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Hyers-Ulam Stability of n^{th} Order Non-Linear Differential Equations with Initial Conditions

Research Article

K.Ravi[1], R.Murali[1], A.Ponmanaselvan[2]* and R.Veerasivsji[2]

1 PG and Research Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, Tamil Nadu, India.

2 PG Department of Mathematics, Sri Sankara Arts and Science College (Autonomous), Enathur, Kanchipuram, Tamil Nadu, India.

Abstract: In this paper, we investigate the Hyers - Ulam stability of a Generalized n^{th} order Non Linear Differential Equation of the form $x^{(n)}(t) - F(t, x(t)) = 0$ with initial conditions $x(a) = x^{'}(a) = x^{''}(a) = ... = x^{(n-1)}(a) = 0$, where $x \in C^n(I)$, $I = [a, b]$, $-\infty < a < b < \infty$ and $|F(t, x(t))| \leq L\left|x^{(n-2)}(t)\right|^{\alpha}$, $\alpha > 0, -\infty < x < \infty$, with $F(t, 0) = 0$. Moreover, we prove the Hyers - Ulam stability of the Emden - Fowler type differential equation of n^{th} order $x^{(n)}(t) - h(t)\ |x(t)|^{\alpha}\ sgn\ x(t) = 0$, with the initial conditions $x(a) = x^{'}(a) = x^{''}(a) = ... = x^{(n-1)}(a) = 0$. Where $x \in C^n(I)$, $I = [a, b]$, $-\infty < a < b < \infty$, $\alpha > 0$, $\alpha \neq 1$ and $h(t)$ is bounded in $\mathbb{R}$.

MSC: 34K20, 26D10, 31K20, 39A10, 34A40, 39B82, 34C20.

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1. Introduction

In 1940, S. M. Ulam [1] gave a wide range of talk before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. It motivated the study of stability problems for various functional equations. Among the problem raised by S. M. Ulam, the following question is concerned about the stability of homomorphisms.

Theorem 1.1. *Let G_1 be a group and let G_2 be a group endowed with a metric ρ. Given $\epsilon > 0$, does there exists a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies*

$$\rho(f(xy), f(x)\ f(y)) < \ \delta,$$

for all $x, y \in G$, then we can find a homomorphism $h : G_1 \to G_2$ exists with

$$\rho(f(x), h(x)) < \epsilon$$

for all $x \in G_1$?.

If the answer is affirmative, we say that the functional equation for homomorphisms is stable. In 1941, Hyers [2] was the first Mathematician, who brilliantly answered the question of Ulam by considering G_1 and G_2 to be Banach spaces. The result of Hyers is stated in the following celebrated Theorem.

* *E-mail: selvaharry@yahoo.com*

Theorem 1.2 (Hyers [2]). *Assume that G_1 and G_2 are Banach spaces. If a function $f : G_1 \to G_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \tag{1}$$

for some $\epsilon > 0$ and for all $x,\ y \in G_1$, then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n}\ f(2^n\ x) \tag{2}$$

exists for each $x \in G_1$ and $A : G_1 \to\ G_2$ is the unique additive function such that

$$\|f(x) - A(x)\|\ \leq \epsilon \tag{3}$$

for all $x \in G_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in G_1$, then A is linear.

Taking the above fact into account, the additive functional equation

$$f(x+y) = f(x) + f(y)$$

is said to have **Hyers - Ulam stability** on (G_1, G_2). In the above Theorem, an additive function A satisfying the inequality (3) is constructed directly from the given function f and it is the most powerful tool to study the stability of several functional equations. In course of time, the Theorem formulated by Hyers was generalized by Aoki, Th. M. Rassias [3–5], for additive mappings. A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation

$$\phi\left(f, x, x^{'},\ x^{''},\ ...\ x^{(n)}\right) = 0 \tag{4}$$

has the Hyers - Ulam stability if for a given $\epsilon > 0$ and a function x such that

$$\left|\phi\left(f, x, x^{'},\ x^{''},\ ...\ x^{(n)}\right)\right| \leq \epsilon,$$

there exists a solution x_a of the differential equation (4) such that

$$|x(t) - x_a(t)|\ \leq\ K(\epsilon)$$

and

$$\lim_{\epsilon\to 0} K(\epsilon) = 0.$$

If we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\varphi(t)$, where $\phi,\ \varphi$ are appropriate functions not depending on x and x_a explicitly, then we say that the corresponding differential equation has the generalized Hyers - Ulam stability or Hyers - Ulam - Rassias stability.

