On Semi $\alpha$-Regular Pre-Semi Closed Sets in Bitopological Spaces

Research Article

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Abstract: In this paper, we introduce a new class of closed sets and open sets in bitopological spaces, namely, $\tau_1\tau_2$-arsps-closed sets and $\tau_1\tau_2$-arsps-open sets and characterize the properties of these sets.

Keywords: $\tau_1\tau_2$-arsps-closed, $\tau_1\tau_2$-arsps-open and $\tau_1$-arsps-open.

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1. Introduction

A triple $(X, \tau_1, \tau_2)$, where $X$ is a non-empty set and $\tau_1$ and $\tau_2$ are topologies on $X$ is called a bitopological space. In 1963, Kelly initiated the study of bitopological spaces. In 1985, Fukutake introduced the concept of g-closed sets in bitopological spaces and after that several authors turned their attention towards generalizations of various concepts of topology by considering bitopological spaces. The aim of this paper is to extend the same concept in bitopological spaces.

2. Preliminaries

Let $(X, \tau_1, \tau_2)$ or simply $X$ denotes a bitopological space. For any subset $A \subseteq X$, $\tau_2$-$\text{int} A$ and $\tau_2$-$\text{cl} A$ denote the interior and closure of a set $A$ with respect to the topology $\tau_2$ respectively. $X \setminus A$ denotes the complement of $A$ in $X$. We shall now require the following known definitions.

Definition 2.1. A subset $A$ of a space $X$ is called

(1). pre-open [9] if $A \subseteq \text{int cl} A$ and pre-closed if $\text{cl int} A \subseteq A$.

(2). semi-open [8] if $A \subseteq \text{cl int} A$ and semi-closed if $\text{int cl} A \subseteq A$.

(3). semi-pre-open [1] if $A \subseteq \text{cl int cl} A$ and semi-pre-closed if $\text{int cl int} A \subseteq A$.

(4). $\alpha$-open [10] if $A \subseteq \text{int cl} A$ and $\alpha$-closed if $\text{cl int} A \subseteq A$.

(5). regular open [14] if $A = \text{int cl} A$ and regular closed if $\text{cl int} A = A$.

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(6). b-open [2] if \( A \subseteq \text{int} \text{cl}A \cup \text{int cl}A \) and b-closed if \( \text{cl} \text{int}A \cap \text{cl} \text{int}A \subseteq A \).

The \( \tau_2 \)-semi-closure (resp. \( \tau_2 \)-pre-closure, resp. \( \tau_2 \)-semi-pre-closure, resp. \( \tau_2 \)-\( \alpha \)-closure, resp. \( \tau_2 \)-\( \alpha \)-b-closure) of a subset \( A \) of \( X \) is the intersection of all semi-closed (resp. pre-closed, resp. semi-pre-closed, resp. \( \alpha \) -closed, resp. b-closed) sets containing \( A \) with respect to the topology \( \tau_2 \) and is denoted by \( \tau_2 \text{-scl}A \) (resp. \( \tau_2 \text{-pcl}A \), resp. \( \tau_2 \text{-spcl}A \), resp. \( \tau_2 \text{-acl}A \), resp. \( \tau_2 \text{-bcl}A \)).

**Definition 2.2.** A subset \( A \) of a space \( X \) is called

1. \( \tau_1 \tau_2 \)-generalized closed [5] (briefly \( \tau_1 \tau_2 \)-g-closed) if \( \tau_2 \text{-cl}A \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \)-open.
2. \( \tau_1 \tau_2 \)-regular generalized closed [7] (briefly \( \tau_1 \tau_2 \)-rg-closed) if \( \tau_2 \text{-cl}A \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \)-regular open.
3. \( \tau_1 \tau_2 \)-\( \alpha \)-generalized closed [11] (briefly \( \tau_1 \tau_2 \)-ag-closed) if \( \tau_2 \text{-acl}A \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \)-open.
4. \( \tau_1 \tau_2 \)-generalized semi-closed [6] (briefly \( \tau_1 \tau_2 \)-gs-closed) if \( \tau_2 \text{-scl}A \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \)-open.
5. \( \tau_1 \tau_2 \)-generalized ab-closed [19] (briefly \( \tau_1 \tau_2 \)-gab-closed) if \( \tau_2 \text{-bcl}A \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \)-\( \alpha \)-open.
6. \( \tau_1 \tau_2 \)-regular generalized b-closed [4] (briefly \( \tau_1 \tau_2 \)-rgb-closed) if \( \tau_2 \text{-bcl}A \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \)-regular open.
7. \( \tau_1 \tau_2 \)-\( g^* \)-semi-closed [18] (briefly \( \tau_1 \tau_2 \)-\( g^* \)-s-closed) if \( \tau_2 \text{-scl}A \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \)-\( ag \)-open.
8. \( \tau_1 \tau_2 \)-strongly generalized closed [12] (briefly \( \tau_1 \tau_2 \)-gs-closed) if \( \tau_2 \text{-cl}A \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \)-\( g \)-open.
9. \( \tau_1 \tau_2 \)-\( g^* \)-preclosed [17] (briefly \( \tau_1 \tau_2 \)-\( g^* \)-p-closed) if \( \tau_2 \text{-pcl}A \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \)-\( g \)-open.

