New Results on Edge Pair Sum Graphs

Research Article

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Abstract: Let $G$ be a $(p,q)$ graph. An injective map $f: E(G) \to \{\pm 1, \pm 2, \ldots, \pm q\}$ is said to be an edge pair sum labeling if the induced vertex function $f^* : V(G) \to Z - \{0\}$ defined by $f^*(v) = \sum_{e \in E_v} f(e)$ is one-one where $E_v$ denotes the set of edges in $G$ that are incident with a vertex $v$ and $f^*(V(G))$ is either of the form $\{\pm k_1, \pm k_2, \ldots, \pm k_\frac{q}{2}\}$ or $\{\pm k_1, \pm k_2, \ldots, \pm k_\frac{q+1}{2}\}$ union $\{\pm 1, \pm 2, \ldots, \pm p\}$ according as $p$ is even or odd. A graph that admits an edge pair sum labeling is called an edge pair sum graph. In this paper we prove that the graphs jelly fish, Y-tree, theta, the subdivision of spokes in wheel SS($W_n$), $P_m + 2K_1$, $C_4 \times P_m$, $P_6 \odot K_m^e$ admit edge pair sum labeling.

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1. Introduction

Throughout this paper we consider finite, simple and undirected graph $G = (V(G), E(G))$ with $p$ vertices and $q$ edges. $G$ is also called a $(p,q)$ graph. We follow the basic notations and terminology of graph theory as in [2]. Ponraj et al. introduced the concept of pair sum labeling in [3]. An injective map $f: V(G) \to \{\pm 1, \pm 2, \ldots, \pm p\}$ is said to be a pair sum labeling of a graph $G(p,q)$ if the induced edge function $f_e : E(G) \to Z - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1, \pm k_2, \ldots, \pm k_\frac{q}{2}\}$ or $\{\pm k_1, \pm k_2, \ldots, \pm k_\frac{q+1}{2}\}$ union $\{\pm 1, \pm 2, \ldots, \pm p\}$ according as $q$ is even or odd. A graph that admits a pair sum labeling is called a pair sum graph. Analogous to pair sum labeling we define a new labeling called an edge pair sum labeling in [5] and further studied in [6-12]. In this paper we prove that the graphs jelly fish, Y-tree, theta, the subdivision of spokes in wheel SS($W_n$), $P_m + 2K_1$, $C_4 \times P_m$, $P_6 \odot K_m^e$ admit edge pair sum labeling. We use the following definitions in the subsequent section.

Definition 1.1. A Y-tree $Y_{n+1}$ is a graph obtained from the path $P_n$ by appending an edge to a vertex of the path $P_n$ adjacent to an end point [4].

Definition 1.2. The jelly fish graph $J(m,n)$ is obtained from a 4-cycle $v_1, v_2, v_3, v_4$ by joining $v_1$ and $v_3$ with an edge and appending $m$ pendent edges to $v_2$ and $n$ pendent edges to $v_4$.

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Definition 1.3. Take $k$ paths of length $l_1, l_2, l_3, \ldots, l_k$ where $k \geq 3$ and $l_i = 1$ for at most one $i$. Identify their end points to form a new graph. The new graph is called a generalized theta graph, and it is denoted by $\Theta(l_1, l_2, l_3, \ldots, l_k)$. In other words, $\Theta(l_1, l_2, l_3, \ldots, l_k)$ consists $k \geq 3$ pair wise internally disjoint paths of length $l_1, l_2, l_3, \ldots, l_k$ that share a pair of common end points $u$ and $v$. If each $l_i (i = 1, 2, \ldots, k)$ is equal to $l$, we will write $\Theta(l^k)$.

2. Main results

Theorem 2.1. For any positive integers $m$ and $n$, the jelly fish graph $J(m,n)$ has an edge pair sum labeling.

Proof. Let $V(J(m,n)) = V_1 \cup V_2$ where $V_1 = \{x, u, y, v\}$ and $V_2 = \{u, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. $E(J(m,n)) = E_1 \cup E_2$, where $E_1 = \{e_1^{\prime} = xu, e_2^{\prime} = uy, e_3^{\prime} = yv, e_4^{\prime} = vx, e_5^{\prime} = xy\}$ and $E_2 = \{e_i = uu_i, e_j = vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. Define $f : E(J(m,n)) \rightarrow \{\pm 1, \pm 2, \ldots, \pm (m + n + 5)\}$ as follows:

Case(i). $m$ and $n$ are odd.