Obloza seems to be the first author who has investigated the Hyers - Ulam stability of linear differential equations [6, 7]. Thereafter, In 1998, C. Alsina and R. Ger [8] were the first authors who investigated the Hyers - Ulam stability of the following differential equation

$$x^{'}(t) - x(t) = 0.$$

They proved in [8] the following Theorem.

Theorem 1.3. *Assume that a differentiable function $f : I \longrightarrow R$ is a solution of the differential inequality*

$$\left\| x^{'}(t) - x(t) \right\| \leq \epsilon.$$

where I is an open sub interval of R . Then there exists a solution $g : I \longrightarrow R$ of the differential equation $x^{'}(t) = x(t)$ such that for any $t \in I$, we have,

$$\| f(t) - g(t) \| \leq \ 3\epsilon.$$

This result of C. Alsina and R. Ger [8] has been generalized by Takahasi [9]. They proved in [9] that the Hyers - Ulam stability holds true for the Banach Space valued for differential equation $y^{'}(t) = \lambda y(t)$. Indeed, the Hyers -Ulam stability has been proved for the first order linear differential equations in more general settings in [10–13]. In 2012, M. N. Qarawai [14] proved the Hyers - Ulam stability of Linear and nonlinear differential equation of second order of the form

$$z^{''} + p(x)z^{'} + q(x)z = h(x)\,|z|^{p}\ \ e^{\frac{\beta-1}{2}\ \int p(x)dx} sgn\ z$$

where $\beta \in (0, 1)$ with the initial conditions. He also proved in [15] the Hyers - Ulam stability of a generalized second order nonlinear differential equation of the form $z^{''} - F(x, z(x)) = 0$ with the initial conditions. In this paper, we investigate the Hyers - Ulam stability of a General n^{th} order Non Linear Differential Equation of the form

$$x^{(n)}(t) - F(t, x(t)) = 0 \tag{5}$$

with the initial conditions

$$x(a) = x^{'}(a) = x^{''}(a) = ... = x^{(n-1)}(a) = 0 \tag{6}$$

where $x \in C^{n}(I),\ I = [a, b],\ -\infty < a < b < \infty$ and

$$|F(t, x(t))| \leq \ L \left| x^{(n-2)}(t) \right|^{\alpha},$$

$\alpha > 0, -\infty < x < \infty$, with $F(t, 0) = 0$. Moreover, we prove the Hyers - Ulam stability of the Emden - Fowler type nonlinear differential equation of n^{th} order

$$x^{(n)}(t) - h(t)\ \ |x(t)|^{\alpha}\ \ sgn\ x(t) = 0 \tag{7}$$

with the initial conditions (6). Where $x \in C^{n}(I),\ I = [a, b],\ -\infty < a < b < \infty,\ \alpha > 0,\ \alpha \neq 1$ and $h(t)$ is bounded in $\mathbb{R}$.

2. Preliminaries

First, we give the definition of the Hyers - Ulam stability property with the initial conditions.

Definition 2.1. *We say that the n^{th} order non linear differential equation (5) has the Hyers - Ulam stability property with the initial conditions (6), if there exists a constant $K > 0$ with the following property, for every $\epsilon > 0,\ x \in C^{n}[a, b]$, if*

$$\left| x^{(n)}(t) - F(t,\ x(t)) \right| \leq \epsilon, \tag{8}$$

and with the initial conditions $x(a) = x^{'}(a) = x^{''}(a) = ... = x^{(n-1)}(a) = 0$, then there exists some $y(t) \in C^{n}[a, b]$ is a solution of (5) satisfying

$$y^{(n)}(t) - F(t,\ y(t)) = 0$$

with the initial conditions $y(a) = y^{'}(a) = y^{''}(a) = ... = y^{(n-1)}(a) = 0,$ *such that*

$$|x(t) - y(t)| \leq K \ \epsilon.$$

We call such K *as a Hyers - Ulam stability constant for the equation (5).*

Definition 2.2. *We say that* n^{th} *order nonlinear differential equation (7) has the Hyers - Ulam stability property with the initial conditions (6), if there exists a positive constant* $K > 0$ *with the following property:*
For every $\epsilon > 0, \ x \in C^n[a,b]$*, if*

