One Point Union of $\alpha$–graceful Graphs

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Abstract: The notation of $\alpha$-labeling of a graph is the natural generalization of graceful labeling and it helps produce new graceful graphs by some graph operations. In this paper, we have proved that one point union of two $\alpha$-graceful graphs is $\alpha$-graceful under some conditions, one point union of an $\alpha$-graceful graph and a graceful graph is graceful. We have also proved that the consecutive one point union of a finite number of $\alpha$-graceful graphs with a graceful graph by merging one vertex under some conditions is also a graceful graph. These results help enlarge the class of graceful graphs.

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1. Introduction

Most graph labeling methods trace their origin $\alpha$-labeling as a graceful labeling with an additional property that there exist a non-negative integer $k(0 \leq k < |E(G)|)$ such that for each edge $e = (x, y) \in E(G)$, $\min\{f(x), f(y)\} \leq k < \max\{f(x), f(y)\}$ in a graph $G$. Here $f$ is a graceful labeling for a graph $G$ and it becomes $\alpha$-labeling if it satisfies above property. A graph $G$ which admits an $\alpha$-labeling, we call $\alpha$-graceful graph. An $\alpha$-graceful graph is necessarily bipartite. This was introduced by Rosa [6]. Kaneria and Jariya [3] defined smooth graceful labeling and semi smooth graceful labeling. In [5] Kaneria, Meera and M. Khoda have proved that smooth graceful labeling, semi smooth graceful labeling are equivalent with $\alpha$-graceful labeling. Kaneria, Viradia and Makadia [4] have proved that the path union of a semi smooth graceful graph, star of a semi smooth graceful graph and cycle of a semi smooth graceful graph are graceful. For a comprehensive bibliography of papers on graph labeling are given in Gallian [2]. In this paper, all graphs are finite, simple and undirected and we consider $G$ as a finite simple, undirected graph with $|V(G)| = p$ vertices and $|E(G)| = q$ edges. For all terminology and standard notation we follow by Harary [1]. The present paper is focused to discuss one point union of some graphs and its $\alpha$-graceful labeling as well as graceful labeling. One point of union of two graphs $G$ and $H$ is the graph $G \cup H$ obtain by merging one vertex of $G$ with one vertex of $H$ as a common vertex of $G \cup H$.

2. Main Results

Theorem 2.1. Let $G_1, G_2$ be $\alpha$-graceful graphs and $|E(G_1)| = q_1, |E(G_2)| = q_2$. Let $f_1, f_2$ be $\alpha$-graceful labeling ($\alpha$-labeling) for $G_1$ and $G_2$ respectively. Let $k_1 (0 \leq k_1 < q_1), k_2 (0 \leq k_2 < q_2)$ be two non-negative integers such that for every $e_i = (x_i, y_i) = (x_i, y_i) \in E(\cup H)$, $\min\{f_i(x), f_i(y)\} \leq k_i < \max\{f_i(x), f_i(y)\}$, where $f_i$ is a graceful labeling for $G_i$. Then the one point union $G \cup H$ is $\alpha$-graceful.

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Theorem 2.3. Let \( f_1 \) be an \( \alpha \)-graceful labeling for \( G_i \), \( \forall \ i = 1, 2, \ldots, l \). Let \( f_1 \) be an \( \alpha \)-graceful labeling for \( G_i \), \( \forall \ i = 1, 2, \ldots, l \) be a non-negative integer such that for every \( e_i = (x_i, y_i) \in E(G_i) \), \( \min\{f_1(x_i), f_1(y_i)\} \leq k_i < \max\{f_1(x_i), f_1(y_i)\} \), \( \forall \ i = 1, 2, \ldots, l \). Let \( H \) be a graceful graph with a graceful labeling function \( f_{i+1} : V(H) \rightarrow \{0, 1, \ldots, q_i+1\} \), where \( q_i+1 = |E(H)| \). Then the consecutive one point union of \( G_1, G_2, \ldots, G_l \) and \( H \) by merging vertex \( v_i \in V(G_i) \) with \( f_i(v_i) = k_i \) and vertex \( w_i \in V(G_{i+1}) \) with \( f_{i+1}(w_i) = 0, \forall \ i = 1, 2, \ldots, l \) (assuming \( G_{i+1} = H \)) is also a graceful graph.
Proof. Take, $V^*(i) = \{u \in V(G_i)/f_i(u) \leq k_i\}$ and $V^*(i) = V(G_i) - V^*(i)$, $\forall i = 1, 2, \ldots, l$. Define a labeling function $h_i : V(\bigcup_{j=1}^{i+1} G_j) \rightarrow \{0, 1, \ldots, \sum_{j=1}^{i+1} q_j\}$, $\forall i = 1, 2, \ldots, l - 1$ as follows.