Label the edges $e_1^{\prime}, e_2^{\prime}, e_3^{\prime}, e_4^{\prime}, e_5^{\prime}$ by $1, 3, 4, 2, 6$. Define $f(e_1) = 5 = -f(e_1^{\prime})$, for $1 \leq i \leq \frac{m-1}{2}$, $f(e_{i+1}) = (5 + i) = -f\left(\frac{e_{i+1}}{2}\right)$ and for $1 \leq i \leq \frac{n-1}{2}$, $f(e_{i+1}^{\prime}) = \frac{m+2+i}{2} = -f\left(\frac{e_{i+1}}{2}\right)$. For each edge label $f$, the induced vertex label $f^*$ is calculated as follows: $f^*(x) = -1$, $f^*(u) = 3 = -f^*(y)$, $f^*(v) = 5 = -f^*(v_1)$. For $1 \leq i \leq \frac{m-1}{2}$, $f^*(u_{i+1}) = (5 + i) = -f^*\left(\frac{u_{i+1}}{2}\right)$ and for $1 \leq i \leq \frac{n-1}{2}$, $f^*(v_{i+1}^{\prime}) = \frac{m+2+i}{2} = -f^*\left(\frac{v_{i+1}^{\prime}}{2}\right)$. Then $f^*(V(J(m,n))) = \{\pm 1, \pm 3, \pm 5, \pm 6, \pm 7, \pm 8, \ldots, \pm (m-2), \pm (m-11), \pm (m-13), \pm (m-15), \ldots, \pm (m+n+5)\}$. Hence $f$ is an edge pair sum labeling.

The example for the edge pair sum labeling of $J(3, 5)$ is shown in Figure 1.

![Figure 1. Edge pair sum labeling of $J(3, 5)$](image)

Case(ii). $m$ and $n$ are even.

Label the edges $e_1^{\prime}, e_2^{\prime}, e_3^{\prime}, e_4^{\prime}, e_5^{\prime}$ by $1, 3, 4, 6$. Define $f(e_1) = 1, f(e_2) = 4, f\left(\frac{e_1}{2}\right) = 2, f\left(\frac{e_2}{2}\right) = 7$, for $1 \leq i \leq \frac{m-2}{2}$, $f(e_{i+1}) = 8 + i = -f\left(\frac{e_{i+1}^{\prime}}{2}\right)$ and for $1 \leq i \leq \frac{n-2}{2}$, $f(e_{i+1}^{\prime}) = \frac{m+1+i}{2} = -f\left(\frac{e_{i+1}}{2}\right)$. For each edge label $f$, the induced vertex label $f^*$ is calculated as follows: $f^*(x) = -2$, $f^*(u) = -4 = -f^*(u_2), f^*(y) = 2 = -f^*(v), f^*(v) = 1 = -f^*(u_1)$, for $1 \leq i \leq \frac{m-2}{2}$, $f^*(u_{i+1}) = 8 + i = -f^*\left(\frac{u_{i+1}}{2}\right)$ and for $1 \leq i \leq \frac{n-2}{2}$, $f^*(v_{i+1}^{\prime}) = \frac{m+1+i}{2} = -f^*\left(\frac{v_{i+1}^{\prime}}{2}\right)$. Then we get $f^*(V(J(m,n))) = \{\pm 1, \pm 2, \pm 4, \pm 7, \pm 9, \pm 10, \pm 11, \ldots, \pm (m-14), \pm (m-16), \pm (m-18), \pm (m-20), \ldots, \pm (m+n+12)\}$. Hence $f$ is an edge pair sum labeling.

Case(iii). $m$ is odd and $n$ is even or $m$ is even and $n$ is odd.

Label the edges $e_1^{\prime}, e_2^{\prime}, e_3^{\prime}, e_4^{\prime}, e_5^{\prime}$ by $3, 4, 1, 1$. Define $f(e_1) = -6$, for $1 \leq i \leq \frac{m-1}{2}$, $f(e_{i+1}) = 6 + i = -f\left(\frac{e_{i+1}}{2}\right)$ and for $1 \leq i \leq \frac{n}{2}$, $f\left(\frac{e_i}{2}\right) = \frac{m+1+i}{2} = -f\left(\frac{e_i^{\prime}}{2}\right)$. For each edge label $f$, the induced vertex label $f^*$ is calculated as follows: $f^*(x) = 3 = -f^*(u), f^*(v) = 1 = -f^*(y), f^*(u_i) = 6$, for $1 \leq i \leq \frac{m-1}{2}$, $f^*(u_{i+1}) = 6 + i = f^*\left(\frac{u_{i+1}}{2}\right)$ and for $1 \leq i \leq \frac{n}{2}$, $f^*(v_i) = \frac{m+1(i+2)}{2} = -f^*\left(\frac{v_i}{2}\right)$. Therefore we get $f^*(V(J(m,n))) = \ldots$. 

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\{±1, ±3, ±7, ±8, ±9, ..., ±(\frac{m+11}{2}), ±(\frac{m+13}{2}), ±(\frac{m+15}{2}), ±(\frac{m+17}{2}), ..., ±(\frac{m+n+11}{2})\} \cup \{-6\}. Hence f is an edge pair sum labeling.

