Common Fixed Point Theorems For A Pair Of Weakly Increasing/Decreasing Self Maps Under $\psi$-Weak Generalized Geraghty Contractions in Partially Ordered Partial b-Metric Spaces

Research Article

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Abstract: In this paper we consider the concept of $\psi$-weak generalized Geraghty contractive pair of weakly increasing/decreasing self mappings in a complete partially ordered partial b-metric space. We study the existence of fixed points for such a pair of weakly increasing/decreasing self mappings in complete partially ordered partial b-metric spaces controlled by $\psi$-weak generalized Geraghty contractive type condition and obtain some fixed point results of G.V.R.Babu et.al [3] in complete partially ordered metric spaces as corollaries. Supporting example is also provided. An open problem is given at the end of the paper.

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1. Introduction and preliminaries

Most of the generalizations of fixed point theorems usually start from Banach [5] contraction principle. But all the generalizations may not be from this principle. In 1973, Geraghty [9] introduced an extension of the contraction in which the contraction constant was replaced by a function having some specified properties. In 1989, Bakkhtin [4] introduced the concept of a b-metric space as a generalization of a metric spaces. In 1993, Czerwik [8] extended many results related to the b-metric spaces. In 1994, Matthews [17] introduced the concept of partial metric space in which the self distance of any point of space may not be zero. In 1996, S.J.O’Neill [22] generalized the concept of partial metric space by admitting negative distances. In 2013, Shukla [28] generalized both the concepts of b-metric and partial metric space by introducing the partial b-metric spaces. Many authors recently studied the existence of fixed points of self maps in different types of metric spaces [1, 2, 14, 21, 26, 29]. Xian Zhang [31] proved a common fixed point theorem for two self maps on a metric space satisfying generalized contractive type conditions. Some authors studied some fixed point theorems in b-metric spaces [18, 24, 25, 32]. After that some authors proved $\alpha - \psi$ versions of certain fixed point theorems in different type metric spaces [13, 19, 24]. Mustafa [20] gave a generalization of Banach contraction principle in complete ordered partial b-metric spaces.

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space by introducing a generalized $\alpha - \psi$ weakly contractive mapping.

In this paper we prove fixed point theorems for $\psi$-weak generalized Geraghty contractive pair of weakly increasing/decreasing self mappings in complete partially ordered partial b-metric spaces satisfying a contractive type condition by considering partial b-metric $p$ as in Definition 1.1 (Shukla [28]) which is more general than that of any partial b-metric and obtained some fixed point results of G.V.R.Babu et.al [3] in complete partially ordered metric space as corollaries A supporting example is given and an open problem is also given at the end of the paper. Shukla [28] introduced the notation of a partial b-metric called a partial b-metric if for all $x, y, z \in X$ the following conditions are satisfied.

(1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$

(2) $p(x, x) \leq p(x, y)$

(3) $p(x, y) = p(y, x)$

(4) $p(x, y) \leq s(p(x, z) + p(z, y)) - p(z, z)$. The pair $(X, p)$ is called a partial b-metric space. The number $s \geq 1$ is called a coefficient of $(X, p)$.

Definition 1.2 ([13]). Let $(X, \leq)$ be a partially ordered set and $T : X \to X$ be a mapping. We say that $T$ is non decreasing with respect to $\leq$ if $x, y \in X$, $x \leq y \Rightarrow Tx \leq Ty$.

Definition 1.3 ([13]). Let $(X, \leq)$ be a partially ordered set. A sequence $\{x_n\} \in X$ is said to be non decreasing with respect to $\leq$ if $x_n \leq x_{n+1} \forall n \in \mathbb{N}$.

Definition 1.4 ([20]). A triple $(X, \leq, p)$ is called an ordered partial b-metric space if $(X, \leq)$ is a partially ordered set and $p$ is a partial b-metric on $X$.

Definition 1.5 ([19]). Define $\Psi = \{\psi : [0, \infty) \to [0, \infty) \mid \psi \text{ is non-decreasing and satisfies (1)}\}$ $\psi$ is continuous and

$$\psi(t) = 0 \Leftrightarrow t = 0 \quad (1)$$

Definition 1.6 ([9]). A self map $f : X \to X$ is said to be a Geraghty contraction if there exists $\beta \in \Omega$ such that $d(f(x), f(y)) \leq \beta d(x, y) \Omega = \{\beta : [0, \infty) \to [0, 1]/\beta(t_n) \to 1 \Rightarrow t_n \to 0 \text{ as } n \to \infty\}$.

Definition 1.7 ([7]). Suppose $(X, \leq)$ is a partially ordered set and $f, g : X \to X$ are self maps. $f$ is said to be $g$-non-decreasing if for $x, y \in X$, $gx \leq gy \Rightarrow fx \leq fy$.

Definition 1.8 ([3]). Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric $d$ such that $(X, d)$ is a metric space. Let $f$ and $g$ be two self mappings on $X$. Suppose there exists $\psi \in \Psi, \beta \in \Omega$ and $L > 0$ such that

$$\psi(d(f(x), f(y))) \leq \beta(M(x, y))M(x, y) + LN(x, y) \quad (2)$$

for all $x, y \in X$ with $gx \geq gy$, where $M(x, y) = \max\{d(gx, gy), d(gx, fx), d(gy, fy), \frac{1}{2}[d(gx, fy) + d(fx, gy)]\}$ and $N(x, y) = \min\{d(gx, gy), d(gx, fy), d(fx, gy)\}$. Then we say that $(f, g)$ is a pair of $\psi$ weak generalized Geraghty contraction maps.
Definition 1.9 ([10]). Two self maps \( f \) and \( g \) of a metric space \((X,d)\) are said to be compatible if
\[
\lim_{n \to \infty} d(fgx_n, gfx_n) = 0
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = u \) for some \( u \in X \).

Definition 1.10 ([11]). Two self maps \( f \) and \( g \) of a metric space \((X,d)\) are said to be weakly compatible if they commute at their coincidence points, that is if \( fu = gu \) for some \( u \in X \), then \( fg u = gfu \).

Definition 1.11 ([23]). Two self maps \( f \) and \( g \) of a metric space \((X,d)\) are said to be reciprocally continuous if
\[
\lim_{n \to \infty} fgx_n = fz \quad \text{and} \quad \lim_{n \to \infty} gfx_n = gz
\]
whenever \( \{x_n\} \) is a sequence in \( X \) with \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = z \). G.V.R.Babu et.al [3] proved the following theorems:

Theorem 1.12 ([3]). Let \((X, \leq)\) be a partially ordered set and suppose that there exists a metric \( d \) such that \((X,d)\) is a complete metric space. Let \( f \) and \( g \) be two self maps on \( X \) such that \( f \) is \( g \)-non-decreasing. Suppose that \((f,g)\) is a pair of generalized Geraghty contraction maps satisfying (2). Assume that

1. \( fX \subseteq gX \)
2. there exists \( x_0 \in X \) such that \( gx_0 \leq fx_0 \)
3. \( g(X) \) is is a closed subset of \( X \).
4. if any non-decreasing \( \{x_n\} \) in \( X \), converges to \( u \), then \( x_n \leq u \) \( \forall \ n \in \mathbb{N} \). Then \( f \) and \( g \) have a coincidence point in \( X \).

