_X \varphi)(Y) = g(\varphi X, Y)\xi - \eta(Y)\phi X \quad (16)
\]

then $M$ is called a Kenmotsu manifold [13]. From the above relations, it follows that

\[
\nabla_X \xi = X - \eta(X)\xi \quad (17)
\]

\[
(\nabla_X \eta)(Y) = g(X, Y)\xi - \eta(X)\eta(Y) \quad (18)
\]

Moreover the curvature tensor $K$ and the Ricci tensor $S$ of the Kenmotsu manifold with respect to the Levi-Civita connection satisfies

\[
K(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (19)
\]

\[
K(\xi, Y)X = \eta(X)Y - g(X, Y)\xi, \quad (20)
\]

\[
K(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \quad (21)
\]

\[
\tilde{S}(\varphi X, \varphi Y) = S(X, Y) + 2n\eta(X)\eta(Y), \quad (22)
\]

\[
S(X, \xi) = -2n\eta(X) \quad (23)
\]

we state the following lemma which will be used in the next section:

**Lemma 2.1** ([13]). Let $M$ be an $\eta$–Einstein Kenmotsu manifold of the form $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$. If $b =$ constant (or, $a =$ constant), then $M$ is an Einstein manifold.
3. M-projectively Flat and Quasi–M–Projectively Flat Kenmotsu Manifolds with Respect to the Semi-symmetric Metric Connection

**Definition 3.1.** A Kenmotsu manifold is said to be $M$–projectively flat with respect to semi-symmetric metric connection if

$$P(X,Y)Z = 0.$$  \hfill (24)

**Definition 3.2.** A Kenmotsu manifold is said to be $M$–projectively flat with respect to semi-symmetric metric connection if

$$g(P(X,Y)Z,\phi W) = 0.$$  \hfill (25)

**Definition 3.3.** A Kenmotsu manifold is said to be an $\eta$-Einstein manifold if its Ricci tensor $S$ of the Levi-Civita connection is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$  \hfill (26)

where $a$ and $b$ are smooth functions on the manifold.

Using (7), (14) and (18) in (6), we obtain

$$R(X,Y)Z = K(X,Y)Z - 3g(Y,Z)X + 3g(X,Z)Y + 2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)Y + 2g(Y,Z)\eta(X)\eta(Y) - 2g(X,Z)\eta(Y)$$  \hfill (27)

using (9) in (27), we obtain

$$R(X,Y,Z,W) = K(X,Y,Z,W) - 3g(Y,Z)g(X,W) + 3g(X,Z)g(Y,W) + 2\eta(Y)\eta(Z)g(X,W) - 2\eta(X)\eta(Z)g(Y,W) + 2g(Y,Z)\eta(W)\eta(X) - 2g(X,Z)\eta(W)\eta(Y)$$  \hfill (28)

Contracting $X$ in (27), we obtain

$$S(Y,Z) = S(Y,Z) - 2(3n - 1)g(Y,Z) + 2(2n - 1)\eta(Y)\eta(Z).$$  \hfill (29)

Substituting $Z = \xi$ in (29) and using (23), (13) and (14), we get

$$S(Y,\xi) = -4n\eta(Y)$$  \hfill (30)

Again contracting $Y$ and $Z$ in (29), we get

$$r = r - 2n(6n - 1).$$  \hfill (31)

where $r$ and $\tilde{r}$ are the scalar curvature with respect to the semi-symmetric metric connection and the Levi-Civita connection respectively.

Assume that $M$ is M-projectively flat Kenmotsu manifold with respect to the connection $\vartriangle$, i.e., $P(X,Y)Z = 0$. Then from (10), we get

$$R(X,Y)Z = \left(\frac{1}{4n}\right)S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY$$  \hfill (32)

putting $Z = \xi$ in (26) and using (13) and (14), we get

$$R(X,Y)\xi = K(X,Y)\xi + \eta(X)Y - \eta(Y)X$$  \hfill (33)
using (19) in (33), we obtain

$$R(X,Y)\xi = 2\{\eta(X)Y - \eta(Y)X\}$$  \hspace{1cm} (34)

putting $Z = \xi$ in (32) and taking inner product with $W$ of (32) and using (34), we get

$$\{\eta(X)g(Y,W) - \eta(Y)g(X,W)\} = \left(\frac{1}{4n}\right)\{\eta(Y)S(X,W) - \eta(X)S(Y,W)\}$$  \hspace{1cm} (35)

putting $Y = \xi$ in (35) and using (14) and (30), we get

$$S(X,W) = -\left(\frac{1}{4n}\right)g(X,W)$$  \hspace{1cm} (36)

Hence (36) leads the following:

**Theorem 3.4.** A $M$-projectively flat Kenmotsu manifold with respect to semi-symmetric metric connection is an Einstein manifold with respect to semi-symmetric metric connection.