**Theorem 2.2.** For \(n \geq 4\), the Y-tree \(G = Y_{n+1}\) is an edge pair sum graph.

**Proof.** Let \(V(G) = \{v, u_i : 1 \leq i \leq n\}\) and \(E(G) = \{e_i = vu_2, e_i = u_iu_{i+1} : 1 \leq i \leq n - 1\}\) are the vertices and edges of the graph G. Define \(f : E(G) \rightarrow \{±1, ±2, ..., ±n\}\) as follows:

**Case(i).** \(n = 4\).

Let \(f(e_1') = -4 = f(e_1), f(e_2) = -1\) and \(f(e_3) = 2\). For each edge label \(f\), the induced vertex label \(f^*\) is calculated as follows: \(f^*(v) = -4 = f^*(u_1), f^*(u_2) = -1 = f^*(u_3)\) and \(f^*(u_4) = 2\). Hence \(f\) is an edge pair sum labeling.

**Case(ii).** \(n = 5\).

Let \(f(e_1') = 4 = -f(e_1), f(e_2) = -2, f(e_3) = -1\) and \(f(e_4) = 3\). For each edge label \(f\), the induced vertex label \(f^*\) is calculated as follows: \(f^*(v) = 4 = f^*(u_1), f^*(u_2) = -2 = f^*(u_4)\) and \(f^*(u_3) = -3 = f^*(u_5)\). Then we get \(f^*(V(G)) = \{±2, ±3, ±4\}\). Hence \(f\) is an edge pair sum labeling.

**Case(iii).** \(n\) is odd, take \(n = 2k + 1, k \geq 3\).

Let \(f(e_1') = -2, f(e_{4k+1}) = -1, f(e_{4k+2}) = 3, f(e_1) = 4 = -f(e_1'), \) for \(1 \leq i \leq k - 2\) \(f(e_{i+1}) = (2k + 1 - 2i)\) and for \(k + 2 \leq i \leq 2k - 1\) \(f(e_{i+1}) = (2k - 1 - 2i)\). For each edge label \(f\), the induced vertex label \(f^*\) is calculated as follows: \(f^*(v) = -4 = -f^*(u_1), f^*(u_2) = (2k - 1), f^*(u_k) = 3 = -f^*(u_{k+1}), f^*(u_{k+2}) = 2 = -f^*(u_{k+3}), f^*(u_n) = -(2k - 1),\) for \(2 \leq i \leq k - 2\) \(f^*(u_{i+1}) = 4(k + 1 - i)\) and \(f^*(u_{k+2}) = 4(k - 1)\). Then the vertex labeling are \(f^*(V(G)) = \{±2, ±3, ±4, ±(2k - 1), ±12, ±16, \ldots, ±4(k - 1)\}\). Hence \(f\) is an edge pair sum labeling.

**Case(iv).** \(n\) is even, take \(n = 2k, k \geq 3\).

Let \(f(e_{4k+1}) = 1, f(e_1) = 2, f(e_{4k-1}) = -5 = f(e_{4k+2}), f(e_1) = 4 = -f(e_1')\), for \(2 \leq i \leq k - 2\) \(f(e_i) = -(2k + 3 - 2i)\) and for \(k + 3 \leq i \leq 2k - 1\) \(f(e_i) = -(2k + 1 - 2i)\). For each edge label \(f\), the induced vertex label \(f^*\) is calculated as follows: \(f^*(v) = -4 = -f^*(u_1), f^*(u_2) = -(2k - 1), f^*(u_k) = 3 = -f^*(u_{k+1}), f^*(u_{k+2}) = 2 = -f^*(u_{k+3}), f^*(u_n) = -(2k - 1),\) for \(3 \leq i \leq k - 1\) \(f^*(u_i) = 4(-k + i - 2)\) and \(f^*(u_{k+2}) = 4(k - 1)\). Then the vertex labeling are \(f^*(V(G)) = \{±3, ±4, ±(2k - 1), ±12, ±16, \ldots, ±(4k - 1)\}\). Hence \(f\) is an edge pair sum labeling. The example for the edge pair sum graph labeling of \(Y_{6+1}\) is shown in Figure 2.

![Figure 2. Edge pair sum labeling of \(Y_{6+1}\)](image)

**Theorem 2.3.** The theta graph \(\Theta(\ell^{|m|})\) is an edge pair sum graph.

**Proof.** Let \(G(V, E) = \Theta(\ell^{|m|})\). Then \(|V(G)| = m(l - 1) + 2|\) and \(|E(G)| = ml\) are the vertices and edges of G. Where \(V(G) = \{u, v, u_i : 1 \leq i \leq m, 1 \leq j \leq l - 1\}\) and \(E(G) = \{e_i : 1 \leq i \leq m, 1 \leq j \leq l\}\).

