Weaker Form of $\delta$-open Sets via Ideals

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Abstract: In this paper, the notions of $\delta_I$-semi-open sets and $\delta_I$-semi-closed sets are introduced and investigated in ideal topological spaces.

MSC: 54A05.

Keywords: $\delta_I$-semi-open sets and $\delta_I$-semi-closed sets.

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1. Introduction

The concept of ideals in topological spaces has been introduced and studied by Kuratowski [12] and Vaidyanathasamy [19]. An ideal $\mathcal{I}$ on a topological space $(X, \tau)$ is a non-empty collection of subsets of $X$ which satisfies (i) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. For a subset $A$ of $X$, $A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$ is called the local function [12] of $A$ with respect to $\mathcal{I}$ and $\tau$. We simply write $A^*$ in case there is no chance for confusion. A Kuratowski closure operator $cl^*(\cdot)$ for a topology $\tau^*(\mathcal{I}, \tau)$ finer than $\tau$ is defined by $cl^*(A) = A \cup A^*$ [19]. Throughout this paper, $(X, \tau, \mathcal{I})$ (or simply $X$), always mean ideal topological space on which no separation axiom is assumed. In this paper we introduce weaker form of $\delta$-open sets in ideal topological spaces.

2. Preliminaries

Definition 2.1 ([18]). A subset $A$ of a topological space $(X, \tau)$ is said to be

(1). regular open if $A = int(cl(A))$,

(2). regular closed if $A = cl(int(A))$.

A is called $\delta$-open [20] if for each $x \in A$, there exists a regular open set $G$ such that $x \in G \subseteq A$. The complement of a $\delta$-open set is called $\delta$-closed. A point $x \in X$ is called a $\delta$-cluster point of $A$ if $\text{int}(cl(V)) \cap A \neq \emptyset$ for each open set $V$ containing $X$. The set of all $\delta$-cluster points of $A$ is called the $\delta$-closure of $A$ and is denoted by $\delta cl(A)$. The $\delta$-interior of $A$ is the union of all regular open sets of $X$ contained in $A$ and it is denoted by $\delta int(A)$.

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Definition 2.2. A subset $A$ of a topological space $(X, \tau)$ is said to be

1. **semi-open** [13] if $A \subseteq \text{cl}(\text{int}(A))$,
2. **pre-open** [14] if $A \subseteq \text{int}(\text{cl}(A))$,
3. **$\alpha$-open** [15] if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$,
4. **$\beta$-open** [2] if $A \subseteq \text{cl}(\text{int}(A))$,
5. **$b$-open** [4] if $A \subseteq \text{int}(\text{cl}(A)) \cup \text{cl}(\text{int}(A))$,
6. **$\delta$-semi-open** [16] if $A \subseteq \text{cl}(\text{int}(\delta(A)))$,
7. **$\delta$-pre-open** [17] if $A \subseteq \text{int}(\text{cl}(\delta(A)))$,
8. **$\delta$-$\beta$-open** [10] if $A \subseteq \text{cl}(\text{int}(\delta(A)))$,
9. **$\alpha$-$\delta$-open** [8] if $A \subseteq \text{int}(\text{cl}(\delta(A)))$.

Definition 2.3. A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be

1. **$I$-open** [3] if $A \subseteq \text{int}(A^*)$,
2. **$\delta$-$I$-open** [1] if $\text{int}(\text{cl}^*(A)) \subseteq \text{cl}^*(\text{int}(A))$,
3. **pre-$I$-open** [6] if $A \subseteq \text{int}(\text{cl}^*(A))$,
4. **semi-$I$-open** [9] if $A \subseteq \text{cl}^*(\text{int}(A))$,
5. **$\alpha$-$I$-open** [9] if $A \subseteq \text{int}(\text{cl}^*(\text{int}(A)))$,
6. **$\beta$-$I$-open** [9] if $A \subseteq \text{cl}(\text{int}(\text{cl}^*(A)))$,
7. **$b$-$I$-open** [5] if $A \subseteq \text{int}(\text{cl}^*(A)) \cup \text{cl}^*(\text{int}(A))$,
8. **$\alpha$-$\delta$-$I$-open** [7] if $A \subseteq \text{int}(\text{cl}^*(\text{int}(\delta(A))))$,
9. **$\beta$-$\delta$-$I$-open** [7] if $A \subseteq \text{cl}^*(\text{int}(\delta(A)))$,
10. **$t$-$I$-set** [10] if $\text{int}(\text{cl}^*(A)) = \text{int}(A)$,
11. **$\delta_3$-$t$-set** [10] if $\text{cl}(\text{int}(\delta(A))) = \text{int}(A)$.