### 3.1. Example for $M$-projectively Flat Kenmotsu Manifold with Respect to Semi-symmetric Metric Connection

Let us consider a 5-dimensional manifold $M = \{(x_1, x_2, y_1, y_2, z) \in R^5 : z \neq 0\}$, where $(x_1, x_2, y_1, y_2, z)$ are the standard coordinates in $R^5$. Let $e_1 = e^{-z}\left(\frac{\partial}{\partial x_1}\right)$, $e_2 = e^{-z}\left(\frac{\partial}{\partial x_2}\right)$, $e_3 = e^{-z}\left(\frac{\partial}{\partial y_1}\right)$, $e_4 = e^{-z}\left(\frac{\partial}{\partial y_2}\right)$, $e_5 = e^{-z}\left(\frac{\partial}{\partial z}\right)$, which are linearly independent vector fields at each point of $M$. Define a Riemannian metric $g$ on $M$ as

$$g = e^{2z}(dx \otimes dx + dy \otimes dy) + \eta \otimes \eta$$

where $\eta$ is the 1-form defined by $\eta(X) = g(X,e_5)$ for any vector $X$ on $M$. Hence, $\{e_1, e_2, e_3, e_4, e_5\}$ is an orthonormal basis of $M$ and $\phi$ be the tensor field of type $(1,1)$ defined as

$$\phi = \sum_{i=1}^{n} \left( X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i} \right) + Z \frac{\partial}{\partial z} = \sum_{i=1}^{n} \left( Y_i \frac{\partial}{\partial x_i} - X_i \frac{\partial}{\partial y_i} \right)$$

Thus, we have $\phi(e_1) = e_3$, $\phi(e_2) = e_4$, $\phi(e_3) = -e_1$, $\phi(e_4) = -e_2$, $\phi(e_5) = 0$. Then by applying linearity of $\phi$ and $g$, we have $\eta(e_5) = 1$, $\phi^2X = -X + \eta(X)e_5$, $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$, for any vector fields $X$, $Y$ on $M$. Hence for $e_5 = \xi$, $M(\phi, \xi, \eta, g)$ defines an almost contact metric manifold. The 1-form $\eta$ is closed. In addition, we have

$$\phi \left( \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} \right) = g \left( \frac{\partial}{\partial x}, \phi \frac{\partial}{\partial y} \right) = g \left( \frac{\partial}{\partial y}, -\frac{\partial}{\partial x} \right) = -e^{2z}$$

Thus, we obtain $\phi = -e^{2z}dx \wedge dy$. Hence $d\phi = -e^{2z}dz \wedge dx \wedge dy = 2\eta \wedge \phi$. Therefore, $M(\phi, \xi, \eta, g)$ is an almost Kenmotsu manifold. It can be seen that $M(\phi, \xi, \eta, g)$ is normal. So, it is a Kenmotsu manifold. Moreover, we get

$$[X_i, \xi] = X_i, \hspace{1cm} \eta(Y, \xi) = Y, \hspace{1cm} [X_i, X_j] = 0,$$

$$[X_i, Y_j] = 0, \hspace{1cm} [X_i, Y_j] = 0, \hspace{1cm} [Y_i, Y_j] = 0, \hspace{1cm} 1 \leq i, j \geq 2$$

The Riemannian connection $\nabla$ of the metric is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y).$$
By using Koszul’s Formula, we get $\nabla_X X_i = \xi$, $\nabla_Y Y_i = \xi$, $\nabla_X X_j = \nabla_Y Y_j = \nabla_X Y_i = \nabla_X Y_j = 0$, $\nabla_X \xi = X_i$, $\nabla_Y \xi = Y_i$, $1 \leq i, j \leq 2$. Therefore, the semi-symmetric metric connection on $M$ is given by $\nabla_X X_i = 0$, $\nabla_Y Y_i = 0$, $\nabla_X X_j = \nabla_Y Y_j = \nabla_X Y_i = \nabla_X Y_j = 0$, $\nabla_X \xi = 2X_i$, $\nabla_Y \xi = 2Y_i$, $1 \leq i, j \geq 2$.