**Case(i).** \(m\) is odd and \(l\) is even.

For \(1 \leq j \leq \frac{m-2}{2}\) \(f(e_j') = l+3-2j, f(e_j^2) = -2, f(e_j^{1+^2}) = -1, \) for \(\frac{m+2}{2} \leq j \leq l\) \(f(e_j') = l-1-2j\) and for \(1 \leq i \leq \frac{m-1}{2}; 1 \leq j \leq l\)
is shown in Figure 3. For each edge label \( f \), the induced vertex label \( f^* \) is calculated as follows:

\[
f^*(u) = l + 1 = -f^*(v), \quad f^*(u_1^{l-1}) = 3, \quad f^*(u_1^l) = -3, \quad f^*(u_1^{l+1}) = -6,
\]

for \( 1 \leq j \leq \frac{l-3}{2} \), \( f^*(u_1^l) = 2l + 4 - 4j \), for \( 1 \leq j \leq \frac{l-4}{2} \), \( f^*(u_1^{l+1}) = -(8 + 4j) \) and for \( 1 \leq i \leq \frac{m-1}{2}; 1 \leq j \leq (l-1) \), \( f^*(u_1^{m+1}) = 4l(i-1) + 6 + 4j = -f^*(u_1^{m+1}) \). Then \( f^*(V(G)) = \{\pm 3, \pm(l+1), \pm 12, \pm 16, \pm 20, \ldots, \pm 2l\} \cup \{\pm (4l(i-1) + 6 + 4j)\} |l \leq i \leq \frac{m-1}{2}; 1 \leq j \leq (l-1)\} \cup \{-6\}. Hence \( f \) is an edge pair sum labeling.

**Case(ii).** \( m \) and \( l \) are odd.

For \( 1 \leq i \leq \frac{m-1}{2}; 1 \leq j \leq l \), \( f(e_i^l) = -l - 2 - 2j \), \( f(e_i^{l+1}) = 2 \), \( f(e_i^{l+2}) = 1 \), \( f(e_i^{l+3}) = -3 \), for \( 1 \leq j \leq \frac{l-3}{2} \), \( f^*(u_1^l) = 2l + 2 + 4j \), for \( 1 \leq j \leq \frac{l-4}{2} \), \( f^*(u_1^{l+1}) = -(2l(i-1) + 4 + 4j) \), for \( 1 \leq j \leq \frac{l-5}{2} \), \( f^*(u_1^{l+2}) = 8 + 4j \) and for \( 1 \leq i \leq \frac{m-1}{2}; 1 \leq j \leq (l-1) \), \( f^*(u_1^{m+1}) = 4l(i-1) + 6 + 4j = -f^*(u_1^{m+1}) \). Then \( f^*(V(G)) = \{\pm 2, \pm 3, \pm l, \pm 12, \pm 16, \ldots, \pm (2l-1)\} \cup \{\pm (4l(i-1) + 6 + 4j)\} |l \leq i \leq \frac{m-1}{2}; 1 \leq j \leq (l-1)\}. Hence \( f \) is an edge pair sum labeling. The example for the edge pair sum graph labeling of \( \Theta(5^{[2]} \) is shown in Figure 3.

**Figure 3.** Edge pair sum graph labeling of \( \Theta(5^{[2]} \)

**Case(iii).** \( m \) is even and \( l \) is odd.

For \( 1 \leq i \leq \frac{m}{2}; 1 \leq j \leq l \), \( f(e_i^l) = 2l(i-1) + 2j \), for \( 1 \leq j \leq \frac{l-3}{2} \), \( f(e_i^{l+1}) = -(l(m-2i) + 2 + 2j) \). For each edge label \( f \), the induced vertex label \( f^* \) is calculated as follows:

\[
f^*(u) = -l - f^*(v), \quad f^*(u_1^l) = -3, \quad f^*(u_1^{l+1}) = 8l(i-1) + 6 + 8j - 8), \quad f^*(u_1^{l+2}) = -l(2m - 8i + 8j - 2) \)

and \( f^*(u_1^{m+1}) = -l(2m - 8i + 8j + 2) \). Then \( f^*(V(G)) = \{\pm (2l(i-1) + 6 + 8j)\} \cup \{8l(i-1) + 10 + (8j - 8)\} \cup \{8l(i-1) + 10 + (8j - 8)\} \cup \{8l(i-1) + 10 + (8j - 8)\} \cup \{8l(i-1) + 10 + (8j - 8)\} \cup \{8l(i-1) + 10 + (8j - 8)\} \cup \{8l(i-1) + 10 + (8j - 8)\} \cup \{-6\}. Hence \( f \) is an edge pair sum labeling.