Lemma 2.4. [11] Let $(X, \tau, I)$ be an ideal topological space and let $A \subseteq X$. Then $U \in \tau \Rightarrow U \cap A^* \subseteq (U \cap A)^*$.

3. **$\delta_I$-semi-open Sets**

Definition 3.1. A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be **$\delta_I$-semi-open** if $A \subseteq \text{cl}^*(\text{int}(\delta(A)))$.

The family of all $\delta_I$-semi-open sets of $(X, \tau, I)$ is denoted by $\delta_I\text{SO}(X)$.

Theorem 3.2. Every $\delta$-open set is $\delta_I$-semi-open.

Theorem 3.3. For a space $(X, \tau, I)$, the following hold:
(1) Every $\delta_I$-semi-open set is semi-open, $\beta$-open and $b$-open.

(2) Every $\delta_I$-semi-open set is $\delta$-semi-open and $\delta$-open.

(3) Every $\delta_I$-semi-open set is $\delta-I$-open, semi-$I$-open, $\beta-I$-open, $b-I$-open and $\beta_I^*$-open.

**Remark 3.4.** The converses of the above theorems need not be true as seen from the following examples.

**Example 3.5.** Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b, c\}, X\}$, and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}\}$. Then $A = \{b\}$ is semi-open, $\beta$-open, $b$-open, $\delta-I$-open, $\delta$-open, semi-$I$-open, $\beta-I$-open, $b-I$-open and $\beta_I^*$-open but it is not $\delta_I$-semi-open.

**Example 3.6.** Let $X = \{a, b, c, d\}$ with topologies $\tau = \{\emptyset, \{d\}, \{a, c\}, \{a, c, d\}, X\}$, and $\mathcal{I} = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Then the set $A = \{b, d\}$ is $\delta$-semi-open but not $\delta_I$-semi-open.

**Example 3.7.** Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$, and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $A = \{a, b\}$ is $\delta_I$-semi-open but not $\delta$-open.

**Remark 3.8.** From the following examples, we see that in a space $(X, \tau, \mathcal{I})$,

(1). The notions of $\delta_I$-semi-open sets and open (resp.$\mathcal{I}$-open) sets are independent.

(2). The notions of $\delta_I$-semi-open sets and pre open (resp.$\delta$-pre open and $\mathcal{I}$-open) sets are independent.

(3). The notions of $\delta_I$-semi-open sets and $\alpha$-open (resp.$\alpha$-$\mathcal{I}$-open) sets are independent.

(4). The notions of $\delta_I$-semi-open sets and $\alpha$-$\mathcal{I}$-open sets (resp.$\mathcal{I}$-sets and $\delta$-$\mathcal{I}$-sets) are independent.

**Example 3.9.** In Example 3.5, the set $A = \{b\}$ is open and $\mathcal{I}$-open but not $\delta_I$-semi-open. In Example 3.7, the set $B = \{a, c\}$ is $\delta_I$-semi-open but not open, $\mathcal{I}$-open, pre open, $\delta$-pre open, pre-$\mathcal{I}$-open and $\delta$-$\mathcal{I}$-set.

**Example 3.10.** In Example 3.5, the set $A = \{b\}$ is pre open, $\delta$-pre open, pre-$\mathcal{I}$-open $\alpha$-open and $\alpha$-$\mathcal{I}$-open but not $\delta_I$-semi-open. In Example 3.7, the set $B = \{a\}$ is $\delta$-$\mathcal{I}$-set and $\mathcal{I}$-set but not $\delta_I$-semi-open.