**Lemma 2.3** ([16]). For a subset \( A \) of \( X \), \( \text{sint(scl}A \setminus A) = \emptyset \).

### 3. On Semi \( \alpha \)-Regular Pre-Semi Closed Sets In Bitopological Spaces

In this section, we introduce the concept of \( \tau_1 \tau_2 \)-sarpSC-closed sets in bitopological spaces and discuss some of the related properties.

**Definition 3.1.** A subset \( A \) of a bitopological space \( (X, \tau_1, \tau_2) \) is called \( \tau_1 \tau_2 \)-\( \alpha \)-regular pre-semi closed (briefly \( \tau_1 \tau_2 \)-sarpSC-closed) if \( \tau_2 \text{-scl}A \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \)-sarpSC-open in \( X \).

**Remark 3.2.** If \( \tau_1 = \tau_2 \) in the above definition, a \( \tau_1 \tau_2 \)-sarpSC-closed is a sarpSC-closed in the sense of point set topology. The set of all \( \tau_1 \tau_2 \)-sarpSC-closed sets in the bitopological space \( (X, \tau_1, \tau_2) \) is denoted by \( \text{SarpSC}(X, \tau_1, \tau_2) \).

**Remark 3.3.** In general, the set of all \( \tau_1 \tau_2 \)-sarpSC-closed sets need not be equal to the set of all \( \tau_2 \tau_1 \)-sarpSC-closed sets as seen in the following example.

**Example 3.4.** Let \( X = \{a,b,c\} \) with the topologies \( \tau_1 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\} \) and \( \tau_2 = \{\emptyset, \{a\}, \{b,c\}, X\} \). Then \( \text{SarpSC}(X, \tau_1, \tau_2) = \{\emptyset, \{a\}, \{c\}, \{a,c\}, \{b,c\}, X\} \) and \( \text{SarpSC}(X, \tau_2, \tau_1) = P(X) \). Therefore \( \text{SarpSC}(X, \tau_1, \tau_2) \neq \text{SarpSC}(X, \tau_2, \tau_1) \).

**Remark 3.5.** Difference between two \( \tau_1 \tau_2 \)-sarpSC-closed sets need not be a \( \tau_1 \tau_2 \)-sarpSC-closed set as seen in the Example 3.6.

**Example 3.6.** From Example 3.4, \( A = \{b,c\} \) and \( B = \{c\} \) are \( \tau_1 \tau_2 \)-sarpSC-closed sets. But \( A \setminus B = \{b\} \) is not \( \tau_1 \tau_2 \)-sarpSC-closed.

**Proposition 3.7.** Let \( (X, \tau_1, \tau_2) \) be a bitopological space and \( A \subseteq X \). Then the following are true.
(1) Every $\tau_2$-semi-closed set is $\tau_1\tau_2$-sarsps-closed.
(2) Every $\tau_2$-regular closed set is $\tau_1\tau_2$-sarsps-closed.
(3) Every $\tau_2$-$\alpha$-closed set is $\tau_1\tau_2$-sarsps-closed.
(4) Every $\tau_1\tau_2$-$g^\#$-semi-closed set is $\tau_1\tau_2$-sarsps-closed.

**Proof.**

(1). Let $A$ be a $\tau_2$-semi-closed subset of $X$. Let $A \subseteq U$ and $U$ is $\tau_1$-arps-open. Since $A$ is $\tau_2$-semi-closed, $\tau_2$-$sclA = A$. Therefore $\tau_2$-$sclA \subseteq U$. Hence $A$ is $\tau_1\tau_2$-sarsps-closed.

