Reliability Analysis of Multi-State Series System

M. Reni Sagayaraj¹, A. Merceline Anita¹, A. Chandra Babu² and N. Thirumoorthi¹*

¹ Department of Mathematics, Sacred Heart College, Vellore, Tamilnadu, India.
² Department of Mathematics, Noorul Islam University, Kanyakumari, Tamilnadu, India.

Abstract: Reliability Analysis considering Multiple Possible States is known as Multi-State Reliability Analysis. Multi-State System Reliability Models allow both the System and its Components to assume more than two levels of performance. Though Multi-State Reliability Models provide more realistic and more precise representation of Engineering System, they are much more complex and present major difficulties in System definition and performance evaluation. This paper presents a new Systematic Approach for the Reliability Analysis of Multi-State Series System which helps to determine the Expected Throughput and Performance-Free Failure Operation (PFFO) of the System. This Approach is applied to a Mathematical Model dealing with a Particular Population, being categorized into four groups based on their Haemoglobin level and the Health Status of the Population under Study is analyzed.

Keywords: Multi-State System, Series System, Descartes Product, Multi-Series System.

1. Introduction

All Systems are designed to perform their intended tasks in a given environment. Some Systems can perform their tasks with various distinctive levels of efficiency usually referred to as Performance rates. A System that can have a finite number of Performance Rates is called a Multi-State System (MSS). Usually a MSS is composed of elements that in their turn can be Multi-State. Actually, a Binary System is the simplest case of a MSS having two distinctive States (perfect functioning and complete failure).

The basic concepts of MSS Reliability were primarily introduced in the mid of the 1970’s by Murchland (1975), El-Neveihi et al. (1978), Barlow and Wu (1978) and Ross (1979). Natvig (1982), Block and Savits (1982) and Hudson and Kapur (1982) extended the results obtained in these works. Since that time MSS Reliability began intensive development. Essential achievements that were attained up to the mid 1980’s were reflected in Natvig (1985) and in El-Neveihi and Prochan (1984) where can be found the state of the art in the field of MSS Reliability at this stage. The history of ideas in MSS reliability theory at next stages are found in Lisnianski, Levitin(2003) and Natvig (2007) [8].


* E-mail: mercelineanita@gmail.com
Proschan, and Sethuran [4] and Ross [6] treat the general case of more than two states. This idea is quite useful since in many real life situations, Components (and/or) System can be in intermediate states [7].

In this Paper, we apply a New Technique for evaluating the Expected Throughput and Probability of Failure Free Operation of Multistate Series System; which is applied to Reliability Models [5]. We study the Health status of a particular Population categorized into different groups based on their haemoglobin level. The paper is organized as follows. In Section 1, we give the Introduction on Multistate Series System, In Section 2, we give the basic definitions and related concepts. In Section 3, we derive the expression for calculating the Expected Throughput of Multi State Series System and in Section 4, we deal with the Application of the proposed model and in Section 5, we draw the conclusion.

2. Basic Definitions and Related Concepts

2.1. Series System

In a Series System, all Components in the System should be operating to maintain the required operation of the System. Thus the failure of any one Component of the System will cause failure of the whole System.

\[
R_s = r_1 \cdot r_2 \cdots r_m = \prod_{i=1}^{m} r_i
\]

2.2. Parallel System

In a Parallel System, the System operates if one or more Components operate and the System fails if all components fail. The Parallel n-Components are represented by the following block diagram [8].

For Constant Failure Rates of Parallel units,

\[
R_p (t) = 1 - (1 - r_1 (t)) (1 - r_2 (t)) \cdots (1 - r_n (t))
\]

Where \( R_p (t) \) is the Parallel System Reliability at time \( t \). \( r_i (t) \) is the Reliability of the \( i^{th} \) component

\[
R_p (t) = 1 - \prod_{i=1}^{n} (1 - r_i (t)).
\]
2.3. Descartes Product of Two Sets

\[
\begin{array}{cccc}
(p_{11}, a_{11}) & (p_{12}, a_{12}) & \ldots & (p_{15}, a_{15}) \\
(p_{21}, a_{21}) & (p_{11}, A_{11}) & \ldots & (P_{25}, A_{51}) \\
(p_{22}, a_{22}) & (p_{22}, A_{22}) & \ldots & (P_{25}, A_{25}) \\
\ldots & \ldots & \ldots & \ldots \\
(p_{27}, a_{27}) & (P_{17}, A_{17}) & \ldots & (P_{27}, A_{27})
\end{array}
\]

