Fuzzyfication of Semigroups

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Abstract: The motivation mainly comes from the Fuzzification of sets that are of importance and interest in Semigroups, Gamma-Semigroups etc. In this paper we characterize the properties related to Fuzzy sub semigroups and Fuzzy ideals using identities on Semigroups.

Keywords: Fuzzy sub semigroups, Fuzzy (left, right) ideals, Regular Semigroups, Fuzzy bi ideal.

1. Introduction

Fuzzy set was introduced by Zadeh.L.A [1] and others have found many applications in the domain of Mathematics and elsewhere. After the introduction of fuzzy sets by Zadeh.L.A, reconsideration of classical mathematics began [2]. Fuzzy set has an important impact over the field of mathematical research in both theory and application. It has found manifold applications in Mathematics and related areas [2, 10, 11].

The concept of Fuzzification in Semigroups was first discussed by Kuroki.N.A. He studied fuzzy (left, right) ideals and fuzzy bi ideals in Semigroups [3–5, 9, 12]. The study of Fuzzy algebraic structures with the introduction of the concept of Semigroups and Fuzzy ideals were studied by Rosenfield.A [6]. Later Dib. K.A studied some basic concepts of fuzzy algebra such as fuzzy (left, right) ideals and fuzzy bi ideals in Semigroups using a new approach of fuzzy spaces and fuzzy groups [7]. Wang XuePing, Mo Zhi-Wen and Liu Wang-Jin discussed about Fuzzy ideals generated by fuzzy point in Semigroups [8]. The results in the present communication are obtained by considering some identities of Semigroups with different techniques.

1.1. Fuzzy Sets Definitions

(a). Fuzzy subset of a non empty set is a collection of objects with each object being assigned a value between 0 and 1 by a membership function

(b). Let X be a non empty set. A fuzzy set μ of the set X is a function μ : X → [0, 1].

(c). Let S be a semigroup. A map A from S to [0, 1] is called a fuzzy set in S.

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(d). Let $F(S)$ denote the set of all fuzzy sets in $S$. For $A, B \in F(S)$, $A \subseteq B$ if and only if $A(x) \leq B(x)$ in the ordering of $[0, 1]$, $\forall x \in S$.

(e). For $A, B \in F(S)$, the product $A \circ B$ is defined as

$$A \circ B(x) = \sup\{\min\{A(y), B(z)\}\} \text{ for } y, z \in S, x = yz$$

$$= 0; \text{ for } y, z \in S, x \neq yz$$

(f). A fuzzy set $A \in F(S)$ is said to be a fuzzy point, if $A = \bigcup x\lambda$, where $0 < \lambda \leq 1$, $x\lambda \in A$ if $x\lambda \subseteq A$ and

$$x\lambda(y) = \lambda, \text{ if } y = x$$

$$= 0, \text{ if } y \neq x, \forall y \in S$$

(g). A fuzzy set $A \in F(S)$ is said to be a fuzzy sub semigroup of $S$ if $A(xy) \geq \min\{A(x), A(y)\} \forall x, y \in S$.

(h). A fuzzy set $A \in F(S)$ is said to be a fuzzy left ideal of $S$ if $A(xy) \geq A(y) \forall x, y \in S$.

(i). A fuzzy set $A \in F(S)$ is said to be a fuzzy right ideal of $S$ if $A(xy) \geq A(x) \forall x, y \in S$.

(j). A fuzzy set $A \in F(S)$ is said to be a fuzzy ideal of $S$ if it is both a fuzzy left and fuzzy right ideal of $S$.

(k). A fuzzy sub semigroup $A \in F(S)$ is said to be a fuzzy bi ideal of $S$ if $A(xyz) \geq \min\{A(x), A(z)\} \forall x, y, z \in S$.

2. Fuzzyfication on Semigroup $S$ with Some Identities: For all $a, b \in S$, $aba = ab$

**Theorem 2.1.** Let $S$ be a regular semigroup and satisfy an identity $aba = ab \forall a, b \in S$ then

(i). $\mu(ab) = \mu(a)$

(ii). $\mu(a^2) = \mu(a)$

(iii). $\mu(a^{n+1}) = \mu(a)$ for any nonempty fuzzy subset $\mu$ of $S$.

**Proof.** Let $S$ be a regular semigroup. Then we know that $axa = a \forall a \in S$ and for some $x \in S$. Given $S$ satisfies $aba = ab \forall a, b \in S$.

(i). Consider

$$\mu(ab) = \mu(abab) \quad \text{[Since ab=aba]}$$

$$= \mu(a(bab)) \quad \text{[Associativity in S]}$$

$$= \mu(aba) \quad \text{[Since bab=ba]}$$

$$\mu(ab) = \mu(a) \quad \text{[Since aba=a].}$$

(ii). Consider $\mu(a^2) = \mu(a.a) \Rightarrow \mu(a^2) = \mu(a)$ [Put $b = a$ from (i) $\mu(ab) = \mu(a)$; proved].
Consider \( \mu(a^{n+1}) \), where \( n = 1, 2, 3, \ldots \). Let us prove by mathematical induction.