(2). Let $A$ be a $\tau_2$-regular closed subset of $X$. Since every $\tau_2$-regular closed set is $\tau_2$-semi-closed and by (1), we have $A$ is $\tau_1\tau_2$-sarsps-closed.

(3). Let $A$ be a $\tau_2$-$\alpha$-closed subset of $X$. Since every $\tau_2$-$\alpha$-closed set is $\tau_2$-semi-closed and by (1), we have $A$ is $\tau_1\tau_2$-sarsps-closed.

(4). Let $A$ be a $\tau_1\tau_2$-$g^\#$-semi-closed subset of a space $X$. Let $A \subseteq U$ and $U$ is $\tau_1$-arps-open. Since every $\tau_1$-arps-open set is $\tau_1$-$\alpha g$-open and since $A$ is $\tau_1\tau_2$-$g^\#$-semi-closed, $\tau_2$-$sclA \subseteq U$. Hence $A$ is $\tau_1\tau_2$-sarsps-closed.

The reverse implications are not true as shown in the Examples 3.8 and 3.9.

**Example 3.8.** Let $X = \{a,b,c\}$ with the topologies $\tau_1 = \{\phi,\{a\},\{b\},\{a,b\},X\}$ and $\tau_2 = \{\phi,\{a\},\{b,c\},X\}$. Then $\{c\}$ is $\tau_1\tau_2$-sarsps-closed, but not $\tau_2$-semi-closed, not $\tau_2$-regular closed and not $\tau_2$-$\alpha$-closed.

**Example 3.9.** Let $X = \{a,b,c\}$ with the topologies $\tau_1 = \{\phi,\{b\},X\}$ and $\tau_2 = \{\phi,\{a\},\{b,c\},X\}$. Then $\{c\}$ is $\tau_1\tau_2$-sarsps-closed, but not $\tau_1\tau_2$-$g^\#$-semi-closed.

**Proposition 3.10.** Let $(X,\tau_1, \tau_2)$ be a bitopological space and $A \subseteq X$. Then the following are true.

(1) Every $\tau_1\tau_2$-sarsps-closed set is $\tau_1\tau_2$-$gs$-closed.
(2) Every $\tau_1\tau_2$-sarsps-closed set is $\tau_1\tau_2$-$gab$-closed.
(3) Every $\tau_1\tau_2$-sarsps-closed set is $\tau_1\tau_2$-$rgb$-closed.

**Proof.**

(1). Let $A$ be a $\tau_1\tau_2$-sarsps-closed subset of a space $X$. Let $A \subseteq U$ and $U$ is $\tau_1$-open. Since every $\tau_1$-open set is $\tau_1$-arps-open and since $A$ is $\tau_1\tau_2$-sarsps-closed, $\tau_2$-$sclA \subseteq U$. Hence $A$ is $\tau_1\tau_2$-$gs$-closed.

(2). Let $A$ be a $\tau_1\tau_2$-sarsps-closed subset of a space $X$. Let $A \subseteq U$ and $U$ is $\tau_1$-$\alpha$-open. Since every $\tau_1$-$\alpha$-open set is $\tau_1$-arps-open and since $A$ is $\tau_1\tau_2$-sarsps-closed, $\tau_2$-$sclA \subseteq U$. But $\tau_2$-$bclA \subseteq \tau_2$-$sclA$. Hence $A$ is $\tau_1\tau_2$-$gab$-closed.

(3). Let $A$ be a $\tau_1\tau_2$-sarsps-closed subset of a space $X$. Let $A \subseteq U$ and $U$ is $\tau_1$-regular open. Since every $\tau_1$-regular open set is $\tau_1$-arps-open and since $A$ is $\tau_1\tau_2$-sarsps-closed, $\tau_2$-$sclA \subseteq U$. But $\tau_2$-$bclA \subseteq \tau_2$-$sclA$. Hence $A$ is $\tau_1\tau_2$-$rgb$-closed.

The reverse implications are not true as shown in the Examples 3.11 and 3.12.
Example 3.11. Let $X = \{a, b, c, d\}$ with the topologies $\tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then $\{a, b, c\}$ is $\tau_1, \tau_2$-gs-closed and $\tau_1, \tau_2$-rgb-closed, but not $\tau_1, \tau_2$-sarps-closed.