$$h_i(w) = h_{i-1}(w), \text{ when } h_{i-1}(w) \leq \sum_{j=1}^{i} k_j$$

$$= h_{i-1}(w) + q_{i+1}, \text{ when } h_{i-1}(w) > \sum_{j=1}^{i} k_j, \forall w \in V(\bigcup_{j=1}^{i} G_j)$$

$$h_i(w') = f_{i+1}(w') + \sum_{j=1}^{i} k_j, \forall w' \in V(G_{i+1}), \text{ by assuming } h_0 = f_1.$$ 

By applying Theorem 2.1, $h_i$ ($i = 1, 2, \ldots, l - 1$) are graceful labeling for $\bigcup_{j=1}^{i+1} G_j$. Moreover, for each $e \in (a, b) \in E(\bigcup_{j=1}^{i+1} G_j)$, $h_i(a)$ is either $h_{i-1}(a) + q_{i+1}$ or $f_{i+1}(a) + \sum_{j=1}^{i+1} k_j$; these both are greater than $\sum_{j=1}^{i+1} k_j$ and $h_i(b)$ is either $h_{i-1}(b)$ or $f_{i+1}(b) + \sum_{j=1}^{i+1} k_j$; these both are less than or equal to $\sum_{j=1}^{i+1} k_j$ (assuming $f_{i+1}(a) - f_{i+1}(b)$, $f_{i}(a) - f_{i}(b)$ are positive and $h_0 = f_1$) for each $i = 1, 2, \ldots, l - 1$.

Thus, $\sum_{j=1}^{i+1} k_j$ is a non-negative integer and for each $e = (a, b) \in E(\bigcup_{j=1}^{i+1} G_j)$, $h_i(b) \leq \sum_{j=1}^{i+1} k_j < h_i(a), \forall i = 1, 2, \ldots, l - 1$.

Therefore, for each $i = 1, 2, \ldots, l - 1$, $h_i : V(\bigcup_{j=1}^{i+1} G_j) \rightarrow \{0, 1, \ldots, \sum_{j=1}^{i+1} q_j\}$ is $\alpha$- graceful labeling for the graph $\bigcup_{j=1}^{i+1} G_j$.

Now define $h_l : V(\bigcup_{j=1}^{l} G_j \cup H) \rightarrow \{0, 1, \ldots, \sum_{j=1}^{l+1} q_j\}$ as follows.

$$h_l(w) = h_{l-1}(w) \text{ when } h_{l-1}(w) \leq \sum_{j=1}^{l} k_j$$

$$= h_{l-1}(w) + q_{l+1}, \text{ when } h_{l-1}(w) > \sum_{j=1}^{l} k_j \forall w \in V(\bigcup_{j=1}^{l} G_j);$$

$$h_l(w') = f_{l+1}(w') + \sum_{j=1}^{l} k_j, \forall w' \in V(H).$$

By applying Theorem 2.2, $h_l$ is a graceful labeling for $\bigcup_{j=1}^{l} G_j \cup H$. So, it is a graceful graph.

Illustration 2.4. One point union of $C_8$, $K_{4, 3}$, and $K_{3, 4} \cup P_7$ is a graceful graph.