So, it can be seen that $R = 0$. Thus, $M(\varphi, \xi, \eta, g)$ is a M-projectively flat Kenmotsu manifold with respect to semi-symmetric metric connection. From above theorem, $M(\varphi, \xi, \eta, g)$ is an Einstein manifold with respect to semi-symmetric metric connection. Next, Substituting $X = \varphi X$ and $Y = \varphi Y$ in (10) and using (12), we get

$$g(P(\varphi X, Y)Z, \varphi W) = \left(\frac{1}{4n}\right) [S(Y, Z)g(\varphi X, \varphi W) - S(\varphi X, Z)g(Y, \varphi W) + g(Y, Z)S(\varphi X, \varphi W) - g(\varphi X, Z)S(Y, \varphi W)]$$

we begin with the following:

**Lemma 3.5.** Let $M$ be a $(2n + 1)$-dimensional Kenmotsu manifold. If $M$ satisfies

$$g(P(\varphi X, Y)Z, \varphi W) = 0, \quad X, Y, Z, W \in \chi(M),$$

then $M$ is an $\eta$-Einstein manifold.

**Proof.** Using (38) in (37), we have

$$\nabla^2 Y, Z, \varphi W = \left(\frac{1}{4n}\right) [S(Y, Z)g(\varphi X, \varphi W) - S(\varphi X, Z)g(Y, \varphi W) + g(Y, Z)S(\varphi X, \varphi W) - g(\varphi X, Z)S(Y, \varphi W)]$$

Again using (27) and (28) in (39), we get

$$\nabla^2 Y, Z, \varphi W = \left(\frac{1}{4n}\right) [S(Y, Z)g(\varphi X, \varphi W) - S(\varphi X, Z)g(Y, \varphi W) + g(Y, Z)S(\varphi X, \varphi W) - g(\varphi X, Z)S(Y, \varphi W)]$$

Let $\{e_1, e_2, e_3, \ldots, e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in $M$, then $\{\varphi e_1, \varphi e_2, \varphi e_3, \ldots, \varphi e_{2n}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (40) and summing over $1, \ldots, 2n$, we get

$$\sum_{i=1}^{2n} K(\varphi e_i, Y, Z, \varphi e_i) = \frac{1}{n} \sum_{i=1}^{2n} g(Y, Z)g(\varphi e_i, \varphi e_i) - \frac{1}{n} \sum_{i=1}^{2n} g(\varphi e_i, Z)g(Y, \varphi e_i)$$

$$\quad - \frac{(2n + 1)}{2n} \sum_{i=1}^{2n} \eta(Y)\eta(Z)g(\varphi e_i, \varphi e_i) - \frac{1}{4n} \sum_{i=1}^{2n} [S(Y, Z)g(\varphi e_i, \varphi e_i)$$

$$\quad - S(\varphi e_i, Z)g(Y, \varphi e_i) + g(Y, Z)S(\varphi e_i, \varphi e_i) - g(\varphi e_i, Z)S(Y, \varphi e_i)]$$

from (41), we get

$$S(Y, Z) = \frac{(10 - n + r)}{2(n + 1)} g(Y, Z) - (\frac{2n + 1}{2(n + 1)}) \eta(Y)\eta(Z)$$

Therefore, $S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z)$, where

$$a = \left(\frac{10 - n + r}{2(n + 1)}\right), \quad b = \left(\frac{2n + 1}{2(n + 1)}\right) \eta(Y)\eta(Z)$$

This result shows that the manifold is an $\eta$-Einstein manifold. This proves the lemma.
In view of Lemma 3.5, we can state the following theorem:

**Theorem 3.6.** If a Kenmotsu manifold is quasi-M-projectively flat with respect to the semi-symmetric metric connection, then the manifold is an \( \eta \)-Einstein manifold.

Since \( a \) and \( b \) are both constant, by Lemma 2.1, we get following:

**Corollary 3.7.** If a Kenmotsu manifold is quasi-M-projectively flat with respect to the semi-symmetric metric connection, then the manifold is an Einstein manifold.