Example 3.12. Let $X = \{a, b, c, d\}$ with the topologies $\tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{a, b\}, X\}$. Then $\{a\}$ is $\tau_1, \tau_2$-gsb-closed, but not $\tau_1, \tau_2$-sarps-closed. The concept $\tau_1, \tau_2$-sarps-closed is independent from the concepts $\tau_1, \tau_2$-sg-closed, $\tau_1, \tau_2$-2g*-closed, $\tau_1, \tau_2$-rg-closed and $\tau_1, \tau_2$-g-closed as seen in the following examples.

Example 3.13. Let $X = \{a, b, c, d\}$ with the topologies $\tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{a, b\}, X\}$. Then

(1). $\{a\}$ is $\tau_1, \tau_2$-sarps-closed but not $\tau_1, \tau_2$-sg-closed and $\{a, b\}$ is $\tau_1, \tau_2$-sg-closed but not $\tau_1, \tau_2$-sarps-closed.

(2). $\{a\}$ is $\tau_1, \tau_2$-sarps-closed but not $\tau_1, \tau_2$-g-closed and $\{a, b\}$ is $\tau_1, \tau_2$-g-closed but not $\tau_1, \tau_2$-sarps-closed.

Example 3.14. Let $X = \{a, b, c, d\}$ with the topologies $\tau_1 = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$. Then $\{a\}$ is $\tau_1, \tau_2$-sarps-closed but not $\tau_1, \tau_2$-rg-closed and $\{a, b\}$ is $\tau_1, \tau_2$-rg-closed but not $\tau_1, \tau_2$-sarps-closed.

Example 3.15. Let $X = \{a, b, c, d\}$ with the topologies $\tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$. Then $\{a\}$ is $\tau_1, \tau_2$-sarps-closed but not $\tau_1, \tau_2$-g*-closed and not $\tau_1, \tau_2$-g*p-closed.

Example 3.16. Let $X = \{a, b, c\}$ with the topologies $\tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, X\}$. Then $\{b\}$ is $\tau_1, \tau_2$-g*-closed and $\tau_1, \tau_2$-g*p-closed, but not $\tau_1, \tau_2$-sarps-closed.

Theorem 3.17. If $A$ and $B$ are $\tau_1, \tau_2$-sarps-closed sets, then $A \cap B$ is also $\tau_1, \tau_2$-sarps-closed.

Proof. Let $A \cap B \subseteq U$ be $\tau_1, \tau_2$-open. Then $A \subseteq U$ and $B \subseteq U$. Since $A$ and $B$ are $\tau_1, \tau_2$-sarps-closed sets, $\tau_2$-scl $A \subseteq U$ and $\tau_2$-scl $B \subseteq U$. Therefore $(\tau_2$-scl $A) \cap (\tau_2$-scl $B) \subseteq U$. But $\tau_2$-scl $(A \cap B) \subseteq (\tau_2$-scl $A) \cap (\tau_2$-scl $B)$ [3]. Therefore $\tau_2$-scl $(A \cap B) \subseteq U$. Hence $A \cap B$ is $\tau_1, \tau_2$-sarps-closed.

Corollary 3.18.

(1). If $A$ is $\tau_1, \tau_2$-sarps-closed and $F$ is $\tau_2$-semi-closed, then $A \cap F$ is $\tau_1, \tau_2$-sarps-closed.

(2). If $A$ is $\tau_1, \tau_2$-sarps-closed and $F$ is $\tau_2$-regular closed, then $A \cap F$ is $\tau_1, \tau_2$-sarps-closed.

Proof.

(1). Since $F$ is $\tau_2$-semi-closed, by Proposition 3.7(i), $F$ is $\tau_1, \tau_2$-sarps-closed. Since $A$ is $\tau_1, \tau_2$-sarps-closed, by Theorem 3.17, $A \cap F$ is $\tau_1, \tau_2$-sarps-closed.

(2). Since $F$ is $\tau_2$-regular-closed, by Proposition 3.7 (2), $F$ is $\tau_1, \tau_2$-sarps-closed. Since $A$ is $\tau_1, \tau_2$-sarps-closed, by Theorem 3.17, $A \cap F$ is $\tau_1, \tau_2$-sarps-closed.