$$\left| x^{(n)}(t) - \ h(t) \ |x(t)|^{\alpha} \ sgn \ x(t) \right| \leq \epsilon, \tag{9}$$

and $x(a) = x^{'}(a) = x^{''}(a) = ... = x^{(n-1)}(a) = 0,$ *then there exists some* $y(t) \in C^n[a,b]$ *is a solution of (7) and satisfying*

$$y^{(n)}(t) - \ h(t) \ |y(t)|^{\alpha} \ sgn \ y(t) = 0$$

with the initial conditions $y(a) = y^{'}(a) = y^{''}(a) = ... = y^{(n-1)}(a) = 0,$ *such that*

$$|x(t) - y(t)| \leq K \ \epsilon.$$

We call such K *as a Hyers - Ulam stability constant for the equation (7).*

3. Hyers - Ulam Stability

Now, in the following Theorem, we prove the Hyers - Ulam stability property of the non linear differential equation of the form (5) with the initial conditions (6).

Theorem 3.1. *Assume that* $x : [a,b] \to R$ *is a* n *times continuously differentiable function such that*

$$|x(t)| \leq \left| x^{'}(t) \right| \leq ... \leq \left| x^{(n-1)}(t) \right|$$

and

$$L < \left(\max_{a \leq t \leq b} |x(t)| \right)^{1-\alpha}$$

then the equation (5) has the Hyers - Ulam stability property with the initial conditions (6).

Proof. Suppose that for every $\epsilon > 0$, and $x \in C^n[a,b]$, be a n times continuously differentiable real valued function on $I = [a,b]$.
Now, we have to prove that there exists a function $y(t) \in C^n[a,b]$ satisfying the equation (5) such that

$$|x(t) - y(t)| \leq K \ \epsilon$$

where K is the Hyers - Ulam stability constant that never depends on ϵ nor on $y(t)$. Since $x(t)$ is a continuous on I, then it keep its sign on some interval $[a,t] \subseteq [a,b]$.
Without loss of generality assume that $x(t) > 0$ on $[a,t]$. Assume that for every $\epsilon > 0$, and $x \in C^n[a,b]$ satisfies the inequality (8) with the initial conditions (6) and that

$$M = \max_{a \leq t \leq b} |x(t)|.$$

From the inequality (8) we have

$$- \epsilon \leq x^{(n)}(t) - F(t,\ x(t)) \leq \epsilon. \tag{10}$$

Since $x(t) > 0$ on $[a, t] \subseteq I$ and $x(t) = x^{'}(t) = ... = x^{(n-2)}(t) = 0$, then by Mean Value Theorem, we have $x^{(n-1)}(t) > 0$ in (a, t).

Multiplying the inequality (10) by $x^{(n-1)}(t) > 0$, we have

$$- \epsilon x^{(n-1)}(t) \leq x^{(n)}(t)\ x^{(n-1)}(t) - F(t,\ x(t))\ x^{(n-1)}(t) \leq \epsilon x^{(n-1)}(t). \tag{11}$$

Then integrating the above equation (11) from a to t, we obtain

$$\int_a^t -\epsilon\ x^{(n-1)}(t)\ dt \leq \int_a^t \left\{x^{(n)}(t) - F(t,\ x(t))\right\} x^{(n-1)}(t)\ dt \leq \int_a^t \epsilon\ x^{(n-1)}(t)\ dt$$

$$\int_a^t -\epsilon\ x^{(n-1)}(t)\ dt \leq \int_a^t \left\{x^{(n)}(t)\ x^{(n-1)}(t) - F(t,\ x(t))\ x^{(n-1)}(t)\right\}\ dt \leq \int_a^t \epsilon\ x^{(n-1)}(t)\ dt$$

$$\int_a^t -\epsilon\ x^{(n-1)}(t)\ dt \leq \int_a^t x^{(n)}(t)\ x^{(n-1)}(t)\ dt - \int_a^t F(t,\ x(t))\ x^{(n-1)}(t)\ dt \leq \int_a^t \epsilon\ x^{(n-1)}(t)\ dt,$$

then we have

$$- 2\ \epsilon\ x^{(n-2)}(t) \leq \left(x^{(n-1)}(t)\right)^2 - 2\int_a^t F(t,\ x(t))\ x^{(n-1)}(t)\ dt \leq\ 2\ \epsilon\ x^{(n-2)}(t). \tag{12}$$