Theorem 1.13 ([3]). In addition to the hypothesis of Theorem 1.12, if \( gu < ggu \) where \( u \) is as in (iv) and \( f \) and \( g \) are weakly compatible then \( f \) and \( g \) have a common fixed point in \( X \).

Theorem 1.14 ([3]). Let \((X, \leq)\) be a partially ordered set and suppose that there exists a metric \( d \) such that \((X,d)\) is a complete metric space. Let \( f \) and \( g \) be two self maps on \( X \), \( f \) is \( g \)-non-decreasing. Suppose that \((f,g)\) is a pair of \( \psi \)-weak generalized Geraghty contraction maps. Assume that

1. \( fX \subseteq gX \)
2. \( f \) and \( g \) are compatible.
3. there exists \( x_0 \in X \) such that \( gx_0 \leq fx_0 \)
4. \( f \) and \( g \) are reciprocally continuous. Then \( f \) and \( g \) have a coincidence point in \( X \).

2. Main Result

In this section we prove coincident point and common fixed point theorems for a pair of weakly increasing/decreasing self maps on partially ordered partial \( b \)-metric spaces by using by partial \( b \)-metric \( p \) of definition 1.1 and obtain Theorems 1.12, 1.13 and 1.14 as corollaries. A supporting example is also given. An open problem is also given at the end. We begin this section with the following definition

Definition 2.1 ([20]). Suppose \((X, \leq)\) is a partially ordered set and \( p \) is a partial \( b \)-metric with \( s \geq 1 \) as the coefficient of \((X, p)\). Then we say that the triplet \((X, \leq, p)\) is a partially ordered partial \( b \)-metric space. We observe that every ordered partial \( b \)-metric space is a partially ordered partial \( b \)-metric space.

Definition 2.2 ([20]). A sequence \( \{x_n\} \) in a partial \( b \)-metric space \((X, p)\) is said to be:

1. convergent to a point \( x \in X \) if \( \lim_{n \to \infty} p(x_n, x) = p(x, x) \)
(2). A Cauchy sequence if \( \lim_{n,m \to \infty} p(x_n, x_m) \) exists and is finite

(3). A partial metric space \((X, p)\) is said to be complete if every Cauchy sequence \( \{x_n\} \) in \(X\) converges to a point \(x \in X\) such that
\[
\lim_{n,m \to \infty} p(x_n, x_m) = \lim_{n \to \infty} p(x_n, x) = p(x, x).
\]

Now we introduce the notions of a pair of weakly increasing/decreasing self maps, compatibility, weak compatibility and reciprocal continuity of two self maps on a partially ordered partial b-metric space.

**Definition 2.3** ([6]). Let \((X, \leq)\) be a partially ordered set and \(S,T:X \to X\) be such that \(Sx \leq TSx\) and \(Tx \leq STx\) \((Sx \geq TSx\) and \(Tx \geq STx)\) \(\forall x \in X\). Then \(S\) and \(T\) are said to be weakly increasing/decreasing mappings.

**Definition 2.4.** A pair of weakly increasing/decreasing self maps \(S,T\) and a self map \(g\) on a partially ordered partial b-metric space \((X, \leq, p)\) are said to be compatible if \(\lim_{m \to \infty} p(Sx_m, gSx_m) = 0 = \lim_{m \to \infty} p(Tx_m, gTx_m)\) whenever sequence \(\{x_n\}\) in \(X\) such that \(\lim_{m \to \infty} p(Sx_m, Tx_m) = \lim_{m \to \infty} p(Tx_m, u) = \lim_{m \to \infty} p(Sx_m, u) = p(u, u) = 0\) and \(\lim_{m \to \infty} p(gx_m, gx_n) = \lim_{m \to \infty} p(gx_n, u) = p(u, u) = 0\) for some \(u \in X\).

**Definition 2.5.** A pair of weakly increasing/decreasing self maps \(S,T\) and a self map \(g\) on a partially ordered partial b-metric space \((X, \leq, p)\) are said to be weakly compatible if they commute at their coincidence points, that is \(Su = Tu = gu\) for some \(u \in X\), then \(Sgu = gSu = Tgu = gTu\).

**Definition 2.6.** A pair of weakly increasing/decreasing self maps \(S,T\) and a self map \(g\) on a partially ordered partial b-metric space \((X, \leq, p)\) are said to be reciprocally continuous if \(\lim_{m \to \infty} p(Sx_m, Tgx_m) = \lim_{m \to \infty} p(Sx_m, S) = \lim_{m \to \infty} p(Tgx_m, Tz) = p(Sz, Tz) = 0\) and \(\lim_{m \to \infty} p(gSx_m, Tgx_m) = \lim_{m \to \infty} p(gSx_m, S) = \lim_{m \to \infty} p(gTx_m, Tz) = p(gz, gz) = 0\) whenever \(\{x_m\}\) is a sequence in \(X\) with \(\lim_{m \to \infty} p(Sx_m, Tx_m) = \lim_{m \to \infty} p(Sx_m, z) = \lim_{m \to \infty} p(Tx_m, z) = p(z, z) = 0\) and \(\lim_{m \to \infty} p(gx_m, gx_n) = \lim_{m \to \infty} p(gx_n, z) = p(z, z) = 0\) for some \(z \in X\).

In the following definition we extend the notion of \(\psi\) - weak generalized Geraghty contraction for a pair of weakly increasing/decreasing self maps \(S,T\) and a self map \(g\) on a partially ordered partial b-metric space \((X, \leq, p)\).

**Definition 2.7.** Let \((X, \leq)\) be a partially ordered set and suppose that there exists a partial b-metric \(p\) such that \((X, p)\) is a partial b-metric space. Let \(S,T\) be a pair of weakly increasing/decreasing self maps and \(g\) be a self mapping on \(X\). Suppose there exists
\[
\psi \in \Psi, \beta \in \Omega \text{ such that } \psi(s)p(Sx, Ty) \leq \beta(\psi(M(x, y)))\psi(M(x, y))
\]
for all \(x,y \in X\) whenever \(gx\) and \(gy\) are comparable, where \(M(x, y) = \max\{p(gx, gy), p(gx, Sx), p(gy, Ty), \frac{1}{2}[p(gx, Ty) + p(Sx, gy)]\}\). Then we say that \(g\) is a pair of \(\psi\) weak generalized Geraghty contraction maps \(S,T\). We also say that \(g\) is a pair of weak generalized Geraghty contraction maps \(S,T\) if \(\psi(t) = t \forall t \in [0, \infty)\).