Remark 3.19. The union of two $\tau_1, \tau_2$-sarps-closed sets need not be $\tau_1, \tau_2$-sarps-closed as seen in the following example.

Example 3.20. From Example 3.13, $A = \{a\}$ and $B = \{b, c\}$ are $\tau_1, \tau_2$-sarps-closed sets. But $A \cup B = \{a, b, c\}$ is not $\tau_1, \tau_2$-sarps-closed.

Theorem 3.21. If a set $A$ is $\tau_1, \tau_2$-sarps-closed then, $\tau_2$-scl $A \setminus A$ does not contain a non empty $\tau_1, \tau_2$-sarps-closed set.
Proof. Suppose that $A$ is $\tau_1 \tau_2$-sarps-closed in $X$. Let $F$ be a $\tau_1$-arps-closed subset of $\tau_2$-scl$A \setminus A$. Then $F \subseteq \tau_2$-scl$A \cap (X \setminus A) \subseteq X \setminus A$ and so $A \subseteq X \setminus F$. Since $A$ is $\tau_1 \tau_2$-sarps-closed and since $X \setminus F$ is $\tau_1$-arps-open, $\tau_2$-scl$A \subseteq X \setminus F$ that implies $F \subseteq X \setminus \tau_2$-scl$A$. As we have already $F \subseteq \tau_2$-scl$A$. It follows that $F \subseteq \tau_2$-scl$A \cap (X \setminus \tau_2$-scl$A) = \emptyset$. Thus $F = \emptyset$. Therefore $\tau_2$-scl$A \setminus A$ does not contain a non empty $\tau_1$-arps-closed set.

Theorem 3.22. Let $A$ be $\tau_1 \tau_2$-sarps-closed. Then $A$ is $\tau_2$-semi-closed if and only if $\tau_2$-scl$A \setminus A$ is $\tau_1$-arps-closed.

Proof. If $A$ is $\tau_2$-semi-closed, then $\tau_2$-scl$A = A$ and so $\tau_2$-scl$A \setminus A = \emptyset$ which is $\tau_1$-arps-closed.

Conversely suppose that $\tau_2$-scl$A \setminus A$ is $\tau_1$-arps-closed. Since $A$ is $\tau_1 \tau_2$-sarps-closed, by Theorem 3.21, $\tau_2$-scl$A \setminus A = \emptyset$. That is $\tau_2$-scl$A = A$ and hence $A$ is $\tau_2$-semi-closed.

Theorem 3.23. If $A$ is $\tau_1 \tau_2$-sarps-closed and $\tau_1$-arps-open, then $A$ is $\tau_2$-semi-closed.

Proof. Since $A$ is $\tau_1 \tau_2$-sarps-closed and $\tau_1$-arps-open, $\tau_2$-scl$A \subseteq A$. Therefore $A = \tau_2$-scl$A$. Hence $A$ is $\tau_2$-semi-closed.

Theorem 3.24. If $A$ is $\tau_1 \tau_2$-sarps-closed and if $A \subseteq B \subseteq \tau_2$-scl$A$, then

1. $B$ is $\tau_1 \tau_2$-sarps-closed.
2. $\tau_2$-scl$B \setminus B$ contains no non empty $\tau_1$-arps-closed set.

Proof. $A \subseteq B \subseteq \tau_2$-scl$A \Rightarrow \tau_2$-scl$A = \tau_2$-scl$B$.

1. Let $B \subseteq U$ and $U$ be $\tau_1$-arps-open. Then $A \subseteq U$. Since $A$ is $\tau_1 \tau_2$-sarps-closed, $\tau_2$-scl$A \subseteq U$. That implies $\tau_2$-scl$B \subseteq U$. This proves (1).

2. Since $B$ is $\tau_1 \tau_2$-sarps-closed, (2) follows from Theorem 3.21.

Theorem 3.25. For every point $x$ of a space $X$, $X \setminus \{x\}$ is $\tau_1 \tau_2$-sarps-closed or $\tau_1$-arps-open.

Proof. Suppose $X \setminus \{x\}$ is not $\tau_1$-arps-open. Then $X$ is the only $\tau_1$-arps-open set containing $X \setminus \{x\}$. This implies $\tau_2$-scl$(X \setminus \{x\}) \subseteq X$. Then by using Definition 3.1, $X \setminus \{x\}$ is $\tau_1 \tau_2$-sarps-closed.