Since we have,

$$|x(t)| \leq \left|x^{'}(t)\right| \leq ... \leq \left|x^{(n-1)}(t)\right|, \tag{13}$$

by using the above inequality (13) in (12), then easilly we arrive that

$$\left(x^{(n-1)}(t)\right)^2 \leq\ 2\int_a^t F(t,\ x(t))\ x^{(n-1)}(t)\ dt +\ 2\ \epsilon\ x^{(n-2)}(t)$$

$$\begin{aligned}
\left|x^{(n-2)}(t)\right|^2 &\leq \left|\ 2\int_a^t F(t,\ x(t))\ x^{(n-1)}(t)\ dt +\ 2\ \epsilon\ x^{(n-2)}(t)\right| \\
&\leq\ 2\int_a^t |F(t,\ x(t))|\ \left|x^{(n-1)}(t)\right|\ dt +\ 2\ \epsilon\ \left|x^{(n-2)}(t)\right| \\
&\leq \int_a^t L\ M^{\alpha-1} x^{(n-2)}(t)^2\ dt +\ 2\ \epsilon\ \left|x^{(n-2)}(t)\right| \\
&\leq\ \ L\ M^{\alpha-1}\left|x^{(n-2)}(t)\right|^2 +\ 2\ \epsilon\ \left|x^{(n-2)}(t)\right| \\
\left|x^{(n-2)}(t)\right|^2\ &\leq\ L\ M^{\alpha-1}\left|x^{(n-2)}(t)\right|^2\ +\ 2\ \epsilon\ \left|x^{(n-2)}(t)\right| \\
|x(t)|\ &\leq \frac{2\ \epsilon}{1 -\ L\ M^{\alpha-1}}.
\end{aligned}$$

Hence we have $|x(t)| \leq K\epsilon$, where

$$K = \frac{2}{1 -\ L\ M^{\alpha-1}},$$

for all $t \geq a$. Obviously, we have $y_0(t) = 0$ is a solution and satisfies of the equation (5) with the initial conditions (6), then we have

$$|x(t) - y_0(t)| \leq\ K\epsilon.$$

Hence the non linear differential equation of the form (5) has the Hyers - Ulam stability property with the initial conditions (6). □

Theorem 3.2. *Suppose that $x(t) \in C^n[a,b]$ satisfies the inequality (8) with the initial conditions (6). If*

$$\left|x^{(n-1)}(t)\right| \leq \left|x^{(n-2)}(t)\right| \leq ... \left|x^{'}(t)\right| \leq |x(t)|$$

and if

$$L > \left(\max_{a\leq t\leq b} |x(t)|\right)^{1-\alpha}$$

then the equation (5) has the Hyers - Ulam stability property with the initial conditions (6).

Proof. By using the Theorem 3. 1, we can similarly prove that the equation (5) has the Hyers - Ulam stability property with the initial conditions (6). □

Theorem 3.3. *Suppose that $x(t) \in C^n(I)$, $I = [a,b]$, be a n times cotinuously differentiable function such that*

$$|x(t)| \leq \left|x^{'}(t)\right| \leq ... \leq \left|x^{(n-1)}(t)\right|$$

and

$$L < \left(\max_{a\leq t\leq b} |x(t)|\right)^{1-\alpha}$$

then the equation (7) has the Hyers - Ulam stability property with the initial conditions (6).

Proof. Suppose that for every $\epsilon > 0$, and $x \in C^n[a,b]$, be a n times continuously differentiable real valued function on $I = [a,b]$.

Now, we have to prove that there exists a function $y(t) \in C^n[a,b]$ satisfying the equation (7) such that

$$|x(t) - y(t)| \leq K\ \epsilon$$

where K is the Hyers - Ulam stability constant that never depends on ϵ nor on $y(t)$. Since $x(t)$ is a continuous on I, then it keep its sign on some interval $[a,t] \subseteq [a,b]$.