For \( n = 1 \), \( \mu(a^2) = \mu(a.a) = \mu(a) \) [proved above]

For \( n = 2 \), \( \mu(a^3) = \mu(a^2.a) = \mu(a.a.a) = \mu(a) \) [Since \( axa = a \), given put \( x = a \)]

\[ \text{In general } \mu(a^{n+1}) = \mu(a.a^n) = \mu(a). \] Thus \( \mu(a^{n+1}) = \mu(a) \).

\[ \square \]

**Theorem 2.2.** Let \( \mu \) be a fuzzy bi ideal in a semigroup \( S \) and \( S \) satisfy an identity \( aba = ab \), \( \forall \ a, b \in S \) then \( \mu \) is a right ideal in \( S \).

**Proof.** Let \( \mu \) be a fuzzy bi ideal in a semigroup \( S \). Then for any \( x, y, z \in S \), we have

\[ \mu(xyz) \geq \min\{\mu(x), \mu(z)\} \quad (1) \]

and is a fuzzy sub semigroup on \( S \). Given \( S \) satisfies the identity \( aba = ab \), \( \forall \ a, b \in S \). To show that \( \mu \) is a fuzzy right ideal in \( S \). i.e., \( \mu(xy) \geq \mu(x) \) \( \forall \ x, y \in S \). Consider

\[ \mu(xy) = \mu(xyx) \] [Given \( aba = ab \), we write \( axa = ax \), \( \forall \ x, y \in S \)]

\[ \geq \min\{\mu(x), \mu(x)\} \] [From (1) put \( z = x \)]

\[ \geq \mu(x). \]

Therefore \( \mu(xy) \geq \mu(x) \) \( \forall \ x, y \in S \). Hence \( \mu \) is a fuzzy right ideal in \( S \). \[ \square \]

**Theorem 2.3.** Let \( \mu \) be a fuzzy left ideal of a semigroup \( S \) and \( S \) satisfy an identity \( aba = ab \), \( \forall \ a, b \in S \) then \( \mu \) is a fuzzy sub semigroup of \( S \).

**Proof.** Let \( \mu \) be a fuzzy left ideal in a semigroup \( S \). Then for all \( x, y \in S \), we have

\[ \mu(xy) \geq \mu(y) \quad (2) \]

Given \( S \) satisfies the identity

\[ aba = ab \], \( \forall \ a, b \in S \) \quad (3)

To prove \( \mu \) is a fuzzy sub semigroup. i.e., \( \mu(xy) \geq \min\{\mu(x), \mu(y)\} \) \( \forall \ x, y \in S \) or \( \mu(xy) \geq \{\mu(x) \Lambda \mu(y)\} \) \( \forall \ x, y \in S \). Now consider

\[ \mu(xy) = \mu(xy) \] [From (3)]

\[ = \mu((xy)x) \] [Associativity in \( S \)]

\[ \mu(xy) \geq \mu(x) \] [From (2)]

From (4) we have

\[ \mu(xy) \wedge \mu(xy) \geq \mu(x) \wedge \mu(xy) \]

\[ \geq \mu(x) \wedge \mu(y) \] [From (2)]

\[ \mu(xy) \geq \mu(x) \wedge \mu(y) \] \( \forall \ x, y \in S \).

Thus \( \mu \) is a fuzzy sub semigroup of \( S \). \[ \square \]
**Theorem 2.4.** Let $\mu$ be a fuzzy left ideal of a semigroup $S$ and $S$ satisfy an identity $aba = ab$, $\forall a, b \in S$ then $\mu(ab) = \mu(ba)$.

**Proof.** Let $\mu$ be a fuzzy left ideal in a semigroup $S$. Then for all $x, y \in S$, we have

$$\mu(ab) \geq \mu(b) \quad \forall a, b \in S$$  \hspace{1cm} (5)

Given $S$ satisfies the identity $aba = ab$, $\forall a, b \in S$. To prove $\mu(ab) = \mu(ba)$, we prove that $\mu(ab) \geq \mu(ba)$ and $\mu(ba) \geq \mu(ab)$.

Now

$$\mu(ab) = \mu(aba) = \mu(a(ba)) \quad [\text{Associativity in } S]$$

$$\geq \mu(ba) \quad [\text{From (5)}]$$

$$\therefore \mu(ab) \geq \mu(ba)$$  \hspace{1cm} (6)

Now

$$\mu(ba) = \mu(bab) = \mu(b(ab)) \quad [\text{Associativity in } S]$$

$$\geq \mu(ab) \quad [\text{From (5)}]$$

$$\mu(ba) \geq \mu(ab)$$  \hspace{1cm} (7)

Thus from (6) and (7) we have $\mu(ab) = \mu(ba)$ $\forall a, b \in S$.

**Theorem 2.5.** Let $\mu$ be a fuzzy left ideal of a semigroup $S$ and $S$ satisfy an identity $aba = ab$, $\forall a, b \in S$ then $\mu \circ \mu \leq \mu$.

**Proof.** Let $\mu$ be a fuzzy left ideal in a semigroup $S$. Given $S$ satisfies the identity $aba = ab$, $\forall a, b \in S$. Then from result 3 we know that

$$\mu(ab) \geq \mu(a) \land \mu(b) \quad \forall a, b \in S$$  \hspace{1cm} (8)

Now

$$\mu \circ \mu(x) = V\{\mu(p) \land \mu(q)\}$$

$$= V\{\mu(p) \land \mu(q)\} \quad [\text{Let } x = ab]$$

$$= \mu(a) \land \mu(b) \leq \mu(ab) \quad [\text{From (8)}]$$

i.e., $\mu \circ \mu(x) \leq \mu(x)$

$$\therefore \mu \circ \mu \leq \mu$$

References