**Case(iv).** \( m \) and \( l \) are even if \( m \geq 4 \).

For \( 1 \leq i \leq \frac{m}{2}; 1 \leq j \leq l \), \( f(e_i^l) = 4l(i-1) + 4j + 3 \), \( f(e_i^{l+1}) = 8l(i-1) + 4j + 1 \), for \( 1 \leq j \leq \frac{l-3}{2} \), \( f^*(u_1^l) = -(l(m-4i + 4j + 4j + 3) \) and \( f^*(u_1^{l+1}) = -(l(m-4i + 4j + 4j + 3) \). For each edge label \( f \), the induced vertex label \( f^* \) is calculated as follows:

\[
f^*(u) = -(2m(l-1)), \quad f^*(u_1^l) = (2m - 8i + 8j - 8) \) and \( f^*(u_1^{m+1}) = -(l(2m - 8i + 8j - 8) \). Then \( f^*(V(G)) = \{\pm (2l(i-1) + 6 + 8j)\} \cup \{8l(i-1) + 10 + (8j - 8)\} \cup \{8l(i-1) + 10 + (8j - 8)\} \cup \{8l(i-1) + 10 + (8j - 8)\} \cup \{-6\}. Hence \( f \) is an edge pair sum labeling.

**Theorem 2.4.** The subdivision of spokes in wheel \( SS(W_n) \) admits edge pair sum labeling.

**Proof.** Let \( V(SS(W_n)) = \{u_0, u_i, v_i : 1 \leq i \leq n\} \) and \( E(SS(W_n)) = \{e_i = u_i u_{i+1} : 1 \leq i \leq (n-1), e_n = u_n u_1, e'_i = u_i v_i \) and \( e''_i = u_i v_i : 1 \leq i \leq n\} \) be the vertices and edges of the graph \( SS(W_n) \). Define the edge labeling \( f : E(SS(W_n)) \to \{\pm 1, \pm 2, \ldots, \pm 3n\} \) by considering the following two cases:

**Case (i)** \( n \) is even.

For \( 1 \leq i \leq n \), \( f(e_i') = -i \) and \( f(e_i'') = -(3n - 2i + 1) \), for \( 1 \leq i \leq n \), \( f(e_i') = n + i \) and \( f(e_i'') = 2n \). For each edge label
Theorem 2.5. The graph $P_m + 2K_1$ is an edge pair sum graph if $m \geq 3$.

Proof. Let $V(P_m + 2K_1) = \{u_0, v_0, u_i : 1 \leq i \leq m\}$ and $E(P_m + 2K_1) = \{e_i = u_0 u_i, e' = v_0 u_i : 1 \leq i \leq m, e'' = u_i u_{i+1} : 1 \leq i \leq m - 1\}$ are the vertices and edges of the graph $P_m + 2K_1$. Define $f : E(P_m + 2K_1) \to \{\pm 1, \pm 2, \ldots, \pm (3m - 1)\}$ as follows:

Case (i) $m$ is even.

Subcase (a). $m = 4$.

For $1 \leq i \leq 4$ $f(e_i) = 2 + 2i = -f(e'_i), f(e''_i) = -2, f(e''_2) = -1$ and $f(e''_3) = 3$. For each edge label $f$, the induced vertex label $f^*$ is calculated as follows: $f^*(u_1) = -2 = -f^*(u_3), f^*(u_2) = -3 = -f^*(u_4), f^*(u_0) = 28 = -f^*(v_0)$. Then $f^*(V(P_m + 2K_1)) = \{\pm 2, \pm 3, \pm 28\}$. Hence $f$ is an edge pair sum labeling. The example for the edge pair sum graph labeling of $P_m + 2K_1$ for $m = 4$ is shown in Figure 6.
Figure 6. Edge pair sum labeling of $P_4 + 2K_1$

Subcase (b). $m$ is even, $m \geq 6$.

Define $f(e''_{\frac{m-1}{2}}) = -2$, $f(e''_{\frac{m}{2}}) = -1$, $f(e''_{2}) = 3$, for $1 \leq i \leq \frac{m}{2} - 2$ $f(e''_i) = m + 1 - 2i$, for $\frac{m}{2} + 2 \leq i \leq m$ $f(e''_i) = m - 1 - 2i$ and for $1 \leq i \leq m$ $f(e_i) = (2 + 2i) = -f(e'_i)$. For each edge label $f$, the induced vertex label $f^*$ is calculated as follows: $f^*(u_1) = m - 1 = -f^*(u_m)$, $f^*(u_{\frac{m}{2}-1}) = 3 = -f^*(u_{\frac{m}{2}})$, $f^*(u_{\frac{m}{2}+1}) = 2 = -f^*(u_{m+2})$, $f^*(u_0) = m^2 + 3m = -f^*(v_0)$, for $2 \leq i \leq \frac{m}{2} - 2$ $f^*(u_i) = 4\left(\frac{m}{2} + 1 - i\right)$ and for $\frac{m}{2} + 3 \leq i \leq m - 1$ $f^*(u_i) = 4\left(\frac{m}{2} - i\right)$. Then $f^*(V(P_m + 2K_1)) = \{\pm 2, \pm 3, \pm (m - 1), \pm (m^2 + 3m), \pm 12, \pm 16, \pm 20, \ldots, \pm 2(m - 2)\}$. Hence $f$ is an edge pair sum labeling.

Case (ii) $m$ is odd.

Subcase (a). $m = 3$.