Without loss of generality assume that $x(t) > 0$ on $[a,t]$. Assume that for every $\epsilon > 0$, and $x \in C^n[a,b]$ satisfies the inequality (9) with the initial conditions (6) and that

$$M = \max_{a\leq t\leq b} |x(t)|.$$

From the inequality (9) we have,

$$-\ \epsilon\ \leq\ x^{(n)} - h(t)\ x(t)^{\alpha}\ \leq\ \epsilon \tag{14}$$

Since $x(t) > 0$ on $[a,t] \subseteq I$ and $x(t) = x^{'}(t) = ... = x^{(n-2)}(t) = 0$, then by Mean Value Theorem, we have $x^{(n-1)}(t) > 0$ in (a,t).

Then multiplying the inequality (14) by $x^{(n-1)}(t) > 0$, we have

$$-\ \epsilon x^{(n-1)}(t) \leq x^{(n)}(t)\ x^{(n-1)}(t) - h(t)\ x(t)^{\alpha}\ x^{(n-1)}(t) \leq \epsilon x^{(n-1)}(t). \tag{15}$$

Then integrating the above equation (15) from a to t, we obtain

$$\int_a^t -\epsilon\ x^{(n-1)}(t)\ dt \leq \int_a^t \left\{x^{(n)}(t) - h(t)\ x(t)^{\alpha}\right\} x^{(n-1)}(t)\ dt \leq \int_a^t \epsilon\ x^{(n-1)}(t)\ dt$$

$$\int_a^t -\epsilon\, x^{(n-1)}(t)\ dt \le \int_a^t \left\{x^{(n)}(t)\ x^{(n-1)}(t) - h(t)\ x(t)^{\alpha}\ x^{(n-1)}(t)\right\}\ dt \le \int_a^t \epsilon\ x^{(n-1)}(t)\ dt$$

$$\int_a^t -\epsilon\, x^{(n-1)}(t)\ dt \le \int_a^t x^{(n)}(t)\ x^{(n-1)}(t)\ dt - \int_a^t h(t)\ x(t)^{\alpha}\ x^{(n-1)}(t)\ dt \le \int_a^t \epsilon\ x^{(n-1)}(t)\ dt,$$

then we have

$$-\,2\ \epsilon\ x^{(n-2)}(t) \le \left(x^{(n-1)}(t)\right)^2 - 2\int_a^t h(t)\ x(t)^{\alpha}\ x^{(n-1)}(t)\ dt \le\ 2\ \epsilon\ x^{(n-2)}(t). \tag{16}$$

Since we have,

$$|x(t)| \le \left|x^{'}(t)\right| \le ... \le \left|x^{(n-1)}(t)\right|, \tag{17}$$

by using the above inequality (17) in (16), then easilly we arrive that

$$\left(x^{(n-1)}(t)\right)^2 \le\ 2\int_a^t h(t)\ x(t)^{\alpha}\ x^{(n-1)}(t)\ dt +\ 2\ \epsilon\ x^{(n-2)}(t)$$

$$\begin{aligned}
\left|x^{(n-2)}(t)\right|^2 &\le \left|\ 2\int_a^t h(t)\ x(t)^{\alpha}\ x^{(n-1)}(t)\ dt +\ 2\ \epsilon\ x^{(n-2)}(t)\right| \\
&\le\ 2\int_a^t |h(t)\ x(t)^{\alpha}|\ \left|x^{(n-1)}(t)\right|\ dt +\ 2\ \epsilon\ \left|x^{(n-2)}(t)\right| \\
&\le\ 2\int_a^t L\ M^{\alpha-1} x^{(n-2)}(t)\ x^{(n-1)}(t)\ dt +\ 2\ \epsilon\ \left|x^{(n-2)}(t)\right| \\
&\le\ L\ M^{\alpha-1}\left|x^{(n-2)}(t)\right|^2 +\ 2\ \epsilon\ \left|x^{(n-2)}(t)\right| \\
\left|x^{(n-2)}(t)\right|^2 &\le\ L\ M^{\alpha-1}\left|x^{(n-2)}(t)\right|^2 +\ 2\ \epsilon\ \left|x^{(n-2)}(t)\right| \\
|x(t)| &\le \frac{2\ \epsilon}{1-\ L\ M^{\alpha-1}}.
\end{aligned}$$