4. On Semi $\alpha$-regular Pre-Semi Open Sets in Bitopological Spaces

In this section, we introduce the concept of $\tau_1 \tau_2$-sarps-open sets in bitopological spaces and discuss some of the related properties.

Definition 4.1. A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1 \tau_2$-semi $\alpha$-regular pre-semi open (briefly $\tau_1 \tau_2$-sarps-open) if its complement is $\tau_1 \tau_2$-sarps-closed.

Remark 4.2. If $\tau_1 = \tau_2$ in the above definition, $\tau_1 \tau_2$-sarps-open is sarps-open in the sense of point set topology. The set of all $\tau_1 \tau_2$-sarps-open sets in the bitopological space $(X, \tau_1, \tau_2)$ is denoted by $\text{SaRPSO}(X, \tau_1, \tau_2)$.

Remark 4.3. In general, the set of all $\tau_1 \tau_2$-sarps-open sets need not be equal to the set of all $\tau_2 \tau_1$-sarps-open sets as seen in the following example.

Example 4.4. Let $X = \{a, b, c\}$ with the topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then $\text{SaRPSO}(X, \tau_1, \tau_2) = \{\emptyset, \{b, c\}, \{a, b\}, \{b\}, \{a\}, X\}$ and $\text{SaRPSO}(X, \tau_2, \tau_1) = \{\emptyset, \{b\}, \{a\}, \{b\}, \{a\}, X\}$. Therefore $\text{SaRPSO}(X, \tau_1, \tau_2) \neq \text{SaRPSO}(X, \tau_2, \tau_1)$.
Remark 4.5. Difference between two $\tau_1\tau_2$-sarps-open sets need not be a $\tau_1\tau_2$-sarps-open set as seen in the Example 4.6.

Example 4.6. From Example 4.4, $A = \{b, c\}$ and $B = \{b\}$ are $\tau_1\tau_2$-sarps-open sets. But $A \setminus B = \{c\}$ is not $\tau_1\tau_2$-sarps-open.

Proposition 4.7. Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A \subseteq X$. Then the following are true.

(1). Every $\tau_2$-semi-open set is $\tau_1\tau_2$-sarps-open.

(2). Every $\tau_2$-regular open set is $\tau_1\tau_2$-sarps-open.

(3). Every $\tau_2$-$\alpha$-open set is $\tau_1\tau_2$-sarps-open.

(4). Every $\tau_1\tau_2$-$g^\#$-semi-open set is $\tau_1\tau_2$-sarps-open.

Proof.

(1). Let $A$ be a $\tau_2$-semi-open subset of a space $X$. Then $X \setminus A$ is $\tau_2$-semi-closed. Since every $\tau_2$-semi-closed set is $\tau_1\tau_2$-sarps-closed, $X \setminus A$ is $\tau_1\tau_2$-sarps-closed. Therefore $A$ is $\tau_1\tau_2$-sarps-open in $X$.

(2). Let $A$ be a $\tau_2$-regular open subset of a space $X$. Then $X \setminus A$ is $\tau_2$-regular closed. Since every $\tau_2$-regular closed set is $\tau_1\tau_2$-sarps-closed, $X \setminus A$ is $\tau_1\tau_2$-sarps-closed. Therefore $A$ is $\tau_1\tau_2$-sarps-open in $X$.

(3). Let $A$ be a $\tau_2$-$\alpha$-open subset of a space $X$. Then $X \setminus A$ is $\tau_2$-$\alpha$-closed. Since every $\tau_2$-$\alpha$-closed set is $\tau_1\tau_2$-sarps-closed, $X \setminus A$ is $\tau_1\tau_2$-sarps-closed. Therefore $A$ is $\tau_1\tau_2$-sarps-open in $X$.

(4). Let $A$ be a $\tau_1\tau_2$-$g^\#$-semi-open subset of a space $X$. Then $X \setminus A$ is $\tau_1\tau_2$-$g^\#$-semi-closed. Since every $\tau_1\tau_2$-$g^\#$-semi-closed set is $\tau_1\tau_2$-sarps-closed, $X \setminus A$ is $\tau_1\tau_2$-sarps-closed. Therefore $A$ is $\tau_1\tau_2$-sarps-open in $X$.

The reverse implications are not true as shown in the Examples 4.8 and 4.9.