**Definition 2.8** ([7]). Suppose \((X, \leq)\) is a partially ordered set and \(S,T,g:X \to X\) are self maps on \(X\). \(S,T\) are said to be \(g\)-non-decreasing if for \(x,y \in X\), \(gx \leq gy \Rightarrow Sx \leq Sy\) and \(Tx \leq Ty\).

Now we state the following useful lemmas, whose proofs can be found in Sastry et. al [26].

**Lemma 2.9.** Let \((X, \leq, p)\) be a complete partially ordered partial b-metric space with coefficient \(s \geq 1\). Let \(\{x_n\}\) be a sequence in \(X\) such that
\[
(1) \lim_{n \to \infty} x_n = x, \lim_{n \to \infty} y_n = y \text{ and } \lim_{n \to \infty} p(x_n, y_n) = 0 \Rightarrow x = y
\]
\[
(2) \lim_{n \to \infty} p(x_n, x_{n+1}) = 0 \text{ and } \lim_{n \to \infty} x_n = x, \lim_{n \to \infty} x_n = y
\]
Then \(\lim_{n \to \infty} p(x_n, x) = \lim_{n \to \infty} p(x_n, y) = p(x, y)\) and hence \(x = y\).
Lemma 2.10.

(1) \( p(x, y) = 0 \Rightarrow x = y \)

(2) \( \lim_{n \to \infty} p(x_n, x) = 0 \Rightarrow p(x, x) = 0 \) and hence \( x_n \to x \) as \( n \to \infty \).

Lemma 2.11. Let \( (X, \leq, p) \) be a partially ordered partial \( b \)-metric space with coefficient \( s \geq 1 \). Let \( \{x_n\} \) be a sequence in \( X \) such that \( \lim_{n \to \infty} p(x_n, x_{n+1}) = 0 \). Then

(1) \( \{x_n\} \) is a Cauchy sequence \( \Rightarrow \lim_{m,n \to \infty} p(x_m, x_n) = 0. \)

(2) \( \{x_n\} \) is not a Cauchy sequence \( \Rightarrow \exists \epsilon > 0 \) and sequences \( \{m_i\}, \{n_i\} \ni m_k > n_k > k \in \mathbb{N}; p(x_{m_k}, x_{n_k}) > \epsilon \) and \( p(x_{n_k}, x_{m_k-1}) \leq \epsilon. \)

Proof.

(1) Suppose \( \{x_n\} \) is a Cauchy sequence then \( \lim_{m,n \to \infty} p(x_m, x_n) \) exists and finite. Therefore \( 0 = \lim_{n \to \infty} p(x_n, x_{n+1}) = \lim_{m,n \to \infty} p(x_m, x_n). \) Therefore \( \lim_{m,n \to \infty} p(x_m, x_n) = 0. \)

(2) \( \{x_n\} \) is not a Cauchy sequence \( \Rightarrow \lim_{m,n \to \infty} p(x_m, x_n) \neq 0 \) if it exists \( \Rightarrow \exists \epsilon > 0 \) and for every \( N \) and \( m, n > N \ni p(x_m, x_n) > \epsilon \) \( \Rightarrow \lim_{n \to \infty} p(x_n, x_{n+1}) = 0 \Rightarrow \exists M \ni p(x_n, x_{n+1}) < \epsilon \ \forall n > M. \) Let \( N_1 > M \) and \( n_1 \) be the smallest such that \( m > n_1 \) and \( p(x_{n_1}, x_m) > \epsilon \) for at least one \( m \). Let \( m_1 \) be the smallest such that \( m_1 > n_1 > N_1 > 1 \) and \( p(x_{n_1}, x_{m_1}) > \epsilon \) so that \( p(x_{n_1}, x_{m_1-1}) \leq \epsilon. \) Let \( N_2 > N_1 \) and choose \( m_2 > n_2 > N_2 > 2 \ni p(x_{n_2}, x_{m_2}) > \epsilon \) and \( p(x_{n_2}, x_{m_2-1}) \leq \epsilon. \) Continuing this process we can get sequences of positive integers \( \{m_k\} \) and \( \{n_k\} \) such that \( m_k > n_k > k \) and \( p(x_{n_k}, x_{m_k}) > \epsilon; p(x_{n_k}, x_{m_k-1}) \leq \epsilon. \)

Now we state our first main result for a pair of weakly increasing self maps:

Theorem 2.12. Let \( (X, \leq, p) \) be a complete partially ordered partial \( b \)-metric space with coefficient \( s \geq 1 \). Let \( S, T \) be a pair of weakly increasing self maps and \( g \) be a self mapping on \( X \). \( S, T \) are \( g \)-non-decreasing. Suppose that \( g \) is a pair of weak generalized Geraghty contraction maps \( S, T \), that is there exist \( \psi \in \Psi \) and \( \beta \in \Omega \) such that \( \psi(s p(Sx, Ty)) \leq \beta(\psi(M(x, y))) \psi(M(x, y)) \) for all \( x, y \in X \) whenever \( gx \) and \( gy \) are comparable, where

\[
M(x, y) = \max \{p(gx, gy), p(gx, Sx), p(gy, Ty), \frac{1}{2s}[p(gx, Ty) + p(Sx, gy)]\}
\]

Assume that

(1) \( S(X), T(X) \subseteq g(X) \)

(2) there exists \( x_0 \in X \) such that \( gx_0 \leq Sx_0 \)

(3) \( g(X) \) is a closed subset of \( X \).

(4) if any non-decreasing sequence \( \{x_n\} \) in \( X \), converges to \( u \), then \( x_n \leq u \ \forall n \geq 0 \)

Then \( S, T \) and \( g \) have a coincidence point in \( X \).
Proof. \( \text{let } x_0 \in X \text{ be as in (ii). If } gx_0 = Sx_0 \text{ then } x_0 \text{ is a coincident point and there is nothing to prove. Now suppose } gx_0 < Sx_0. \) By (i) \( \exists x_1 \in X \text{ such that } gx_1 = Sx_1. \) Since \( gx_0 < Sx_0 = gx_1 \) and \( S \) is \( g^- \)-non decreasing, we have \( Sx_0 \leq Tgx_0 \Rightarrow Sx_0 \leq Tx_1. \) Since \( S(X), T(X) \subseteq g(X) \) and \( Tx_1 \in T(X) \subseteq g(X), \) there exists \( x_2 \in X \) such that \( gx_2 = Tx_1 \) and \( gx_1 \leq gx_2. \) Continuing this process, we can find sequence \( \{x_n\} \) with \( Sx_{2n} = gx_{2n+1} \) and \( Tx_{2n+1} = gx_{2n+2} \) for \( n = 0, 1, 2, 3, \ldots \) Further, since \( gx_1 \leq gx_2 \text{ and } S, T \) are weakly increasing \( g^- \)-non decreasing, we have \( Tx_1 \leq Sx_2 \) so that \( gx_2 \leq gx_3. \)