**Example 3.11.** In Example 3.7, the set $B = \{a, b\}$ is $\delta_I$-semi-open but not $\alpha$-open and $\alpha$-$\mathcal{I}$-open. Moreover, the set $C = \{b, c\}$ is $\delta_I$-semi-open but not $\mathcal{I}$-set.

**Remark 3.12.** If $A$ is $\delta_I$-semi-open and open, then it is $\alpha$-open and $\alpha$-$\mathcal{I}$-open.

**Theorem 3.13.** If $A_\alpha \in \delta_I SO(X)$ for each $\alpha \in \Delta$, then $\bigcup_{\alpha \in \Delta} \{A_\alpha : \alpha \in \Delta\} \in \delta_I SO(X)$.

**Proof.** Let $A_\alpha$ be $\delta_I$-semi-open for each $\alpha \in \Delta$.

Then, we have $A_\alpha \subseteq \text{cl}^*(\text{ints}(A_\alpha))$. Thus

$$\bigcup_{\alpha \in \Delta} A_\alpha \subseteq \bigcup_{\alpha \in \Delta} \text{cl}^*(\text{ints}(A_\alpha)) = \bigcup_{\alpha \in \Delta} \text{ints}(A_\alpha) \cup (\text{ints}(A_\alpha))^* \subseteq \text{ints}(\bigcup_{\alpha \in \Delta} A_\alpha) \cup (\text{ints}(\bigcup_{\alpha \in \Delta} A_\alpha))^* = \text{cl}^*(\text{ints}(\bigcup_{\alpha \in \Delta} A_\alpha))$$

This shows that $\bigcup_{\alpha \in \Delta} \{A_\alpha : \alpha \in \Delta\} \in \delta_I SO(X)$. □

**Remark 3.14.** From the following example, we observe that the intersection of two $\delta_I$-semi-open sets need not be $\delta_I$-semi-open.

**Example 3.15.** In Example 3.7, the sets $A = \{a, b\}$ and $B = \{a, c\}$ are $\delta_I$-semi-open sets but $A \cap B = \{a\}$ is not $\delta_I$-semi-open.

**Theorem 3.16.** For a subset $A$ of a space $(X, \tau, \mathcal{I})$,
(1) If \( \mathcal{I} = \emptyset \), then \( A \) is \( \delta_2 \)-semi-open if and only if \( A \) is \( \delta \)-semi-open.

(2) If \( \mathcal{I} = P(X) \), then \( A \) is \( \delta_2 \)-semi-open if and only if \( A \) is \( \delta \)-open.

**Proof.** The proof of (1) follows from the fact that \( A^*(\{\emptyset\}) = cl(A) \) and (2) follows from the fact that \( A^*(P(X)) = \{\emptyset\} \).

**Corollary 3.17.** If \( \mathcal{I} = P(X) \) and \( A \) be \( \delta_2 \)-semi-open then \( A \) is \( \alpha \)-open (resp. \( \delta \)-pre open and \( \alpha \)-open).

**Theorem 3.18.** A subset \( A \) of a space \((X, \tau, \mathcal{I})\) is \( \delta_2 \)-semi-open if and only if \( cl^*(A) = cl^*(\text{ints}_A(A)) \).

**Proof.** Let \( A \) be \( \delta_2 \)-semi-open, we have \( A \subseteq cl^*(\text{ints}_A(A)) \). Then \( cl^*(A) \subseteq cl^*(\text{ints}_A(A)) \). Hence \( cl^*(A) = cl^*(\text{ints}_A(A)) \). Conversely, \( A \subseteq cl^*(A) = cl^*(\text{ints}_A(A)) \). Thus \( A \) is \( \delta_2 \)-semi-open.

**Theorem 3.19.** A subset \( A \) of a space \((X, \tau, \mathcal{I})\) is \( \delta_2 \)-semi-open if and only if there exists a \( \delta \)-open set \( U \) such that \( U \subseteq A \subseteq cl^*(U) \).