Hence we have $|x(t)| \le K\epsilon$, where

$$K = \frac{2}{1-\ L\ M^{\alpha}},$$

for all $t \ge a$. Obviously, we have $y_0(t) = 0$ is a solution and satisfies of the equation (5) with the initial conditions (6), then we have

$$|x(t) - y_0(t)| \le\ K\epsilon.$$

Hence the non linear differential equation of the form (7) has the Hyers - Ulam stability property with the initial conditions (6). □

Theorem 3.4. *Suppose that $x(t) \in C^n[a,b]$ satisfies the inequality (9) with the initial conditions (6). If*

$$\left|x^{(n-1)}(t)\right| \le \left|x^{(n-2)}(t)\right| \le ... \left|x^{'}(t)\right| \le |x(t)|$$

and if

$$L > \left(\max_{a\le t\le b} |x(t)|\right)^{1-\alpha}$$

then the equation (7) has the Hyers - Ulam stability property with the initial conditions (6).

Proof. By using the Theorem 3. 3, we can similarly prove that the equation (7) has the Hyers - Ulam stability property with the initial conditions (6). □

Example 3.5. *Consider the equation*

$$x^{(n)}(t) - \epsilon^n \ sin\left(\frac{t}{\epsilon^n+1}\right) \ x(t)^{\frac{1}{n}} = 0 \tag{18}$$

and the inequality

$$\left|x^{(n)}(t) - \epsilon^n \ sin\left(\frac{t}{\epsilon^n+1}\right) \ x(t)^{\frac{1}{n}}\right| \ \leq \ \epsilon \tag{19}$$

where $0 \ \leq \ t \ \leq \ 1.$

Solution. For a given $\epsilon \ > \ 0$, then $x(t) \ = \ \epsilon^n \ t$ satisfies the inequality (19) and the conditions of the Theorem 3. 3. Therefore the equation (refe1) has the Hyers - Ulam stability property.

4. A Special Case on Emden - Fowler Type Equation

Let us consider the special case of the equation (7) when $\alpha = 1$, we have

$$x^{(n)}(t) - h(t) \ x(t) = 0 \tag{20}$$

with the initial conditions

$$x(a) = x^{'}(a) = x^{''}(a) = ... = x^{(n-1)}(a) = 0$$

and the inequality

$$\left|x^{(n)}(t) - h(t) \ x(t)\right| \ \leq \ \epsilon \tag{21}$$

where $x \in C^n(I), \ I = [a,b], \ -\infty < a < b < \infty$, and $h(t)$ is bounded in $\mathbb{R}$.

Theorem 4.1. *Suppose that* $x : [a,b] \rightarrow \ \mathbb{R}$ *is a n times continuous differentiable function and*

$$|x(t)| \leq \left|x^{'}(t)\right| \leq ... \leq \left|x^{(n-1)}(t)\right|$$

then the equation (20) has the Hyers - Ulam stability property if $L < 1$.

Proof. Suppose that $\epsilon > 0$ and that x is a n times continuously differentiable real - valued function on $[a,b]$. Now we have to prove that there exists a function $y(t) \ \in C^n(I)$, where $I = [a,b]$ satisfying the equation (20) such that

$$|x(t) - y(t)| \ \leq K \ \epsilon$$

where K is a constant that never depends on ϵ nor on $y(t)$. Since $x(t)$ is a continuous function on I then it keep its sign on some interval $[a,t] \subseteq \ I$. Then without loss of generality assume that $x(t) > 0$ on the interval $[a,b] \subseteq \ I$. Suppose that $\epsilon > 0$, and $x(t) \ \in \ C^n[a,b]$ satisfies the inequality (21) with the initial conditions (6). Hence, we have from the inequality (21) we have

$$-\epsilon \leq x^{(n)}(t) - F(t, \ x(t)) \leq \epsilon. \tag{22}$$

Since $x(t) > 0$ on $[a,t] \subseteq I$ and $x(t) = x^{'}(t) = ... = x^{(n-2)}(t) = 0$, then by Mean Value Theorem, we have $x^{(n-1)}(t) > 0$ in (a,t).