Define $f(e'_1) = -1$, $f(e'_2) = 2$ and for $1 \leq i \leq 3$ $f(e_i) = 2 + 2i = -f(e'_i)$. For each edge label $f$, the induced vertex label $f^*$ is calculated as follows: $f^*(u_1) = -1 = -f^*(u_2)$, $f^*(u_3) = 2$ and $f^*(u_0) = 18 = -f^*(v_0)$. Then $f^*(V(P_m + 2K_1)) = \{\pm 1, \pm 18\} \cup \{2\}$. Hence $f$ is an edge pair sum labeling.

Subcase (b). $m$ is odd, $m \geq 5$.

Define $f(e''_{\frac{m-1}{2}}) = 1$, $f(e''_{\frac{m}{2}-1}) = 2$, $f(e''_{\frac{m-3}{2}}) = -5 = -f(e''_{\frac{m+3}{2}})$, for $1 \leq i \leq \frac{m-5}{2}$ $f(e''_i) = -(m + 2 - 2i)$, for $\frac{m-5}{2} \leq i \leq m - 1$ $f(e'_i) = (-m + 2 + 2i)$ and for $1 \leq i \leq m$ $f(e'_i) = -(2 + 2i) = -f(e_i)$. For each edge label $f$, the induced vertex label $f^*$ is calculated as follows: $f^*(u_1) = -m = -f^*(u_m)$, $f^*(u_{\frac{m-1}{2}}) = -3 = -f^*(u_{\frac{m+1}{2}})$, $f^*(u_{\frac{m-3}{2}}) = 6$, for $2 \leq i \leq \frac{m-3}{2}$ $f^*(u_i) = 2(-m + 3 + 2i)$, for $\frac{m-3}{2} \leq i \leq m - 1$ $f^*(u_i) = -2(m - 1 - 2i)$ and $f^*(u_0) = m^2 + 3m = -f^*(v_0)$. Then $f^*(V(P_m + 2K_1)) = \{\pm 3, \pm (m^2 + 3m), \pm m, \pm 12, \pm 16, \pm 20, \ldots, \pm 2(m - 1)\} \cup \{6\}$. Hence $f$ is an edge pair sum labeling. The example for the edge pair sum graph labeling of $P_m + 2k_1$ for $m = 7$ is shown in Figure 7.

Figure 7. Edge pair sum labeling of $P_7 + 2K_1$

\[\square\]

Theorem 2.6. The graph $C_4 \times P_m$ is an edge pair sum graph.

Proof. Let $V(C_4 \times P_m) = \{u_{ij} : 1 \leq i \leq m, 1 \leq j \leq 4\}$ and $E(C_4 \times P_m) = \{e_{ij} = u_{ij}u_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq 3; e_{ij} =
$u_iu_j : 1 \leq i \leq m; e_{ij} = u_{ij}u_{i+1,j} : 1 \leq i \leq m, 1 \leq j \leq 4$ are the vertices and edges of the graph $C_4 \times P_m$. Define $f : E(C_4 \times P_m) \to \{\pm 1, \pm 2, \ldots, \pm (8m - 4)\}$ as follows:

**Theorem 2.7.** The graph $P_n \odot K'_m$ is an edge pair sum graph if $m$ is odd.

**Proof.** Let $V(P_n \odot K'_m) = \{u_i, v_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(P_n \odot K'_m) = \{e_i = u_iu_{i+1} : 1 \leq i \leq (n - 1), e_{ij} = u_{ij}v_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$ be the vertices and edges of the graph $P_n \odot K'_m$. Define $f : (E(P_n \odot K'_m)) \to \{\pm 1, \pm 2, \ldots, \pm (mn + n - 1)\}$ as follows:

**Case (i) $n$ is even.**

Subcase (a). $n = 2$.

Define $f(e_1) = 2$, $f(e_{11}) = 1$, $f(e_{12}) = -3$, for $1 \leq i \leq 2$ and $2 \leq j \leq m+1$, $f(e_{ij}) = 2 + \frac{m+1}{2}(i-1) + j = -f(e_{i,m-1+2j})$. For each edge label $f$, the induced vertex label $f^*$ is calculated as follows: $f^*(u_{11}) = -7 = -f^*(u_{13}), f^*(u_{12}) = 8 = f^*(u_{14}), f^*(u_{21}) = -13 = f^*(u_{23}) and f^*(u_{22}) = 4 = f^*(u_{24})$. Then we get $f^*(V(C_4 \times P_m)) = \{\pm 4, \pm 7, \pm 8, \pm 13\}$. Hence $f$ is an edge pair sum labeling.