**Proof.** Suppose that \( A \) is \( \delta_2 \)-semi-open. Then we have \( A \subseteq cl^*(\text{ints}_A(A)) \). Put \( U = \text{ints}_A(A) \). We have \( U \) is \( \delta \)-open and \( U \subseteq A \subseteq cl^*(U) \). Conversely, let \( U \) be \( \delta \)-open set such that \( U \subseteq A \subseteq cl^*(U) \). Thus \( cl^*(\text{ints}_U(U)) \subseteq cl^*(\text{ints}_A(A)) \) and so \( A \subseteq cl^*(U) \subseteq cl^*(\text{ints}_A(A)) \). Therefore \( A \) is \( \delta_2 \)-semi-open.

**Corollary 3.20.** If a set \( A \) is \( \delta_2 \)-semi-open, then there exists a \( \delta \)-open set \( U \) such that \( U \subseteq A \subseteq cl(A) \).

**Proposition 3.21.** If \( U \) and \( V \) are \( \delta \)-open sets and \( A \) is \( \delta_2 \)-semi-open set such that \( U \cap V = \emptyset \) then \( A \cap U = \emptyset \).

**Proof.** Since \( U \) is \( \delta \)-open and \( U \cap V = \emptyset \), we have \( cl^*(V) \subseteq U^c \). Thus \( A \subseteq U^c \). Hence \( A \cap U = \emptyset \).

**Theorem 3.22.** If \( A \) is an \( \delta_2 \)-semi-open set in \((X, \tau, \mathcal{I})\) and \( A \subseteq B \subseteq cl^*(A) \) then \( B \) is \( \delta_2 \)-semi-open.

**Proof.** Since \( A \) is \( \delta_2 \)-semi-open, then there exists a \( \delta \)-open set \( U \) such that \( U \subseteq A \subseteq cl^*(U) \). Then, we have \( U \subseteq A \subseteq B \subseteq cl^*(A) \subseteq cl^*(U) \) and hence \( U \subseteq B \subseteq cl^*(U) \). By Proposition 3.19, we obtain \( B \) is \( \delta_2 \)-semi-open.

**Theorem 3.23.** If \( A \in \delta_2 SO(X) \) and \( B \) is \( \delta \)-open then \( A \cap B \in \delta_2 SO(X) \).

**Proof.** Let \( A \in \delta_2 SO(X) \) and \( B \) be \( \delta \)-open. Then \( A \subseteq cl^*(\text{ints}_A(A)) \). By Lemma 2.4, we have \( A \cap B \subseteq cl^*(\text{ints}_A(A)) \cap B = \text{ints}_A(A) \cap B \cup [(\text{ints}_A(A))^c \cap B] \subseteq \text{ints}_A(A) \cap B \cup [(\text{ints}_A(A))^c \cap B]^c = \text{ints}_A(A \cap B) \cup [(\text{ints}_A(A))^c \cap B]^c = cl^*(\text{ints}_A(A \cap B)) \) Thus \( A \cap B \subseteq \delta_2 SO(X) \).

**Definition 3.24.** A subset \( A \) of an ideal topological space \((X, \tau, \mathcal{I})\) is said to be \( \delta_2 \)-semi-closed if its complement is \( \delta_2 \)-semi-open.

**Theorem 3.25.** A subset \( A \) of a space \((X, \tau, \mathcal{I})\) is \( \delta_2 \)-semi-closed if and only if \( \text{ints}(cl^*\text{ints}_A(A)) \subseteq A \).

**Theorem 3.26.** If a subset \( A \) of a space \((X, \tau, \mathcal{I})\) is \( \delta_2 \)-semi-closed then \( \text{ints}(cl^*(A)) \subseteq A \).

**Proof.** Since \( A \) is \( \delta_2 \)-semi-closed, \( X - A \in \delta_2 SO(X) \). Now, we have \( X - A \subseteq cl^*(\text{ints}(X - A)) \subseteq cl(\text{ints}(X - A)) = X - int_3(\text{ints}(X - A)) \subseteq X - int_3(cl^*(A)) \). Therefore, \( int_3(cl^*(A)) \subseteq A \).