Then multiplying the inequality (22) by $x^{(n-1)}(t) > 0$, we have

$$-\epsilon x^{(n-1)}(t) \leq x^{(n)}(t) \ x^{(n-1)}(t) - h(t) \ x(t) \ x^{(n-1)}(t) \leq \epsilon x^{(n-1)}(t). \tag{23}$$

Then integrating the above equation (15) from a to t, we obtain

$$\int_a^t -\epsilon\, x^{(n-1)}(t)\ dt \leq \int_a^t \left\{x^{(n)}(t) - h(t)\ x(t)\right\} x^{(n-1)}(t)\ dt \leq \int_a^t \epsilon\, x^{(n-1)}(t)\ dt$$

$$\int_a^t -\epsilon\, x^{(n-1)}(t)\ dt \leq \int_a^t \left\{x^{(n)}(t)\ x^{(n-1)}(t) - h(t)\ x(t)\ x^{(n-1)}(t)\right\}\ dt \leq \int_a^t \epsilon\, x^{(n-1)}(t)\ dt$$

$$\int_a^t -\epsilon\, x^{(n-1)}(t)\ dt \leq \int_a^t x^{(n)}(t)\ x^{(n-1)}(t)\ dt - \int_a^t h(t)\ x(t)\ x^{(n-1)}(t)\ dt \leq \int_a^t \epsilon\, x^{(n-1)}(t)\ dt,$$

then we have

$$-\,2\ \epsilon\ x^{(n-2)}(t) \leq \left(x^{(n-1)}(t)\right)^2 - 2\int_a^t h(t)\ x(t)\ x^{(n-1)}(t)\ dt \leq\ 2\ \epsilon\ x^{(n-2)}(t). \tag{24}$$

Since we have,

$$|x(t)| \leq \left|x^{'}(t)\right| \leq ... \leq \left|x^{(n-1)}(t)\right|, \tag{25}$$

by using the above inequality (25) in (24), then easily we arrive that

$$\left(x^{(n-1)}(t)\right)^2 \leq\ 2\int_a^t h(t)\ x(t)\ x^{(n-1)}(t)\ dt +\ 2\ \epsilon\ x^{(n-2)}(t)$$

$$\begin{aligned}
\left|x^{(n-2)}(t)\right|^2 &\leq \left|\ 2\int_a^t h(t)\ x(t)\ x^{(n-1)}(t)\ dt +\ 2\ \epsilon\ x^{(n-2)}(t)\right| \\
&\leq\ 2\int_a^t |h(t)\ x(t)|\ \left|x^{(n-1)}(t)\right|\ dt +\ 2\ \epsilon\ \left|x^{(n-2)}(t)\right| \\
&\leq\ 2\int_a^t L\ x^{(n-2)}(t)\ x^{(n-1)}(t)\ dt +\ 2\ \epsilon\ \left|x^{(n-2)}(t)\right| \\
&\leq\ \ L\ M^{\alpha-1}\left|x^{(n-2)}(t)\right|^2 +\ 2\ \epsilon\ \left|x^{(n-2)}(t)\right| \\
\left|x^{(n-2)}(t)\right|^2 &\leq\ L\ \left|x^{(n-2)}(t)\right|^2 +\ 2\ \epsilon\ \left|x^{(n-2)}(t)\right| \\
|x(t)| &\leq \frac{2\ \epsilon}{1-\ L}.
\end{aligned}$$

Hence we have $|x(t)| \leq K\epsilon$, where

$$K = \frac{2}{1-\ L},$$

for all $t \geq a$. Obviously, we have $y_0(t) = 0$ is a solution and satisfies of the equation (20) with the initial conditions (6), then we have

$$|x(t) - y_0(t)| \leq\ K\epsilon.$$

Hence the non linear differential equation of the form (20) has the Hyers - Ulam stability property with the initial conditions (6). □

Remark 4.2. *Suppose that $x(t)$ be a n times continuously differentiable function and satisfies the inequality (21) with the initial condition (6). If*

$$\left|x^{(n-1)}(t)\right| \leq \left|x^{(n-2)}(t)\right| \leq ... \left|x^{'}(t)\right| \leq |x(t)|$$

then, if $L < 1$ we will easily obtain the Hyers - Ulam stability property for the equation (20) has with the initial conditions (6) by the similar proof of the Theorem 4. 1.