\(\vdots\) By induction, we get \( gx_n \leq gx_{n+1} \forall n = 0, 1, 2, 3, \ldots \) Suppose \( n \) is odd and \( gx_n = gx_{n+1} \Rightarrow gx_{n+1} = Tx_n = gx_n \Rightarrow x_n \) is a coincident point of \( T \) and \( g \) in \( X. \) Suppose \( n \) is even and \( gx_n = gx_{n+1} \Rightarrow gx_{n+1} = Sx_n = gx_n \Rightarrow x_n \) is a coincident point of \( S \) and \( g \) in \( X. \) Suppose \( n \) is odd and \( x_n \) is a coincident point of \( T \) and \( g \) in \( X. \) Then \( gx_n = gx_{n+1} \Rightarrow gx_{n+1} = Tx_n = gx_n \) and assume that \( gx_{n+1} \neq gx_{n+2} \) we have

\[
\psi(spx_{n+2}, gx_{n+1}) = \psi(spx_{n+1}, Tx_n))
\]

\[
\leq \beta(\psi(M(x_{n+1}, x_n)))\psi(M(x_{n+1}, x_n)), \text{ where } M(x_{n+1}, x_n)
\]

\[
= \max \{p(gx_{n+1}, gx_n), p(gx_{n+1}, Sx_{n+1}), p(gx_n, Tx_n), \frac{1}{2\delta} [p(gx_{n+1}, Tx_n) + p(Sx_{n+1}, gx_n)]\}
\]

\[
= \max \{p(gx_{n+1}, gx_n), p(gx_{n+1}, gx_{n+2}), p(gx_n, gx_{n+1}), \frac{1}{2\delta} [p(gx_{n+1}, gx_{n+1}) + p(gx_{n+2}, gx_n)]\}
\]

\[
\leq \max \{p(gx_{n+1}, gx_n), p(gx_{n+1}, gx_{n+2}), \frac{1}{2\delta} [p(gx_{n+1}, gx_{n+2}) + s(p(gx_{n+2}, gx_{n+1}) + p(gx_{n+1}, gx_{n+1}))]\}
\]

\[
= \max \{p(gx_{n+1}, gx_n), p(gx_{n+1}, gx_{n+2})\}
\]

\(\Rightarrow sp(gx_{n+2}, gx_{n+1}) < p(gx_{n+2}, gx_{n+1}), \text{ a contradiction.}\)

\(\vdots\) \(gx_{n+1} = gx_{n+2}\)

\(\vdots\) \(gx_{n+2} = Sx_{n+1} = gx_{n+1}\)

\(\vdots\) \(x_{n+1} = x_n \text{ is a coincident point of } S \) and \( g \) in \( X. \) \(\vdots\) \(x_n \text{ is a coincident point of } T \) and \( g \) in \( X \) then \( x_n \text{ is a coincident point of } S \) and \( g \) in \( X. \) Similarly by considering \( n \) to be even \( x_n \text{ is a coincident point of } S \) and \( g \) in \( X, \) then \( x_n \text{ is a coincident point of } T \) and \( g \) in \( X. \) Let \( n \) be odd and we may assume that \( gx_{n+1} \neq gx_{n+2} \forall n \in \mathbb{N}. \) Then we have \( p(gx_{n+2}, gx_{n+1}) > 0, \) therefore by (4),

\[
\psi(spx_{n+2}, gx_{n+1}) = \psi(spx_{n+1}, Tx_n))
\]

\[
\leq \beta(\psi(M(x_{n+1}, x_n)))\psi(M(x_{n+1}, x_n)), \text{ where } M(x_{n+1}, x_n)
\]

\[
= \max \{p(gx_{n+1}, gx_n), p(gx_{n+1}, Sx_{n+1}), p(gx_n, Tx_n), \frac{1}{2\delta} [p(gx_{n+1}, Tx_n) + p(Sx_{n+1}, gx_n)]\}
\]

\[
= \max \{p(gx_{n+1}, gx_n), p(gx_{n+1}, gx_{n+2}), p(gx_n, gx_{n+1}), \frac{1}{2\delta} [p(gx_{n+1}, gx_{n+1}) + p(gx_{n+2}, gx_n)]\}
\]

\[
\leq \max \{p(gx_{n+1}, gx_n), p(gx_{n+1}, gx_{n+2}), \frac{1}{2\delta} [p(gx_{n+1}, gx_{n+2}) + s(p(gx_{n+2}, gx_{n+1}) + p(gx_{n+1}, gx_{n+1}))]\}
\]

\[
= \max \{p(gx_{n+1}, gx_n), p(gx_{n+1}, gx_{n+2})\}
\]

Suppose

\[
p(gx_{n+1}, gx_n) \leq p(gx_{n+1}, gx_{n+2}) (5)
\]
Then $M(x_{n+1}, x_n) = p(gx_{n+1}, gx_n)$

\[
\therefore \psi(sp(gx_{n+2}, gx_{n+1})) \leq \beta(\psi(p(gx_{n+2}, gx_{n+1}))\psi(p(gx_{n+2}, gx_{n+1})) < \psi(p(gx_{n+2}, gx_{n+1}))
\]

\[
\Rightarrow sp(gx_{n+1}, gx_n) < p(gx_{n+2}, gx_{n+1}), \text{ a contradiction.}
\]

\[
\therefore M(x_{n+1}, x_n) = p(gx_{n+1}, gx_n)
\]

\[
\therefore \psi(p(gx_{n+2}, gx_{n+1})) \leq \psi(sp(gx_{n+2}, gx_{n+1}))
\]

\[
\leq \beta(\psi(p(gx_{n+1}, gx_n))\psi(p(gx_{n+1}, gx_n))
\]

\[
< \psi(p(gx_{n+1}, gx_n))
\]

\[
\Rightarrow p(gx_{n+2}, gx_{n+1}) \leq sp(gx_{n+2}, gx_{n+1}) < p(gx_{n+1}, gx_n)
\]

\[
\therefore \text{sequence } \{\psi(p(gx_{n+1}, gx_n))\} \text{ is strictly decreasing and converges to } r \text{ (say). Also sequence } p(gx_{n+1}, gx_n) \text{ is strictly}
\]

\[
\therefore r = \psi(\lambda)
\]

Suppose $r \neq 0$

\[
\therefore \psi(p(gx_{n+2}, gx_{n+1})) \leq \beta(\psi(p(gx_{n+1}, gx_n)) < 1
\]

taking limits as $n \to \infty$

\[
\therefore \lim_{n \to \infty} \beta(\psi(p(gx_{n+1}, gx_n))) = 1 \Rightarrow \lim_{n \to \infty} \psi(p(gx_{n+1}, gx_n)) = 0
\]

\[
\therefore r = 0 \Rightarrow \psi(\lambda) = 0 \Rightarrow \lambda = 0
\]