Here, take $V(C_8) = \{v_1, v_2, \ldots, v_8\}$, $V(K_{4, 3}) = \{u_1, u_2, u_3, u_4, w_1, w_2, w_3\}$. Define a $\alpha$-labeling function $f_1 : V(C_8) \rightarrow \{0, 1, \ldots, 8\}$ as follows.

$$f_1(v_i) = 9 - i, \forall i = 1, 3, 5, 7$$

$$= \frac{i - 2}{2}, \forall i = 2, 4$$

$$= \frac{i}{2}, \forall i = 6, 8 \text{ is an } \alpha - \text{labeling for } C_8 \text{ and } K_1 = 4.$$ 

Define a $\alpha$-labeling function $f_2 : V(K_{4, 3}) \rightarrow \{0, 1, \ldots, 12\}$ as follows.

$$f_2(u_i) = i - 1, \forall i = 1, 2, 3, 4$$

$$= 12 - 4(j - 1), \forall j = 1, 2, 3$$

is also an $\alpha$-labeling for $K_{4, 3}$ and $k_2 = 3$. $K_{3, 4} \cup P_7$ is a graceful graph with graceful labeling function $f_3$ given in following Figure 1.
Figure 1. graceful labeling function $f_3$ for $K_{3,4} \cup P_7$.

It is obvious that $h_1 : V(C_8 \cup K_{4,3}) \rightarrow \{0, 1, \ldots, 20\}$ by merging the vertex $v_8$ with the vertex $u_1$ as $f_1(v_8) = k_1 = 4$ and $f_2(u_1) = 0$ defined by

$$h_1(u_i) = f_1(v_i) + 12 = 21 - i, \quad \forall \ i = 1, 3, 5, 7$$
$$= f_2(v_i) \quad \forall \ i = 2, 4, 6, 8$$
$$h_1(u_i) = f_2(u_i) + 4 = i + 3 \quad \forall \ i = 1, 2, 3, 4$$
$$h_1(w_j) = f_2(w_j) + 4 = 16 - 4(j - 1) \quad \forall \ j = 1, 2, 3;$$

is an $\alpha$-graceful labeling for $C_8 \cup K_{4,3}$ with the non-negative integer $k_1 + k_2 = 7$. $h_2 : V(C_8 \cup K_{4,3} \cup K_{3,4} \cup P_7) \rightarrow \{0, 1, \ldots, 38\}$ by merging the vertex $u_4$ with vertex $v$ of $K_{3,4} \cup P_7$ where $f_3(v) = 0$, as $h_1(u_4) = 7$ defined by

$$h_2(v_i) = h_1(v_i) + 18 = 39 - i, \quad \forall \ i = 1, 3, 5, 7$$
$$= h_1(v_i) = f_1(v_i) \quad \forall \ i = 2, 4, 6, 8$$
$$h_2(u_i) = h_1(u_i) = f_2(u_i) + 4 \quad \forall \ i = 1, 2, 3, 4$$
$$h_2(w_j) = h_1(w_j) = f_2(w_j) + 4 \quad \forall \ j = 1, 2, 3$$
$$h_2(w') = f_3(w') + 7, \quad \forall \ w' \in V(K_{3,4} \cup P_7);$$

is a graceful labeling for required graph $C_8 \cup K_{4,3} \cup (K_{3,4} \cup P_7)$ by merging the vertex $v_8$ with the vertex $u_1$ and the vertex $u_4$ with the vertex of $K_{3,4} \cup P_7$ whose vertex label is 0 under $f_3$. Such $\alpha$-graceful labeling for $C_8 \cup K_{4,3}$ and graceful labeling for the required graph $C_8 \cup K_{4,3} \cup (K_{3,4} \cup P_7)$ are shown in following figure - 2, 3 respectively.

Figure 2. $\alpha$-graceful labeling of $C_8 \cup K_{4,3}$ [here $k = 7$]
Figure 3. Graceful labeling for one point union of graphs $C_8, K_{4,3}$ and $(K_{3,4} \cup P_7)$.

References