**Case (ii) $m \geq 3$.**

Define $f(e_{11}) = 1 = -f(e_{13}), f(e_{12}) = 2 = -f(e_{14}), f(e_{21}) = -4 = -f(e_{23}), f(e_{22}) = 3 = -f(e_{24}), f(e'_{11}) = -6 = f(e'_{13})$ and $f(e'_{12}) = 5 = f(e'_{14})$. For each edge label $f$, the induced vertex label $f^*$ is calculated as follows: $f^*(u_{11}) = -7 = f^*(u_{13}), f^*(u_{12}) = 8 = f^*(u_{14}), f^*(u_{21}) = -13 = -f^*(u_{23}) and f^*(u_{22}) = 4 = -f^*(u_{24})$. Then we get $f^*(V(C_4 \times P_m)) = \{\pm 4, \pm 7, \pm 8, \pm 13\}$. Hence $f$ is an edge pair sum labeling.

**Figure 8.** Edge pair sum labeling of $C_4 \times P_4$

**Theorem 2.7.** The graph $P_n \odot K'_m$ is an edge pair sum graph if $m$ is odd.
calculated as follows: \( f^*(u_1) = 2 = -f^*(v_2), f^*(u_2) = 3 = f^*(u_4), f^*(v_1) = 4 = f^*(v_3), f^*(v_2) = 6 = -f^*(v_4) \) and for \( 1 \leq i \leq n \) and \( 2 \leq j \leq \frac{m+1}{2} \), \( f^*(v_{ij}) = 5 + \frac{m-1}{2}(i - 1) + j = -f^*(v_{i, \frac{m-1}{2}+j+1}) \). Then we get \( f^*(V(P_n \ominus K_m^c)) = \{ \pm 2, \pm 3, \pm 4, \pm 6 \} \cup \{ \pm 5 + \frac{m-1}{2}(i - 1) + j \} | 1 \leq i \leq n, 2 \leq j \leq \frac{m+1}{2} \}. \) Hence \( f \) is an edge pair sum labeling. The example for the edge pair sum graph labeling of \( P_n \ominus K_m^c \) for \( n = 4 \) and \( m = 3 \) is shown in Figure 9.

Figure 9. Edge pair sum labeling of \( P_4 \ominus K_3^c \)

Subcase(c). \( n \geq 6. \)

Define \( f(e_{\frac{n-1}{2}}) = -2, f(e_{\frac{n}{2}}) = -1, f(e_{\frac{n+1}{2}}) = 3 \), for \( 1 \leq i \leq \frac{n-3}{2} f(e_{i}) = n+1-2i \), for \( \frac{n}{2} + 2 \leq i \leq n - 1 f(e_{i}) = n - 1 - 2i \), for \( 1 \leq i \leq \frac{n-1}{2} f(e_{i}) = n + 2i - 1 \), for \( e_{\frac{n+1}{2}} = 2n - 2 = -f(e_{\frac{n}{2}}), f(e_{\frac{n-1}{2}+1}) = 4 = -f(e_{\frac{n-1}{2}+2}) \), for \( 1 \leq i \leq \frac{n-1}{2} f(e_{i+1}) = (3n-1)+\frac{m-1}{2}(i-1)+j = -f(e_{\frac{n-1}{2}+i+1}) \). For each edge label \( f \), the induced vertex label \( f^* \) is calculated as follows: \( f^*(u_1) = 2n = -f^*(u_n), f^*(u_{\frac{n-2}{2}}) = 2n + 1 = -f^*(u_{\frac{n+2}{2}}), f^*(u_{\frac{n+1}{2}}) = 6 = -f^*(u_{\frac{n-1}{2}+2}), \) for \( 2 \leq i \leq \frac{n-2}{2} f^*(u_{i}) = 3n - (2i-3), \) for \( \frac{n-1}{2} \leq i \leq \frac{n-1}{2} f^*(u_{\frac{n-1}{2}+1}) = (2n + 5 + 2i), \) for \( 1 \leq i \leq \frac{n-1}{2} f^*(e_{i}) = n + 2i - 1, f^*(e_{\frac{n+1}{2}}) = 4 = -f^*(e_{\frac{n-1}{2}+2}), f^*(e_{\frac{n+1}{2}}) = 2n - 2 = -f^*(e_{\frac{n-1}{2}}), \) for \( 1 \leq i \leq \frac{n-2}{2} \), \( f^*(e_{\frac{n-1}{2}+1}) = (2n - 2i - 3) \), and for \( 1 \leq i \leq n \) and \( 1 \leq j \leq \frac{m+1}{2} \), \( f^*(e_{i,j+1}) = (3n - 1) + \frac{m-1}{2}(i-1) + j = -f^*(e_{i, \frac{m-1}{2}+j+1}) \). From the above we get \( f^*(V((P_n \ominus K_m^c)) = \{ \pm 2, \pm 3, \pm (2n-2), \pm 2n, \pm (2n+1), \pm (n+1), \pm (n+3), \pm (n+5), ..., \pm (2n-5), \pm (3n-1), \pm (3n-3), \pm (3n-5), ..., \pm (2n+7) \} \cup \{ \pm 3(n-1) + \frac{m-1}{2}(i-1) + j \} | 1 \leq i \leq n, 2 \leq j \leq \frac{m+1}{2} \}. \) Hence \( P_n \ominus K_m^c \) is an edge pair sum graph.