**Corollary 3.27.** Let \( A \) be a subset of a space \((X, \tau, \mathcal{I})\) such that \( X - \text{ints}(cl^*(A)) = cl^*(\text{ints}(X - A)) \). Then \( A \) is \( \delta_2 \)-semi-closed if and only if \( \text{ints}(cl^*(A)) \subseteq A \).

**Proof.** **Necessity.** This is an immediate consequence of Theorem 3.26. **Sufficiency.** Let \( \text{ints}(cl^*(A)) \subseteq A \). Then \( X - A \subseteq X - [\text{ints}(cl^*(A))] = cl^*(\text{ints}(X - A)) \). Thus \( X - A \) is \( \delta_2 \)-semi-open and so \( A \) is \( \delta_2 \)-semi-closed.
Theorem 3.28. A set $A$ is $\delta_\tau$-semi-closed if and only if there exists a $\delta$-closed set $C$ such that $\text{int}^*(C) \subseteq A \subseteq C$.

Proof. Obvious from definition and Theorem 3.19.

Theorem 3.29. If $U$ is $\delta$-open and $V \in \delta_\tau SO(X, \tau, \tau)$ then $U \cap V \in \delta_\tau SO(U, \tau|U, \tau|U)$.

Proof. Since $U$ is $\delta$-open, we have $\delta \text{int}_\tau(A) = \text{int}_\tau(A)$ for any subset $A$ of $U$.

Now, $U \cap V \subseteq U \cap \text{cl}^*(\text{int}_\tau(V)) = U \cap [\text{int}_\tau(V) \cup (\text{int}_\tau(V))^*] = [(U \cap \text{int}_\tau(V)) \cup [U \cap (\text{int}_\tau(V))^*] \cap U \subseteq [U \cap (\text{int}_\tau(V))^* \cap U \cup [U \cap (\text{int}_\tau(V))^*] = [U \cap (\text{int}_\tau(U \cap V))] \cup [U \cap (\text{int}_\tau(U \cap V))^*] = [\text{int}_\tau(U \cap V)] \cup [\text{int}_\tau(U \cap V))^*] \cap U) = \text{cl}_\tau(\text{int}_\tau(U \cap V)).$ Thus $U \cap V \in \delta_\tau SO(U, \tau|U, \tau|U)$.

Remark 3.30. Intersection of two $\delta_\tau$-semi-closed sets is $\delta_\tau$-semi-closed in $(X, \tau, \tau)$.

Example 3.31. Union of two $\delta_\tau$-semi-closed sets need not be $\delta_\tau$-semi-closed as seen from this example. In Example 3.7, the sets $A = \{b\}$ and $B = \{c\}$ are $\delta_\tau$-semi-closed sets but $A \cup B = \{b, c\}$ is not $\delta_\tau$-semi-closed.

Definition 3.32. Let $A$ be a subset of an ideal bitopological space $(X, \tau_1, \tau_2, \tau)$ and $x$ be a point of $X$. Then

1. $x$ is called an $\delta_\tau$-semi-cluster point of $A$ if $A \cap U \neq \emptyset$ for every $U \in \delta_\tau SO(X)$.

2. The family of all $\delta_\tau$-semi-cluster points of $A$ is called $\delta_\tau$-semi-closure of $A$ and is denoted by $\text{scl}_\delta(A)$.

Theorem 3.33. For subsets $A, B \subseteq (X, \tau, \tau)$, the following hold:

1. $\text{scl}_\delta(A) = \bigcap\{F \subseteq X : A \subseteq F$ and $F$ is $\delta_\tau$-semi-closed\}.

2. $\text{scl}_\delta(A)$ is the smallest $\delta_\tau$-semi-closed subset of $X$ containing $A$.

3. If $A \subseteq B$, then $\text{scl}_\delta(A) \subseteq \text{scl}_\delta(B)$.

4. $A$ is $\delta_\tau$-semi-closed if and only if $A = \text{scl}_\delta(A)$.