In the similar lines we can discuss the case when $n$ is even and arrive the same conclusions.

\[
\therefore r = 0 \Rightarrow \psi(\lambda) = 0 \Rightarrow \lambda = 0
\]

Now we claim sequence $\{gx_n\}$ is a Cauchy sequence. Assume that $\{gx_n\}$ is not a Cauchy sequence. Then by lemma 2.11

\[
\exists \epsilon > 0 \text{ and sequences } \{m_k\}, \{n_k\}; m_k > n_k > k \text{ such that } p(gx_{m_k}, gx_{n_k}) \geq \epsilon \text{ and } p(gx_{m_k-1}, gx_{n_k}) < \epsilon.
\]

Let us observe the following cases:

Case(i): Let $m_k$ is even and $n_k$ is odd

\[
\therefore \psi(se) \leq \psi\{sp(gx_{m_k}, gx_{n_k})\} = \psi\{sp(Tx_{m_k-1}, Sx_{n_k-1})\}
\]

\[
\leq \beta(\psi(M(x_{m_k-1}, x_{n_k-1}))\psi(M(x_{m_k-1}, x_{n_k-1})) \text{ where } M(x_{m_k-1}, x_{n_k-1})
\]

\[
= \max[p(gx_{m_k-1}, gx_{n_k-1}), p(gx_{n_k-1}, Sx_{n_k-1}), p(gx_{m_k-1}, Tx_{m_k-1})
\]

\[
\frac{1}{2s}[(p(gx_{m_k-1}, Sx_{n_k-1}) + p(Tx_{m_k-1}, gx_{n_k-1})]
\]

\[
= \max[p(gx_{m_k-1}, gx_{n_k-1}), p(gx_{n_k-1}, gx_{m_k}), p(gx_{m_k-1}, gx_{m_k})
\]

\[
\frac{1}{2s}[(p(gx_{m_k-1}, gx_{n_k}) + p(gx_{m_k-1}, gx_{n_k-1})]
\]

\[
\leq \max[p(gx_{m_k-1}, gx_{n_k-1}), p(gx_{n_k-1}, gx_{m_k}), p(gx_{m_k-1}, gx_{m_k})
\]

\[
\frac{1}{2s}[(sp(gx_{m_k-1}, gx_{n_k-1}) + sp(gx_{n_k-1}, gx_{m_k}) - p(gx_{n_k-1}, gx_{n_k}) + sp(gx_{m_k-1}, gx_{n_k-1})
\]

\[
+ sp(gx_{m_k}, gx_{m_k-1}) - p(gx_{m_k-1}, gx_{n_k-1})]
\]
Case (ii): Let \( m_k \) be odd and \( n_k \) be odd

\[
\therefore \psi(s) \leq \beta(\psi(M(x_{m_k-1}, x_{n_k-1}))) < \psi(s + s\eta + \eta) < \psi(s + s\eta + \eta)
\] (13)

(\text{This being for large } k \text{ and true for every } \eta > 0). \text{ Since } \psi \text{ is continuous, then we get for large } k,\ \psi(s) \leq \lim_{k \to \infty} \beta(\psi(M(x_{m_k-1}, x_{n_k-1}))) \psi(s) \leq \psi(s). \text{ Therefore } \lim_{k \to \infty} \beta(\psi(M(x_{m_k-1}, x_{n_k-1}))) = 1. \text{ Therefore } \lim_{k \to \infty} M(x_{m_k-1}, x_{n_k-1}) = 0 \Rightarrow \psi(s) \leq 0 \Rightarrow \psi(s) = 0 \Rightarrow s = 0, \text{ a contradiction.}

Case (ii): Let \( m_k \) be odd and \( n_k \) be odd

\[
\therefore \psi(s) \leq \beta(\psi(M(x_{m_k-1}, x_{n_k-1}))) \psi(s + s\eta + \eta) < \psi(s + s\eta + \eta)
\] (13)

Suppose \( M(x_{m_k-1}, x_{n_k}) = p(x_{m_k-1}, x_{n_k}) < \epsilon \). But

\[
\epsilon \leq p(x_{m_k}, x_{n_k}) \leq sp(x_{m_k}, x_{n_k}+1) + sp(x_{n_k}+1, x_{n_k}) - p(x_{n_k}+1, x_{n_k})
\]

\[
\leq sp(x_{m_k}, x_{n_k}+1) + s\eta \text{ where } \eta > 0 \Rightarrow p(x_{n_k}+1, x_{n_k}) < \eta
\] (15)

\[
\Rightarrow \epsilon - s\eta \leq p(x_{m_k}, x_{n_k}+1)
\] (16)

\[
\therefore \psi(\epsilon - s\eta) \leq \psi(\psi(s) + \psi(s) \psi(p(x_{m_k-1}, x_{n_k}))) \psi(p(x_{m_k-1}, x_{n_k}))
\]

\[
\leq \beta(\psi(p(x_{m_k-1}, x_{n_k}))) \psi(p(x_{m_k-1}, x_{n_k}))
\]

\[
< \psi(p(x_{m_k-1}, x_{n_k})) < \psi(\epsilon)
\] (17)

Allowing \( k \to \infty \), then \( \eta \to 0 \) and \( \psi \) is continuous. \( \therefore \psi(s) \leq \lim_{k \to \infty} \beta(\psi(p(x_{m_k-1}, x_{n_k})))\psi(s) \leq \psi(s) \) and \( \lim_{k \to \infty} p(x_{m_k-1}, x_{n_k}) = \epsilon. \therefore \lim_{k \to \infty} \beta(\psi(p(x_{m_k-1}, x_{n_k}))) = 1 \Rightarrow \lim_{k \to \infty} p(x_{m_k-1}, x_{n_k}) = 0 \Rightarrow \epsilon = 0, \text{ a contradiction. Suppose } M(x_{m_k-1}, x_{n_k}) = 1 \text{ [\( p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k}) \)]}. \text{ On the other hand}

\[
p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k}) \leq sp(x_{n_k}, x_{m_k}+1) + sp(x_{m_k}+1, x_{m_k}) - p(x_{m_k}+1, x_{m_k}) + sp(x_{n_k}, x_{m_k}+1) + sp(x_{n_k}, x_{m_k}+1) + sp(x_{n_k}+1, x_{n_k})
\]

\[
\leq 2sp(x_{n_k}, x_{m_k}+1) + 2s\eta \leq 2s\epsilon + 2s\eta.
\]
where \( p(gx_{n_k+1}, gx_{n_k}) \leq \eta \) and \( p(gx_{m_k}, gx_{m_k-1}) \leq \eta \) for some \( \eta > 0 \) for large \( k \)

\[
\therefore \frac{1}{2^n} [p(gx_{m_k}, gx_{n_k}) + p(gx_{m_k-1}, gx_{n_k+1})] \leq \epsilon + \eta. \tag{18}
\]