### 4. \( \xi \)-M-Projectively Flat and \( \varphi \)-M-Projectively Flat Kenmotsu Manifolds With Respect to Semi-Symmetric Metric Connection

Let \( W^* \) be the Weyl conformal curvature tensor of a \((2n+1)\)-dimensional manifold \( M \). Since at each point \( p \in M \) the tangent space \( \chi_p(M) \) can be decomposed into the direct sum \( \chi_p(M) = \varphi(\chi_p(M)) \oplus L(\xi_p) \), where \( L(\xi_p) \) is an \( 1\)-dimensional linear subspace of \( \chi_p(M) \) generated by \( \xi_p \). Then we have a map

\[
W^* : \chi_p(M) \times \chi_p(M) \to \varphi(\chi_p(M)) \oplus L(\xi_p),
\]

Let us consider the following particular cases:

1. \( W^* : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \to L(\xi_p) \), i.e., the projection of the image of \( W^* \) in \( \varphi(\chi_p(M)) \) is zero.
2. \( W^* : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \to \varphi(\chi_p(M)) \), i.e., the projection of the image of \( W^* \) in \( L(\xi_p) \) is zero.

\[
W^*(X,Y)\xi = 0 \quad (43)
\]

3. \( W^* : \varphi(\chi_p(M)) \times \varphi(\chi_p(M)) \times \varphi(\chi_p(M)) \to L(\xi_p) \), i.e., when \( W^* \) is restricted to \( \varphi(\chi_p(M)) \times \varphi(\chi_p(M)) \times \varphi(\chi_p(M)) \), the projection of the image of \( W^* \) in \( \varphi(\chi_p(M)) \) is zero. This condition is equivalent to

\[
\varphi^2 W^*(\varphi X, \varphi Y) \varphi Z = 0 \quad (44)
\]

Here the cases 1, 2 and 3 are conformally symmetric, \( \xi \)-conformally flat and \( \varphi \)-conformally flat respectively. The cases (1) and (2) were considered in [4] and [22] respectively, the case (3) was considered in [23] for the case \( M \) is a K-contact manifold. Analogous to the definition of \( \xi \)-conformally flat and \( \varphi \)-conformally flat, we give the following definitions:

**Definition 4.1.** A Kenmotsu manifold with respect to the semi-symmetric metric connection is said to be \( \xi \)-M-projectively flat if

\[
P(X,Y)\xi = 0 \quad (45)
\]

**Definition 4.2.** A Kenmotsu manifold is said to be \( \varphi \)-M-projectively flat with respect to the semi-symmetric metric connection if

\[
g(\varphi X, \varphi Y) \varphi Z, \varphi W = 0 \quad (46)
\]

where \( X, Y, Z, W \in \chi(M) \).
Putting $Z = \xi$ in (26) and using (13) and (14), we get

$$R(X,Y)\xi = K(X,Y)\xi + \eta(X)Y - \eta(Y)X$$  \hspace{1cm} (47)$$

using (19) in (47), we get

$$R(X,Y)\xi = 2K(X,Y)\xi$$  \hspace{1cm} (48)$$

Putting $Z = \xi$ in (10), we have

$$P(X,Y)\xi = R(X,Y)\xi - \left(\frac{1}{4n}\right)[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY$$  \hspace{1cm} (49)$$

Using (29) and (48) in (49), we get

$$P(X,Y)\xi = 0$$  \hspace{1cm} (50)$$

This leads the following:

**Theorem 4.3.** If a Kenmotsu manifold admits a semi-symmetric metric connection, then the Kenmotsu manifold is $\xi$-M-projectively flat with respect to the semi-symmetric metric connection.

Putting $Y = \varphi Y$ and $Z = \varphi Z$ in (37), we obtain

$$g(P(\varphi X, \varphi Y)\varphi Z, \varphi W) = g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) - \left(\frac{1}{4n}\right)[S(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - S(\varphi X, \varphi Z)g(\varphi Y, \varphi W) + g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W)]$$  \hspace{1cm} (51)$$

Using (13), (14), (26) and (28) in (51), we get

$$g(P(\varphi X, \varphi Y)\varphi Z, \varphi W) = g(K(\varphi X, \varphi Y)\varphi Z, \varphi W) - \left(\frac{1}{n}\right)g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) + \left(\frac{1}{n}\right)g(\varphi X, \varphi Z)g(\varphi Y, \varphi W) - \left(\frac{1}{4n}\right)[S(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - S(\varphi X, \varphi Z)g(\varphi Y, \varphi W) + g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W)]$$  \hspace{1cm} (52)$$