Remark 4.3. *It should be noted that if $x(t) < 0$ on $(a,t]$ and $x(a) = 0$, hence $x^{(n-1)}(t) < 0$ on (a,x), then in the proofs of the Theorem 3. 1, 3. 3 and 4. 1, we can multiply by $x^{(n-1)}(t) < 0$ in the inequalities (11), ((15), (22)), we obtain the inequality*

$$\epsilon\ x^{(n-1)}(t)\ \leq\ x^{(n-1)}(t)\ x^{(n)}(t) - x^{(n-1)}(t)\ F(t,x(t))\ \leq\ -\epsilon\ x^{(n-1)}(t).$$

Then we can apply the similar process used in the above Theorems we get the Hyers - Ulam stability property for the equations (5), (7) and (20).

5. An Additional Case on Hyers - Ulam Stability

In this section, we consider the Hyers - Ulam stability of the following nonlinear differential equation

$$x^{(n)}(t) = \pi(t,x(t)) \tag{26}$$

with the initial conditions (6), where $x(t)$ is a n times continuously differentiable function on $[a,b]$ and $-\infty < a < b < \infty$ and $\pi(t,x(t))$ is a continuous function for $x \in [a,b]$ and $x\ \in\ \mathbb{R}$ such that

$$|\pi(t,x(t))\ -\ \pi(t,y(t))|\ \leq\ L\ |x(t)\ -\ y(t)|\,.$$

Theorem 5.1. *Assume that $\epsilon > 0$ and $x(t)$ be n times continuously differentiable function. If*

$$\frac{L\ (b-a)^2}{2}\ \leq\ 1,$$

then the equation (26) has the Hyers - Ulam stability property with the initial conditions (6).

Proof. Let us assume that $\epsilon > 0$ and $x(t)$ be a n times continuously differentiable real - valued function on an interval $[a,b]$ and satisfying the inequality

$$-\,\epsilon\ \leq\ x^{(n)}(t) - \pi(t,x(t))\ \leq\ \epsilon. \tag{27}$$

Now we have to prove that there exists a function $y(t)\ \in\ C^n[a,b]$ satisfying the equation (26) such that

$$|x(t) - y(t)|\ \leq\ K\ \epsilon$$

where K is a constant that never depend on ϵ nor on $y(t)$. Now, we integrate the inequality (27) with respect to t, we obtain

$$\int_a^t \int_a^u -\epsilon\ dt\ \leq\ \int_a^t \int_a^u \left(x^{(n)}(t) - \pi(s,x(s))\right)\ dt\ \leq\ \int_a^t \int_a^u \epsilon\ dt$$

then we have

$$-\,\frac{\epsilon\ (b-a)^2}{2}\ \leq\ -\frac{\epsilon\ (t-a)^2}{2}\ \leq\ x^{(n-2)}(t) - \int_a^t \int_a^u \pi(s,y(s))\ ds\ \leq\ \frac{\epsilon\ (t-a)^2}{2}\ \leq \frac{\epsilon\ (b-a)^2}{2}. \tag{28}$$

But it is clear that

$$y^{(n-2)}(t) = \int_a^t \int_a^u \pi(s,y(s))\ ds$$

is a solution of the equation

$$y^{(n)}(t) = \pi(t, y(t))$$

satisfying the initial conditions (6). Now let us find the difference

$$|x(t) - y(t)| \leq \left| x^{(n-2)}(t) - \int_a^t \int_a^u \pi(s, x(s))\ ds \right| + \left| \int_a^t \int_a^u \pi(s, y(s))\ ds - \int_a^t \int_a^u \pi(s, x(s))\ ds \right|.$$

Since the function $\pi(t, x(t))$ satisfies the Lipschitz condition, and from the inequality (28), we arrive

$$|x(t) - y(t)| \leq \frac{\epsilon\ (b-a)^2}{2} + L\ \frac{(b-a)^2}{2}\ |y(t) - x(t)|.$$

From which it follows that

$$|x(t) - y(t)| \leq \frac{\epsilon\ (b-a)^2}{2 - L\ (b-a)^2},$$

where $2 > L\ (b-a)^2$. Hence the nonlinear differential equation (26) is stable in the sense of Hyers - Ulam. This completes the proof of the Theorem. □

6. Conclusion

We establish the Hyers - Ulam Stability of a generalized nth order nonlinear differential equation and also proved the Hyers - Ulam stability of a Emden - Fowler type equation of nth order. It will be more useful to readers can apply more and more various nonlinear problems for finding the stability of the solutions in the sense of Hyers - Ulam.

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