Therefore,

\[
M(x_{m_k-1}, x_{n_k}) = \frac{1}{2^n} [p(gx_{m_k}, gx_{n_k}) + p(gx_{m_k-1}, gx_{n_k+1})] \leq \epsilon + \eta
\]

Therefore from (16), (17) and (18)

\[
\epsilon - s\eta \leq \sup_{n,m} p(gx_n, gx_{n+1})
\]

\[
\therefore \psi(\epsilon - s\eta) \leq \psi(\sup_{n,m} p(gx_n, gx_{n+1}))
\]

\[
\leq \beta(\psi(M(x_{m_k-1}, x_{n_k}))) \psi(M(x_{m_k-1}, x_{n_k}))
\]

\[
\leq \psi(M(x_{m_k-1}, x_{n_k}))
\]

\[
\leq \psi(\epsilon + \eta)
\]

Allowing \( k \to \infty \), then \( \eta \to 0 \).

\[
\therefore \psi(\epsilon) \leq \lim_{k \to \infty} \beta(\psi(M(x_{m_k-1}, x_{n_k}))) \lim_{k \to \infty} \psi(M(x_{m_k-1}, x_{n_k})) \leq \psi(\epsilon)
\]

and \( \lim_{k \to \infty} M(x_{m_k-1}, x_{n_k}) = \epsilon. \therefore \lim_{k \to \infty} \beta(\psi(M(x_{m_k-1}, x_{n_k}))) = 1 \Rightarrow \lim_{k \to \infty} M(x_{m_k-1}, x_{n_k}) = 0 \Rightarrow \epsilon = 0 \), a contradiction.

Similarly the other two cases can be discussed. Therefore \( \{gx_n\} \) is a Cauchy sequence. Therefore \( \{gx_n\} \to gy \) for some \( y \in X \) by (iii). Also

\[
0 = \lim_{n,m \to \infty} p(gx_n, gx_m) = \lim_{n \to \infty} p(gx_n, gy) = p(gy, gy)
\tag{19}
\]

Now by (iv) of the hypothesis \( gx_n \leq gy \ \forall \ n \in \mathbb{N} \). Therefore \( gx_{n+1} \leq gy \Rightarrow Tgx_n \leq Ty \) and \( Sx_n \leq Sy \ \forall \ n \in \mathbb{N} \) (since \( S, T \) are \( g \)- non-decreasing). Let \( n \) be even \( \psi \{sp(Sx_n, Ty)\} \leq \beta(\psi(M(x_n, y)))\psi(M(x_n, y)) \), where

\[
M(x_n, y) = \max\{p(gx_n, gy), p(gy, Ty), p(gx_n, Sx_n)\frac{1}{2^n}[p(gx_n, Ty) + p(Sx_n, gy)]\}
\]

\[
= \max\{p(gx_n, gy), p(gy, Ty), p(gx_n, gx_{n+1})\frac{1}{2^n}[p(gx_n, Ty) + p(gx_{n+1}, gy)]\}
\]

\[
= p(gy, Ty) \quad \text{for large } n. \tag{20}
\]

Now

\[
\lim_{n \to \infty} \beta(\psi(M(x_n, y))) = 1 \Rightarrow \lim_{n \to \infty} \psi(M(x_n, y)) = 0
\]

\[
\Rightarrow \psi(p(gy, Ty)) = 0 \quad \text{by (20)} \tag{21}
\]

\[
\Rightarrow p(gy, Ty) = 0 \Rightarrow gy = Ty \quad \text{(by Lemma 2.10 (i)). Therefore } y \text{ is a coincident point of } T \text{ and } g. \quad \text{Suppose } \exists \lambda \text{ such that}
\]

\[
\beta(\psi(M(x_n, y))) = \lambda, \quad \text{for infinitely many } n \tag{22}
\]

\[
\therefore 0 \leq \lambda < 1, \psi(sp(Sx_n, Ty)) \leq \lambda\psi(M(x_n, y)) \leq \lambda\psi(p(gy, Ty)) < \psi(p(gy, Ty))
\]

\[
\Rightarrow sp(Sx_n, Ty) < p(gy, Ty) \Rightarrow \lim_{n \to \infty} \sup sp(Sx_n, Ty) \leq p(gy, Ty) \tag{23}
\]
Now

\[ p(gy,Ty) \leq sp(gy,gx_{n+1}) + sp(gx_{n+1},Ty) - p(gx_{n+1},gx_{n+1}) \]
\[ \leq sp(gy,gx_{n+1}) + sp(gx_{n+1},Ty) \]
\[ \Rightarrow p(gy,Ty) - sp(gy,gx_{n+1}) \leq sp(Sx_n,Ty) \]
\[ \Rightarrow p(gy,Ty) \leq \lim_{n \to \infty} \inf sp(Sx_n,Ty) \] (24)

Therefore \( \lim_{n \to \infty} \sup sp(Sx_n,Ty) \leq p(gy,Ty) \leq \lim_{n \to \infty} \inf sp(Sx_n,Ty) \). Therefore \( \lim_{n \to \infty} sp(Sx_n,Ty) = p(gy,ty) \).

\[ \therefore \psi(p(gy,Ty)) = \psi(\lim_{n \to \infty} sp(Sx_n,Ty)) \]
\[ = \lim_{n \to \infty} \psi(sp(Sx_n,Ty)) \quad (\text{since } \psi \text{ is continuous}) \]
\[ \leq \lambda \psi(p(gy,Ty)) \]
\[ \Rightarrow \psi(p(gy,Ty)) = 0 \Rightarrow p(gy,Ty) = 0 \Rightarrow gy = Ty \] (25)

Therefore \( y \) is a coincident point of \( T \) and \( g \). Let \( n \) be odd. Interchanging the roles of \( S \) and \( T \) in the above discussion we can conclude \( y \) is a coincident point of \( S \) and \( g \). Hence \( y \) is a coincident point of a pair of weakly increasing self maps \( S, T \) and \( g \).

Now we state and prove our second main result.

**Theorem 2.13.** Let \((X,\leq,p)\) be a complete partially ordered partial \( b \)-metric space with coefficient \( s \geq 1 \). Let \( S,T \) be a pair of weakly increasing self maps and \( g \) be a self mapping on \( X \). \( S,T \) are \( g \)-non-decreasing. Suppose that \( g \) is a pair of weak generalized Geraghty contraction maps \( S,T \), that is there exist \( \psi \in \Psi \) and \( \beta \in \Omega \) such that \( \psi(sp(Sx,Ty)) \leq \beta(\psi(M(x,y)))\psi(M(x,y)) \) for all \( x,y \in X \) whenever \( gx \) and \( gy \) are comparable, where

\[ M(x,y) = \max\{p(gx,gy),p(gx,Sx),p(gy,Ty),\frac{1}{2s}[p(gx,Ty) + p(Sx,gy)]\} \] (26)

Assume that

1. \( S(X), T(X) \subseteq g(X) \).
2. there exists \( x_0 \in X \) such that \( gx_0 \leq Sx_0 \).
3. \( g(X) \) is a closed subset of \( X \).
4. if any non-decreasing \( \{x_n\} \) in \( X \), converges to \( y \), then that \( x_n \leq y \ \forall \ n \geq 0 \).