Example 4.8. Let $X = \{a, b, c\}$ with the topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then $\{a, b\}$ is $\tau_1\tau_2$-sarps-open, but not $\tau_2$-semi-open, not $\tau_2$-regular open and not $\tau_2$-$\alpha$-open.

Example 4.9. Let $X = \{a, b, c\}$ with the topologies $\tau_1 = \{\emptyset, \{b, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then $\{a, b\}$ is $\tau_1\tau_2$-sarps-open, but not $\tau_1\tau_2$-$g^\#$-semi-open.

Proposition 4.10. Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A \subseteq X$. Then the following are true.

(1). Every $\tau_1\tau_2$-sarps-open set is $\tau_1\tau_2$-$gs$-open.

(2). Every $\tau_1\tau_2$-sarps-open set is $\tau_1\tau_2$-$gab$-open.

(3). Every $\tau_1\tau_2$-sarps-open set is $\tau_1\tau_2$-$rgb$-open.

Proof.

(1). Let $A$ be a $\tau_1\tau_2$-sarps-open subset of a space $X$. Then $X \setminus A$ is $\tau_1\tau_2$-sarps-closed. Since every $\tau_1\tau_2$-sarps-closed set is $\tau_1\tau_2$-$gs$-closed, $X \setminus A$ is $\tau_1\tau_2$-$gs$-closed. Therefore $A$ is $\tau_1\tau_2$-$gs$-open in $X$.

(2). Let $A$ be a $\tau_1\tau_2$-sarps-open subset of a space $X$. Then $X \setminus A$ is $\tau_1\tau_2$-sarps-closed. Since every $\tau_1\tau_2$-sarps-closed set is $\tau_1\tau_2$-$gab$-closed, $X \setminus A$ is $\tau_1\tau_2$-$gab$-closed. Therefore $A$ is $\tau_1\tau_2$-$gab$-open in $X$. 

32
(3). Let $A$ be a $\tau_{12}-sarps$-open subset of a space $X$. Then $X \setminus A$ is $\tau_{12}-sarps$-closed. Since every $\tau_{12}-sarps$-closed set is $\tau_{12}$-rgb-closed, $X \setminus A$ is $\tau_{12}$-rgb-closed. Therefore $A$ is $\tau_{12}$-rgb-open in $X$.

The reverse implications are not true as shown in the Examples 4.11 and 4.12.

Example 4.11. Let $X = \{a, b, c, d\}$ with the topologies $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b, c\}, X\}$. Then $\{d\}$ is $\tau_{12}$-gs-open and $\tau_{12}$-rgb-open, but not $\tau_{12}$-sarps-open.

Example 4.12. Let $X = \{a, b, c, d\}$ with the topologies $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a, b\}, \{a, b, c\}, X\}$. Then $\{b, c, d\}$ is $\tau_{12}$-gab-open, but not $\tau_{12}$-sarps-open. The concept $\tau_{12}$-sarps-open is independent from the concepts $\tau_{12}$-$\alpha g$-open, $\tau_{12}$-$g^*p$-open, $\tau_{12}$-$r g$-open and $\tau_{12}$-g-open as seen in the following examples.

Example 4.13. Let $X = \{a, b, c, d\}$ with the topologies $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. Then

(1). $\{b, c, d\}$ is $\tau_{12}$-sarps-open but not $\tau_{12}$-$\alpha g$-open and $\{d\}$ is $\tau_{12}$-$\alpha g$-open but not $\tau_{12}$-sarps-open.

(2). $\{b, c, d\}$ is $\tau_{12}$-sarps-open but not $\tau_{12}$-g-open and $\{d\}$ is $\tau_{12}$-g-open but not $\tau_{12}$-sarps-open.

Example 4.14. Let $X = \{a, b, c, d\}$ with the topologies $\tau_1 = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$. Then $\{b, c, d\}$ is $\tau_{12}$-sarps-open but not $\tau_{12}$-rg-open and $\{c, d\}$ is $\tau_{12}$-rg-open but not $\tau_{12}$-sarps-open.

Example 4.15. Let $X = \{a, b, c, d\}$ with the topologies $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$. Then $\{b, c\}$ is $\tau_{12}$-sarps-open but not $\tau_{12}$-$g^*p$-open and $\tau_{12}$-$g^*p$-open.