Using (46) in (52), we get

$$g(K(\varphi X, \varphi Y)\varphi Z, \varphi W) = \left(\frac{1}{n}\right)g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) + \left(\frac{1}{n}\right)g(\varphi X, \varphi Z)g(\varphi Y, \varphi W) - \left(\frac{1}{4n}\right)[S(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - S(\varphi X, \varphi Z)g(\varphi Y, \varphi W) + g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W)]$$  \hspace{1cm} (53)$$

Let $\{e_1, e_2, e_3, \ldots, e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in $M$, then $\{\varphi e_1, \varphi e_2, \varphi e_3, \ldots, \varphi e_{2n}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (53) and summung over $= 1, \ldots, 2n$, we get

$$\sum_{i=1}^{2n} g(K(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) = \frac{1}{n} \sum_{i=1}^{2n} g(\varphi Y, \varphi Z)g(\varphi e_i, \varphi e_i) + \frac{1}{n} \sum_{i=1}^{2n} g(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i) - \frac{1}{4n} \sum_{i=1}^{2n} [S(\varphi Y, \varphi Z)g(\varphi e_i, \varphi e_i) + S(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i) - g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i)]$$  \hspace{1cm} (54)$$
From (54), we get
\[ S(\varphi Y, \varphi Z) = \frac{10n - 4 + r}{2(n + 1)} g(\varphi Y, \varphi Z) \] (55)

Using (15) and (22) in (55) we get
\[ S(Y, Z) = \frac{10n - 4 + r}{2(n + 1)} g(Y, Z) - \frac{4n^2 + 14n - 4 + r}{2(n + 1)} \eta(Y)\eta(Z). \] (56)

Therefore, \( S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z). \) Where
\[ a = \frac{10n - 4 + r}{2(n + 1)}, \quad b = -\frac{4n^2 + 14n - 4 + r}{2(n + 1)} \]

This leads the following:

**Theorem 4.4.** If a Kenmotsu manifold is \( \varphi \)-\( M \)-projectively flat with respect to the semi-symmetric metric connection, then the manifold is an \( \eta \)-Einstein manifold.

Since \( a \) and \( b \) are both constant, by Lemma 2.1 we get following:

**Corollary 4.5.** If a Kenmotsu manifold is \( \varphi \)-\( M \)-projectively flat with respect to the semi-symmetric metric connection, then the manifold is an Einstein manifold.

### 5. Kenmotsu Manifold with Respect to the Semi-Symmetric Metric Connection Satisfying \( P.S = 0 \)

In this Section we consider Kenmotsu Manifold with respect to the semi-symmetric metric connection \( M^{2n+1} \) satisfying condition
\[ (P(U, Y).S)(Z, X) = 0 \]

Then we have
\[ S(P(U, Y)Z, X) + S(Z, P(U, Y)X) = 0 \] (57)

Putting \( U = \xi \) in (57), we get
\[ S(P(\xi, Y)Z, X) + S(Z, P(\xi, Y)X) = 0 \] (58)

Putting \( X = \xi \) and using (28) and (29) in (10), we obtain
\[ P(\xi, Y)Z = R(\xi, Y)Z - \left( \frac{1}{4n} \right) [S(Y, Z)\xi - 2(5n - 1)g(Y, Z)\xi + 2(2n + 1)\eta(Y)\eta(Z)\xi] \] (59)

Again putting \( X = \xi \) in (26) and using (20), we get
\[ R(\xi, Y)Z = 2[\eta(Z)Y - g(Y, Z)\xi] \] (60)

Using (28), (29), (59) and (60) in (58), we obtain
\[ S(Y, Z) = \left( \frac{2(11n + 7)}{3} \right) g(Y, Z) + \left( \frac{4(1 - 2n)}{3} \right) \eta(Y)\eta(Z) \] (61)

Therefore, \( S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z). \) Where
\[ a = \left( \frac{2(11n + 7)}{3} \right), \quad b = \left( \frac{4(1 - 2n)}{3} \right) \]

This leads the following:
Theorem 5.1. If a Kenmotsu manifold with respect to the semi-symmetric metric connection satisfying $P.S = 0$, then the manifold is an $\eta$–Einstein manifold.

Since $a$ and $b$ are both constant, by Lemma 2.1, we get following:

Corollary 5.2. If a Kenmotsu manifold with respect to the semi-symmetric metric connection satisfying $P.S = 0$, then the manifold is an Einstein manifold.

References