We present the first polynomial \( \varphi_1(z) \) as a set of pairs \( \{(p_{11}, a_{11}), (p_{12}, a_{12}), \ldots, (p_{15}, a_{15})\} \) and the second polynomial \( \varphi_2(z) \) as the set of pairs \( \{(p_{21}, a_{21}), (p_{22}, a_{22}), \ldots, (p_{25}, a_{25})\} \), where \( p_{ij} \) and \( a_{ij} \) are corresponding coefficients and are corresponding powers of polynomials in unfolded form. Here \( P_{jk} \) is found as \( P_{jk} = P_{j1} \times P_{2k} \) and \( A_{jk} \) is found as \( A_{jk} = a_{j1} + a_{2k} \). In this case, we can keep a polynomial from of specific type: powers of product of two terms, say \( Z_a \) and \( Z_b \), will be presented by some transforms over powers of individual terms, namely,

\[
p_a z^a \otimes p_b z^b = p_a p_b z^{f(a,b)}
\]

where \( f \) is an arbitrary given function. Assume that unit \( k \) is characterized by following discrete distribution of its operational parameter

\[
X_k : P(X_k = x_{kj}) = p_{kj}.
\]

Then we can characterize the distribution of the operational parameter of unit \( k \) with the following vector of pairs.

\[
Q_k = \{(p_{k1}, x_{k1}), (p_{k2}, x_{k2}), \ldots, (p_{ks(k)}, x_{ks(k)})\}
\]

\[
= \{(p_{kj}, x_{kj}), 1 \leq j \leq s(k)\},
\]

where \( s(k) \) is number of different values of random variable \( X_k \). Interaction of operational parameters of two units \( X_k \) and \( X_i \) can be written as

\[
Q_k \otimes Q_i = \{(p_{kj}, x_{kj}), 1 \leq j \leq s(k)\} \otimes \{(p_{li}, x_{li}), 1 \leq l \leq s(i)\}
\]

\[
= (p_{kj} \times p_{li}, f(x_{kj}, x_{li}), j = 1, 2, \ldots, s(k), l = 1, 2, \ldots, s(i)).
\]

Interaction of Operational parameters of \( N \)-units can be written as

\[
\otimes_{f}(Q_1, \ldots, Q_K, \ldots, Q_N) = \prod_{i=1}^{N} P_{i,m(i)} f(x_{1,m(i)} \ldots x_{N,m(i)})
\]

for all combinations of \( m(i) \) where \( 1 \leq m(i) \leq s(i) \).

3. Multistate Series System

Consider a Simple Multistate Series System consisting of \( n \)-different units, where each unit has different states and the Expected Throughput of Multi State Series System changes randomly due to external and internal causes. The Fig. 3 below represents a Multi State System Configuration [7].
Reliability Analysis of Multi-State Series System

First unit:

\[ p_{11} = \Pr\{v = X_1\} = x_1 \]
\[ p_{12} = \Pr\{v = X_2\} = x_2 \]
\[ p_{13} = \Pr\{v = X_3\} = x_3 \]
\[ p_{14} = \Pr\{v = X_4\} = x_4 \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

\[ p_{1n} = \Pr\{v = X_n\} = x_n \]

Second unit:

\[ p_{21} = \Pr\{v = X_1\} = y_1 \]
\[ p_{22} = \Pr\{v = X_2\} = y_2 \]
\[ p_{23} = \Pr\{v = X_3\} = y_3 \]
\[ p_{24} = \Pr\{v = X_4\} = y_4 \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

\[ p_{2n} = \Pr\{v = X_n\} = y_n \]

where \( X_1, X_2, \ldots, X_n \) denotes different states \( X_i > X_j \) for \( j > i \). The Entire System is characterized by minimum value of its units’ through puts, that is, we have to use the operator \( \otimes_{\min} \) because

\[ f^{(v)}_{\text{SERIES}}(x_k, x_j) = \min(x_k, x_j) \]

Let us consider the following recurrence procedure. Consider the interaction of parameters of units 1 and 2. Take a Descartes product in the form of the following table.