Further if \( S,T \) and \( g \) are weakly compatible and if \( gy \leq ggy \ \forall \ y \in X \), then \( S,T \) and \( g \) have a common fixed point in \( X \).

**Proof.** We have by Theorem 2.12, \( \{gx_n\} \) is a Cauchy sequence, which is non-decreasing that converges to \( gy \) and \( gy = Sgy = Ty \). Therefore \( \lim_{n,m \to \infty} p(gx_n,gx_m) \) exists and is equal to 0. As sequence \( \{gx_n\} \to gy \) implies 0 = \( \lim_{n,m \to \infty} p(gx_n,gx_m) = \lim_{n \to \infty} p(gx_n,gy) = p(gy,gy) \). Since \( S,T \) and \( g \) are weakly compatible, we have \( Sgy = gSy = Tgy = gTy \). Let

\[ gy = Sfy = Ty = u \quad (\text{say}) \] (27)
\[ \therefore Tgy = gTy = gu \] (28)
If \( y = u \), then \( u = Su = Tu = gu \Rightarrow u \) is a common fixed point of \( T \) and \( g \) in \( X \). Let \( y \neq u \Rightarrow gy \neq gu \Rightarrow p(gy, gu) \neq 0 \) (by Lemma 2.10 (i)). We have from (26),

\[
\psi(sp(gy, gu)) = \psi(sp(Sy, Tu)) \leq \beta(\psi(M(y, u)))\psi(M(y, u))
\]

where \( M(y, u) = \max\{p(gy, gu), p(gy, Sy), p(gu, Tu), \frac{1}{2s}[p(gy, Tu) + p(Sy, gu)]\} \)

\[
= p(gy, gu) \quad \text{(by (27) and Lemma 2.10 (i))}
\]

\[
\therefore \psi(sp(gy, gu)) \leq \beta(\psi(M(y, u)))\psi(M(y, u))
\]

\[
\Rightarrow \psi(sp(gy, gu)) \leq \beta(\psi(p(gy, gu)))\psi(p(gy, gu))
\]

\[
\Rightarrow \psi(p(gy, gu)) \leq \psi(sp(gy, gu)) < \psi(p(gy, gu)) \quad \text{if} \quad \psi(p(gy, gu)) > 0, \quad \text{a contradiction.}
\]

Therefore \( \psi(p(gy, gu)) = 0 \Rightarrow p(gy, gu) = 0 \). Therefore \( gy = gu \). Therefore By (27) and (28) \( u = Su = Tu = gu \). Therefore \( u \) is a common fixed point of \( S, T \) and \( g \) in \( X \).

Now we state and prove our third main result.

**Theorem 2.14.** Let \((X, \leq, p)\) be a complete partially ordered partial \( b \)-metric space with coefficient \( s \geq 1 \). Let \( S, T \) be a pair of weakly increasing self maps and \( g \) be a self mapping on \( X \). \( S, T \) are \( g \)-non-decreasing. Suppose that \( g \) is a pair of weak generalized Geraghty contraction maps \( S, T \), that is there exist \( \psi \in \Psi \) and \( \beta \in \Omega \) such that \( \psi(sp(Sx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)) \) for all \( x, y \in X \) whenever \( gx \) and \( gy \) are comparable, where

\[
M(x, y) = \max\{p(gx, gy), p(gx, Sx), p(gy, Ty), \frac{1}{2s}[p(gx, Ty) + p(Sx, gy)]\}
\]

(29)

Assume that

(1). \( S(X), T(X) \subseteq g(X) \)

(2). \( S, T \) and \( g \) are compatible

(3). there exists \( x_0 \in X \) such that \( gx_0 \leq Sx_0 \)

(4). \( S, T \) and \( g \) are reciprocally continuous.

Then \( S, T \) and \( g \) have a coincidence point in \( X \).

**Proof.** We have by Theorem 2.12, \( \{gx_n\} \) is a Cauchy sequence, which is non-decreasing that converges to \( z \) (say). Therefore \( \lim_{m,n \to \infty} p(gx_n, gx_m) \) exists and is equal to 0. As sequence \( \{gx_n\} \to z \) implies \( \lim_{m,n \to \infty} p(gx_n, gx_m) = \lim_{n \to \infty} p(gx_n, z) = p(z, z) = 0 \).

For \( n \) is even

\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} gx_{n+1} = z
\]

\[
\therefore \lim_{n \to \infty} gx_n = \lim_{n \to \infty} Sx_n = z
\]

For \( n \) is odd

\[
\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} gx_{n+1} = z
\]

\[
\therefore \lim_{n \to \infty} gx_n = \lim_{n \to \infty} Tx_n = z
\]
\[
\therefore \lim_{n \to \infty} gx_n = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Sx_n = z
\]

Since \(S, T\) and \(g\) are reciprocally continuous,

\[
\lim_{n \to \infty} Sgx_n = Sz \quad \text{and} \quad \lim_{n \to \infty} gSx_n = gz
\]

Also since \(S, T\) and \(g\) are compatible,

\[
\lim_{n \to \infty} p(Sgx_n, gSx_n) = 0 = p(Tgx_n, gTx_n).
\]

Then by Lemma 2.10 (i), we get

\[
Sz = gz \quad \text{and} \quad Tz = gz.
\]

Hence \(z\) is a coincidence point of \(S, T\) in \(X\).

The following corollaries can be established for the Theorems 2.12, 2.13 and 2.14

**Corollary 2.15.** Let \((X, \leq, p)\) be a complete partially ordered partial \(b\)-metric space with coefficient \(s \geq 1\). Let \(S, T : X \to X\) be a pair of weakly increasing self maps under \(\psi\)-weak generalized Geraghty contraction and there exists \(x_0 \in X\) such that \(x_0 \leq Sx_0\). If any non decreasing sequence \(\{x_n\}\) in \(X\) converges to \(u\), then we assume that \(x_n \leq u \forall n \geq 0\). Then \(S, T\) have a fixed point in \(X\).

**Proof.** Follows from the theorem 2.12 by choosing \(g = I_x\).

**Corollary 2.16.** Let \((X, \leq, p)\) be a complete partially ordered partial \(b\)-metric space with coefficient \(s \geq 1\). Let \(S, T : X \to X\) be a pair of weakly increasing self maps under weak generalized Geraghty contraction and there exists \(x_0 \in X\) such that \(x_0 \leq Sx_0\). If any non decreasing sequence \(\{x_n\}\) in \(X\) converges to \(u\), then we assume that \(x_n \leq u \forall n \geq 0\). Then \(S, T\) has a fixed point.

