Some Results For Semi Derivations of Prime Near Rings  

Research Article

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Abstract: Let $R$ be a prime near ring. By a semi-derivation associated with a function $g : R \rightarrow R$ we mean an additive mapping $f : R \rightarrow R$ such that $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$ and $f(g(x)) = g(f(x))$ for all $x, y \in R$. In this paper we try to generalize some properties of prime rings with derivations to the prime near rings with semi-derivations.

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1. Introduction

In [1], I.S. Chang, K.W. Jun and Y.S. Jung prove that if there exist a derivation $D$ on a non-commutative 2-torsion free prime ring $R$ such that the mapping $x \rightarrow [aD(x), x]$ is commuting on $R$, then $a = 0$ or $D = 0$. We proved that this conclusion for semi-derivations of prime near rings as follows in Theorem 2.1. In [2], K. Kaya, O.Golbasi, N. Aydin proved that if $R$ is a prime ring of characteristic different from 2, $D$ is a nonzero derivation of $R$, then $(D(R), a) = 0$, if and only if, $((R, a)) = 0$. We also proved that this conclusion for semi-derivations of prime near rings as follows in Theorem 2.2.

1.1. Left Near Ring

A Left near ring is a set $R$ with two operations $+$ and · such that $(R, +)$ is a group and $(R, ·)$ is a semi group satisfying the left distributive law: $x(y + z) = xy + xz$ for all $x, y, z$ in $R$.

1.2. Derivation

An additive mapping $D$ from $R$ to $R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$.

1.3. Semi-Derivation

Let $R$ be an associative ring. An additive mapping $f : R \rightarrow R$ is called a semi-derivation associated with a function $g : R \rightarrow R$ if, for all $x, y \in R$:

1. $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$;
2. $f(g(x)) = g(f(x))$.

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1.4. Prime

A ring $R$ is prime if $aRb = 0$ implies that $a = 0$ or $b = 0$.

1.5. Center

Let $R$ be a ring with center $C(R)$. For any $x, y \in R; [x, y], (x, y)$ will denote $xyyx, xy + yx$ respectively.

1.6. Characteristic of a Ring

A ring $R$ is to be n torsion free if $nx = 0, x \in R$ implies $x = 0$. A mapping $F : R \to R$ is said to be commuting on $R$ if $[F(x), x] = 0$ holds for all $x \in R$, and is said to be centralizing on $R$ if $[F(x), x] \in C(R)$ holds for all $x \in R$. We make extensive use of the basic commutator identities: $(xy, z) = (x, z)y + [y, z]$.

2. Main Section

**Theorem 2.1.** Let $R$ be a non-commutative 2-torsion free prime near ring and $f$ is a semi-derivation of $R$ with $g : R \to R$ is an onto endomorphism. If the mapping $x \to [af(x), x]$ is commuting on $R$, then $a = 0$ or $f = 0$.

**Proof.** We assume that $a$ be a non zero element of $R$. Then by [2], Theorem 1, the mapping $x \to af(x)$ is commuting on $R$. Thus we have

$$[af(x), x] = 0,$$

for all $x \in R$. By linearizing (1), we have

$$[af(x), y] + [af(y), x] = 0$$

for all $x, y \in R$. From this relation it follows that

$$a[f(x), y] + [a, y]f(x) + a[f(y), x] + [a, x]f(y) = 0,$$

for all $x, y \in R$. Replacing $y$ by $yx$ in (2) and using (1), we get

$$= a[f(x), y]x + [a, y]f(x)x + a[f(y), x]x + [a, x]f(y)x + ag(y)[f(x), x]$$

$$= a[g(y), x]f(x) + [a, x]g(y)f(x)$$

for all $x, y \in R$. Right multiplication of (3) by $x$ gives

$$a[f(x), y]x + [a, y]f(x)x + a[f(y), x]x + [a, x]f(y)x = 0$$

for all $x, y \in R$. Subtracting (5) from (4), we obtain

$$ag(y)[f(x), x] + a[g(y), x]f(x) + [a, x]g(y)f(x) = 0$$

for all $x, y \in R$. Taking $ag(y)$ instead of $g(y)$ in (6), we have

$$0 = a^2 g(y)[f(x), x] + a^2 [g(y), x]f(x) + a[a, x]g(y)f(x) + [a, x]ag(y)f(x)$$