<table>
<thead>
<tr>
<th>State : 1</th>
<th>State : 2</th>
<th>State : 3</th>
<th>…</th>
<th>State : n</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1, X_1 )</td>
<td>( y_1, X_1 )</td>
<td>( x_2, X_2 )</td>
<td>( y_2, X_2 )</td>
<td>( x_3, X_3 )</td>
</tr>
</tbody>
</table>

\[ (x_1)(y_1) = x_1 y_1 \]
\[ (x_2)(y_1) = x_2 y_1 \]
\[ (x_3)(y_1) = x_3 y_1 \]
\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

\[ (x_1)(y_n) = x_1 y_n \]
\[ (x_2)(y_n) = x_2 y_n \]
\[ (x_3)(y_n) = x_3 y_n \]
\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
Third unit:

\[ p_{31} = Pr\{v = X_1\} = z_1 \]
\[ p_{32} = Pr\{v = X_2\} = z_2 \]
\[ p_{33} = Pr\{v = X_3\} = z_3 \]
\[ p_{34} = Pr\{v = X_4\} = z_4 \]
\[ \ldots \]
\[ p_{3n} = Pr\{v = X_n\} = z_n \]

Here and below the sum of all probabilities is not exactly equal to 1 due to rounding of results of multiplication of corresponding probabilities. The new equivalent unit has to be combined with the third unit.

<table>
<thead>
<tr>
<th>State : 1</th>
<th>State : 2</th>
<th>State : 3</th>
<th>...</th>
<th>State : n</th>
</tr>
</thead>
<tbody>
<tr>
<td>((y_1, X_1))</td>
<td>((y_2, X_2))</td>
<td>((y_3, X_3))</td>
<td>...</td>
<td>((y_n, X_n))</td>
</tr>
</tbody>
</table>

These results allow calculating the expected value of the level \(E[v]\).

\[
E[v] = z_1y_1X_1 + [(z_1, y_1) + (z_2, y_2)]X_2 + [(z_1, y_1) + (z_2, y_2) + (z_3, y_3)]X_3 + \ldots + [(z_1, y_1) + (z_2, y_2) + (z_3, y_3) + \ldots + (z_n, y_{n-1})]X_n
\]

\[
= [z_1y_1X_1 + \sum_{i=1}^{2} z_iy_iX_2 + \sum_{i=1}^{3} z_iy_iX_3 + \sum_{i=1}^{4} z_iy_iX_4 + \ldots + \sum_{i=1}^{n-2} z_iy_iX_{n-2} + \sum_{j=1}^{n-3} z_jy_{n-1}X_{n-1} + \sum_{i=1}^{n-1} z_iy_iX_n]X_n
\]

4. The Example for a SSM (Series system model)

We analyze the problem below to check the health status of the Population under study. Consider a Simple example of Multistate Series System Consisting of four different groups of people, where each group has people with four states of different levels of haemoglobin in their blood. The Hb level varies due to various internal & external causes.
Considered structure of group of peoples HB level.

**First unit:**

\[ p_{11} = Pr\{Hb = X_1\} = 0.7 \]
\[ p_{12} = Pr\{Hb = X_2\} = 0.2 \]
\[ p_{13} = Pr\{Hb = X_3\} = 0.1 \]
\[ p_{14} = Pr\{Hb = X_4\} = 0 \]

**Second unit:**

\[ p_{21} = Pr\{Hb = X_1\} = 0.8 \]
\[ p_{22} = Pr\{Hb = X_2\} = 0.15 \]
\[ p_{23} = Pr\{Hb = X_3\} = 0.05 \]
\[ p_{24} = Pr\{Hb = X_4\} = 0 \]

**Third unit:**

\[ p_{31} = Pr\{Hb = 18\} = 0.56 \]
\[ p_{32} = Pr\{Hb = 17\} = 0.105 + 0.03 + 0.16 = 0.295 \]
\[ p_{33} = Pr\{Hb = 16\} = 0.035 + 0.01 + 0.005 + 0.015 + 0.08 = 0.145 \]
\[ p_{34} = Pr\{Hb = 14\} = 0 \]

Here and below the sum of all probabilities is not exactly equal to 1 due to rounding of results of multiplication of corresponding probabilities. The new equivalent unit has to be combined with the third unit.
These results allow calculating the expected value of the hemoglobin level $E[Hb]$.

$$E[Hb] = 0.448(18) + [0.084 + 0.04425 + 0.236](17) + [0.028 + 0.01475 + 0.00725 + 0.02175 + 0.116](16)$$

$$= 17.26025$$

PFFO of the system is obtained by some chosen criteria of failure. For instance, if a failure criterion is $V < 17$ then PFFO is equal to

$$P\{V > 17\} = 0.448 + 0.084 + 0.04425 + 0.236$$

$$= 0.81225$$

81.225 % of the population are having good range of haemoglobin level.

5. Conclusion

This Paper offers ideas for using Reliability Principles to determine the Health States of the Population and take measures to improve Reliability. Applying the lessons from Reliability Engineering to Health Care System, holds the promise of moving our Blood System to new levels of Hb Consistency and Quality.

References