Case (ii) \( n \) is odd.

Subcase (a). \( n = 3. \)

Define \( f(e_{1}) = -4, f(e_{2}) = -2 = -f(e_{1}), f(e_{2}) = 3 \) and \( f(e_{3}) = 1 \), for \( 1 \leq i \leq n \) and \( 2 \leq j \leq \frac{m+1}{2} \), \( f(e_{i}) = 3 + \frac{m-1}{2}(i - 1) + j = -f(e_{i, \frac{m-1}{2}+j}) \). For each edge label \( f \), the induced vertex label \( f^* \) is calculated as follows: \( f^*(u_1) = -2 = -f^*(v_1), f^*(u_2) = -3 = -f^*(v_2), f^*(u_3) = -1 = -f^*(v_3) \), for \( 1 \leq i \leq n \) and \( 2 \leq j \leq \frac{m+1}{2} \), \( f^*(v_{ij}) = 3 + \frac{m-1}{2}(i - 1) + j = -f^*(v_{i, \frac{m-1}{2}+j+1}), f^*(V(P_n \ominus K_m^c)) = \{ \pm 1, \pm 2, \pm 3 \} \cup \{ \pm 3 + \frac{m-1}{2}(i - 1) + j \} | 1 \leq i \leq n, 2 \leq j \leq \frac{m+1}{2} \}. \) Hence \( f \) is an edge pair sum labeling. The example for the edge pair sum graph labeling of \( P_n \ominus K_m^c \) for \( n = 3 \) and \( m = 3 \) is shown in Figure 10.

Figure 10. Edge pair sum labeling of \( P_3 \ominus K_3^c \)

Subcase(b). \( n \geq 5. \)

Define \( f(e_{\frac{n-1}{2}}) = 2, f(e_{\frac{n+1}{2}}) = 1, f(e_{\frac{n+3}{2}+1}) = -4 = -f(e_{\frac{n+3}{2}+2}), f(e_{\frac{n+3}{2}+2}) = -3, f(e_{11}) = -(n+2) = f(e_{n+1}), \) for \( 1 \leq i \leq \frac{n-3}{2} \), \( f(e_{i}) = -(n+2-2i), \) for \( \frac{n+3}{2} \leq i \leq n-1 \) \( f(e_{i}) = -(n+2+2i), \) for \( 2 \leq i \leq \frac{n-3}{2} \), \( f(e_{1}) = -(n+2i-1), \) for \( 1 \leq i \leq \frac{n-5}{2} \), \( f(e_{n+3+i+1}) = 2(n-1-i) \) and for \( 1 \leq i \leq n \) and \( 1 \leq j \leq \frac{m+1}{2} \), \( f(e_{i,j+1}) = n + 2 + (m-1)(i - 1) + 2j = -f(e_{i, \frac{m+1}{2}+j+1}). \) For
each edge label \( f \), the induced vertex label \( f^* \) is calculated as follows: \( f^*(u_1) = -(2n + 2) = -f^*(u_n), f^*(v_{11}) = -(n + 2), f^*(u_{n+1}) = -7 = -f^*(u_{n+1}), f^*(u_{n+2}) = 3 = -f^*(v_{n+3}), \) for \( 1 \leq i \leq \frac{n-5}{2} \), \( f^*(u_{1+i}) = -(3n + 3 - 2i), \) for \( 1 \leq i \leq \frac{n-5}{2} \), \( f^*(u_{n+2+i}) = (3n+3-2i), \) for \( 2 \leq i \leq \frac{n-3}{2} \), \( f^*(v_{n+3}) = -(n+2i-1), f^*(v_{n+1}) = -4 = -f^*(v_{n+1}), f^*(v_{n+2}) = n+2, \) for \( 1 \leq i \leq \frac{n-5}{2} \), \( f^*(v_{n+3+i}) = 2(n-1-i), \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq \frac{n-1}{2} \), \( f^*(v_{i+j}) = n+2+(m-1)(i-1)+2j = -f^*(v_{i+n+j}). \) From the above labeling we get \( f^*(V((P_n \odot K^c_n))) = \{ \pm 3, \pm 4, \pm (2n+2), \pm 2n, \pm (3n+1), \pm (3n-1), \pm (3n-3), \ldots, \pm (2n+8), \pm (2n-4), \pm (2n-6), \pm (2n-8), \ldots, \pm (n+3) \} \cup \{ \pm ((n+2)+(m-1)(i-1)+2j) | 1 \leq i \leq n, 1 \leq j \leq \frac{n-1}{2} \}. \) Hence \( P_n \odot K^c_n \) is an edge pair sum graph.

**Remark 2.8.** Let \( G(p,q) \) is an edge pair sum graph. Then \( G \odot K^c_n \) is also an edge pair sum graph if \( n \) is even. This is already proved in [5].

References