(7)
for all \(x, y \in R\). Left multiplication of (6) by a leads to

\[
a^2g(y)[f(x), x] + a^2[g(y), x]f(x) + a[a, x]g(y)f(x) = 0
\]

for all \(x, y \in R\). Subtracting (8) from (7), we get

\[
[a, x]ag(y)f(x) = 0
\]

for all \(x, y \in R\). Since \(R\) is prime, we obtain that for any \(x \in R\) either \(f(x) = 0\) or \([a, x] = 0\). It means that \(R\) is the union of its additive subgroups \(P = x \in R : f(x) = 0\) and \(Q = x \in R : [a, x]a = 0\). Since a group cannot be the union of two proper subgroups, we find that either \(P = R\) or \(Q = R\). If \(P = R\), then \(f = 0\). If \(Q = R\), then this implies that \([a, x]a = 0\), for all \(x \in R\). Let us take \(xy\) instead of \(x\) in this relation. Then we get \([a, x]y = 0\), for all \(x \in R\). Since \(a \in R\) is nonzero and \(R\) is prime, we obtain \(a \in C(R)\). Thus by this and (1), the relation (6) reduces to \(a[g(y), x]f(x) = 0\), for all \(x, y \in R\). Since \(g\) is onto, we see that \(az[u, x]f(x) = 0\), for all \(x, u, z \in R\). Now by primeness of \(R\), we obtain that \([u, x]f(x) = 0\), for all \(x, u, w \in R\). Replacing \(u\) by \(uw\), we get \([u, x]w = 0\), for all \(x, u, w \in R\). Again using the fact that a group cannot be the union of two proper subgroups, it follows that \(f = 0\), since \(R\) is noncommutative. Hence we see that, in any case, \(f = 0\).

This completes the proof.

\[\square\]

**Theorem 2.2.** Let \(R\) be a prime near ring of characteristic different from 2, \(f\) is a nonzero semi-derivation of \(R\), with associated endomorphism \(g\) and \(a \in R\). If \(g \neq 1\) (I is an identity map of \(R\)), then \((f(R), a) = 0\) if, and only, if \(f((R, a)) = 0\).

**Proof.** Suppose \((f(R), a) = 0\). Firstly, we will prove that \(f(a) = 0\). If \(a = 0\) then \(f(a) = 0\). So we assume that \(a \neq 0\). By our hypothesis, we have \((f(x), a) = 0\), for all \(x \in R\). From this relation, we get

\[
(f(xa), a) = (f(x)g(a) + xf(a), a)
\]

and so,

\[
[x, a]f(a) = 0
\]

for all \(x \in R\). Now, replacing \(x\) by \(xy\) in (9), we get

\[
[R, a]Rf(a) = 0
\]

The primeness of \(R\) implies that \(a \in C(R)\) or \(f(a) = 0\). Now suppose that \(a \in C(R)\). Then we obtain that

\[
0 = (f(a), a) = f(a)a + af(a) = 2af(a).
\]

Since the characteristic of \(R\) is different from 2, \(af(a) = 0\). Since we assumed that \(0 \in a\) and \(R\) is prime ring, we get \(f(a) = 0\).

Hence we have

\[
f((r, a)) = f(ra + ar) = (f(r), a) + (g(r), f(a)),
\]

for all \(x \in R\). This yields that \(f((R, a)) = 0\).

Conversely, for all \(x \in R\),

\[
f((ax, a)) = f(a(x, a) + [a, a]x)
\]

\[
= f(a(x, a)) + g(a)f((x, a)).
\]
By hypothesis we have

\[ f(a)(x, a) = 0 \]  

(12)

for all \( x \in R \). Replacing \( x \) by \( xy \) in (11), we get

\[ 0 = f(a)(xy, a) = f(a)x[y, a] + f(a)(x, a)y = f(a)x[y, a]. \]

This implies that \( f(a)R[a, R] = 0 \). For the primeness of \( R \), we have either \( f(a) = 0 \) or \( a \in C(R) \). If \( f(a) = 0 \), then we have

\[ 0 = f((r), a) = (f(r), a) + (f(a), g(r)) = (f(r), a), \]

for all \( r \in R \). This yields that \((f(R), a) = 0\). If \( a \in C(R) \), then we have

\[ 0 = f((a, a)) = 2f(a)(a + g(a)). \]

Since the characteristic of \( R \) is different from 2, we obtain \( f(a)(a + g(a)) = 0 \). Since \( R \) is prime we have \( f(a) = 0 \) or \( a + g(a) = 0 \). But since \( g \) is different from \( \mp I \) we find that \( f(a) = 0 \). Finally, \( (f(R), a) = 0 \) implies the required result. 

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