**Proof.** Follows from the theorem 2.12 by choosing \(g = I_x\) and \(\psi(t) = t\).

**Corollary 2.17.** Let \((X, \leq, p)\) be a complete partially ordered partial \(b\)-metric space with coefficient \(s \geq 1\). Let \(S, T : X \to X\) be a pair of weakly increasing self maps under \(\psi\)-weak generalized Geraghty contraction and there exists \(x_0 \in X\) such that \(x_0 \leq Sx_0\) and \(S, T\) are continuous. Then \(S, T\) has a fixed point.

**Proof.** Follows from the theorem 2.12 by choosing \(g = I_x\).

**Corollary 2.18.** Let \((X, \leq, p)\) be a complete partially ordered partial \(b\)-metric space with coefficient \(s \geq 1\). Let \(S, T : X \to X\) be a pair of weakly increasing self maps under weak generalized Geraghty contraction and there exists \(x_0 \in X\) such that \(x_0 \leq Sx_0\) and \(S, T\) is non decreasing and continuous. Then \(S, T\) has a fixed point.

**Proof.** Follows from the theorem 2.12 by choosing \(g = I_x\) and \(\psi(t) = t\).

Now we give an example in support of Theorem 2.12

**Example 2.19.** Let \(X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{10}\}\) with usual ordering. Define

\[
p(x, y) = \begin{cases} 
0 & \text{if } x = y \\
1 & \text{if } x \neq y \in \{0, 1\} \\
|x - y| & \text{if } x, y \in \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{10}\} \\
4 & \text{otherwise}
\end{cases}
\]
Clearly, \((X, \leq, p)\) is a partially ordered partial b - metric space with coefficient \(s = \frac{3}{10}\) (P.Kumam et.al [16]). Define \(T : X \rightarrow X\) by \(T1 = T^{1}_{2} = T^{1}_{3} = T^{1}_{4} = T^{1}_{5} = \frac{1}{2}\); \(T0 = T^{1}_{6} = T^{1}_{7} = T^{1}_{8} = T^{1}_{9} = \frac{1}{4}\) \(\Rightarrow T(X) = \{\frac{1}{2}, \frac{1}{4}\}\). Define \(S : X \rightarrow X\) by \(S1 = S^{1}_{2} = S^{1}_{3} = S^{1}_{4} = S^{1}_{5} = S^{1}_{6} = S^{1}_{7} = S^{1}_{8} = S^{1}_{9} = S^{1}_{10} = \frac{1}{2}\) \(\Rightarrow S(X) = \{\frac{1}{2}\}\)

\[g(x) = \begin{cases} \frac{1}{2n-1} & \text{if } 2 \leq n \leq 5 \\ \frac{1}{2} & \text{if } 6 \leq n \leq 10 \\ 0 & \text{if } n = 0,1 \end{cases}\]

\[0 \leq g(x) \leq 1, \text{ for } x \in X\]

Therefore \(T(X), S(X) \subset g(X) \subset X\) and \(g(x) \leq g(y) \Rightarrow T(x) \leq T(y)\) and \(S(x) \leq S(y)\).

Therefore, \(S, T\) are \(g\)-non decreasing. Define \(\psi : [0, \infty) \rightarrow [0, \infty)\) by \(\psi(t) = \frac{1}{t}\) and

\[\beta(t) = \begin{cases} \frac{1}{t+1} & \text{if } t \in (0, \infty) \\ 0 & \text{if } t = 0 \end{cases}\]

For \(x, y \in X\) \(\Rightarrow 0 \leq x = \frac{1}{m} \leq \frac{1}{m}\) and \(0 \leq y = \frac{1}{n} \leq \frac{1}{n}\), the following cases can be observed

1. For \(x, y \in X\) \(\Rightarrow \psi(sp(Sx,Ty)) = 0\) or \(\frac{1}{3} \leq \frac{2}{3} = \beta(\psi(M(x,y)))\psi(M(x,y))\) where \(M(x,y) = \max\{p(gx,gy),p(gx,Sx),p(gy,Ty),\frac{1}{2}[p(gx,Ty) + p(Sx,gy)]\} = 4\).

2. For \(1 \leq m \leq 5\) and \(6 \leq n \leq 10\), \(\Rightarrow \psi(sp(Sx,Ty)) \leq \frac{1}{5} \leq \frac{2}{5} = \beta(\psi(M(x,y)))\psi(M(x,y))\) where \(M(x,y) = 4\).

3. For \(6 \leq m \leq 10\) and \(1 \leq n \leq 5\), \(\Rightarrow \psi(sp(Sx,Ty)) = 0 \leq \frac{1}{3} = \beta(\psi(M(x,y)))\psi(M(x,y))\) where \(M(x,y) = 4\).

4. For \(6 \leq m \leq 10\) and \(6 \leq n \leq 10\), \(\Rightarrow \psi(sp(Sx,Ty)) \leq \frac{1}{5} \leq \frac{2}{5} = \beta(\psi(M(x,y)))\psi(M(x,y))\) where \(M(x,y) = 4\).

\(T^{1}_{2} = \frac{1}{2} = g^{1}_{2} \Rightarrow Tg^{1}_{2} = \frac{1}{2} = gT^{1}_{2} \Rightarrow T\) and \(g\) are weakly compatible at \(\frac{1}{2} \in X\). Also \(S^{1}_{2} = \frac{1}{2} = g^{1}_{2} \Rightarrow Sg^{1}_{2} = \frac{1}{2} = gS^{1}_{2} \Rightarrow S\) and \(g\) are weakly compatible at \(\frac{1}{2} \in X\). Clearly \(gT^{1}_{2} = \frac{1}{2} < \frac{1}{4} = f^{1}_{2}\). Let \(x_{0} = \frac{1}{m} \Rightarrow gx_{0} < Tx_{0} = \frac{1}{4} = g^{1}_{2} = gx_{1} \Rightarrow Sx_{1} = S^{1}_{2} = \frac{1}{2} = g^{1}_{2} = gx_{2} \Rightarrow Tx_{2} = T^{1}_{2} = \frac{1}{2} = g^{1}_{2} = gx_{2} \Rightarrow T\). Therefore \(\frac{1}{2} \in X\) is a fixed point of \(T\). Also since \(S^{1}_{2} = \frac{1}{2}\), Therefore \(\frac{1}{2} \in X\) is a fixed point of \(S\). Therefore \(\frac{1}{2} \in X\) is a unique common fixed point of \(S, T\). The hypothesis and conclusions of of Theorem 2.12 satisfied.

We observe that Theorems 1.12, 1.13 and 1.14 are corollaries of our main results.

**Open Problem:** Are the Theorems 2.12, 2.13, 2.14 and their corollaries true if continuity of \(\psi\) is dropped?.

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**References**


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