5. $\text{scl}_\delta(\text{scl}_\delta(A)) = \text{scl}_\delta(A)$.

6. $\text{scl}_\delta(A \cap B) \subseteq \text{scl}_\delta(A) \cap \text{scl}_\delta(B)$.

7. $\text{scl}_\delta(A) \cup \text{scl}_\delta(B) \subseteq \text{scl}_\delta(A \cup B)$.

Proof. (1). Suppose that $x \notin \text{scl}_\delta(A)$. Then there exists $U \in \delta_\tau SO(X)$ such that $U \cap A = \emptyset$. Then, we have $U^c$ is $\delta_\tau$-semi-closed set containing $A$ and $x \notin U^c$. Thus $x \notin \bigcap\{F \subseteq X : A \subseteq F$ and $F$ is $\delta_\tau$-semi-closed\}. Conversely, suppose there exists $F \in \delta_\tau SC(X)$ such that $A \subseteq F$ and $x \notin F$. Then $F^c$ is $\delta_\tau$-semi-open set containing $x$, we have $F^c \cap A = \emptyset$. Thus $x \notin \text{scl}_\delta(A)$. Hence $\text{scl}_\delta(A) = \bigcap\{F \subseteq X : A \subseteq F$ and $F$ is $\delta_\tau$-semi-closed\}. The other proofs are obvious.

Definition 3.34. Let $A$ be a subset of an ideal topological space $(X, \tau, \tau)$ and $x$ be a point of $X$. Then

1. $x$ is called an $\delta_\tau$-semi-interior point of $A$ if there exists $U \in \delta_\tau SO(X)$ such that $x \in U \subseteq A$.

2. The family of all $\delta_\tau$-semi-interior points of $A$ is called $\delta_\tau$-semi-interior of $A$ and is denoted by $\text{sint}_\delta(A)$.

Theorem 3.35. For subsets $A, B \subseteq (X, \tau, \tau)$, the following hold:

1. $\text{sint}_\delta(A) = \bigcup\{F \subseteq X : F \subseteq A$ and $F$ is $\delta_\tau$-semi-open\}.

2. $\text{sint}_\delta(A)$ is the largest $\delta_\tau$-semi-open subset of $X$ contained in $A$. 

(3). If $A \subseteq B$, then $sint_s(A) \subseteq sint_s(B)$.

(4). $A$ is $\delta_s$-semi-open if and only if $A = sint_s(A)$.

(5). $sint_s(sint_s(A)) = sint_s(A)$.

(6). $sint_s(A \cap B) \subseteq sint_s(A) \cap sint_s(B)$.

(7). $sint_s(A) \cup sint_s(B) \subseteq sint_s(A \cup B)$.

Proof. (1). Let $x \in \bigcup \{F \subseteq X : F \subseteq A \text{ and } F \text{ is } \delta_s\text{-semi-open}\}$. Then, there exists $F \in \delta_s\text{-SO}(X)$ such that $x \in F \subseteq A$ and hence $x \in sint_s(A)$. This shows that $\bigcup \{F \subseteq X : F \subseteq A \text{ and } F \text{ is } \delta_s\text{-semi-open}\} \subseteq sint_s(A)$. Let $x \in sint_s(A)$. Then there exists $F \in \delta_s\text{-SO}(X)$ such that $x \in F \subseteq A$, we obtain $x \in \bigcup \{F \subseteq X : F \subseteq A \text{ and } F \text{ is } \delta_s\text{-semi-open}\}$. This shows that $sint_s(A) \subseteq \bigcup \{F \subseteq X : F \subseteq A \text{ and } F \text{ is } \delta_s\text{-semi-open}\}$. Therefore, we obtain $sint_s(A) = \bigcup \{F \subseteq X : F \subseteq A \text{ and } F \text{ is } \delta_s\text{-semi-open}\}$. The other proofs are obvious.

Theorem 3.36. For a subset $A \subseteq (X, \tau, I)$, the following hold:

(1). $scl_s(X - A) = X - sint_s(A)$.

(2). $sint_s(X - A) = X - scl_s(A)$.

References