Example 4.16. Let $X = \{a, b, c\}$ with the topologies $\tau_1 = \{\emptyset, \{a\}, \{a, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then $\{a, c\}$ is $\tau_{12}$-$g^*p$-open and $\tau_{12}$-$g^*p$-open, but not $\tau_{12}$-sarps-open.

Theorem 4.17. If $A$ and $B$ are $\tau_{12}$-sarps-open sets of $X$, then $A \cup B$ is also a $\tau_{12}$-sarps-open set in $X$.

Proof. Let $A$ and $B$ be $\tau_{12}$-sarps-open sets of $X$. Then $X \setminus A$ and $X \setminus B$ are $\tau_{12}$-sarps-closed sets of $X$. By Theorem 3.17, $(X \setminus A) \cap (X \setminus B)$ is $\tau_{12}$-sarps-closed in $X$. But $(X \setminus A) \cap (X \setminus B) = X \setminus (A \cup B)$. Therefore $A \cup B$ is $\tau_{12}$-sarps-open in $X$.

Remark 4.18. The intersection of two $\tau_{12}$-sarps-open sets need not be $\tau_{12}$-sarps-open as shown in the following example.

Example 4.19. From Example 4.13, $A = \{a, d\}$ and $B = \{b, d\}$ are $\tau_{12}$-sarps-open sets. But their intersection $A \cap B = \{d\}$ is not a $\tau_{12}$-sarps-open set.

Theorem 4.20. Every singleton point set in a bitopological space $X$ is either $\tau_{12}$-sarps-open or $\tau_{12}$-arps-closed.

Proof. Let $X$ be a bitopological space. Let $x \in X$. Then by Theorem 3.25, $X \setminus \{x\}$ is $\tau_{12}$-sarps-closed or $\tau_{12}$-arps-open. Therefore $\{x\}$ is $\tau_{12}$-sarps-open or $\tau_{12}$-arps-closed.

Theorem 4.21. A set $A \subseteq X$ is $\tau_{12}$-sarps-open if and only if $F \subseteq \tau_{2}$-sint$A$ whenever $F \subseteq A$ and $F$ is $\tau_{12}$-arps-closed.
Proof. Let $A \subseteq X$ be $\tau_{12}$-sarps-open. Let $F$ be $\tau_{1}$-arps-closed and $F \subseteq A$. Then $X \setminus A \subseteq X \setminus F$ where $X \setminus F$ is $\tau_{1}$-arps-open. Since $X \setminus A$ is $\tau_{12}$-sarps-closed, $\tau_{2}$-scl $(X \setminus A) \subseteq X \setminus F$ and hence $X \setminus \tau_{2}$-sint$A \subseteq X \setminus F$ that implies $F \subseteq \tau_{2}$-sint$A$.

Conversely assume that $F \subseteq \tau_{2}$-sint$A$ whenever $F \subseteq A$ and $F$ is $\tau_{1}$-arps-closed. Let $X \setminus A \subseteq U$ and $U$ be $\tau_{1}$-arps-open. Then $X \setminus U \subseteq A$. Since $X \setminus U$ is $\tau_{1}$-arps-closed, by assumption $X \setminus U \subseteq \tau_{2}$-sint$A$. That implies $X \setminus (\tau_{2}$-sint$A) \subseteq U$. That is $\tau_{2}$-scl $(X \setminus A) \subseteq U$. Therefore $X \setminus A$ is $\tau_{12}$-sarps-closed. Hence $A$ is $\tau_{12}$-sarps-open.

Theorem 4.22. If $\tau_{2}$-sint$A \subseteq B \subseteq A$ and $A$ is $\tau_{12}$-sarps-open, then $B$ is $\tau_{12}$-sarps-open.

Proof. Let $A$ be $\tau_{12}$-sarps-open and $\tau_{2}$-sint$A \subseteq B \subseteq A$. Then $X \setminus A \subseteq X \setminus B \subseteq X \setminus \tau_{2}$-sint$A$ that implies $X \setminus A \subseteq X \setminus B \subseteq \tau_{2}$-scl $(X \setminus A)$. Since $X \setminus A$ is $\tau_{12}$-sarps-closed, by Theorem 3.24 (i), $X \setminus B$ is $\tau_{12}$-sarps-closed. This proves that $B$ is $\tau_{12}$-sarps-open.

Theorem 4.23. If $A \subseteq X$ is $\tau_{12}$-sarps-closed, then $\tau_{2}$-$scl A \setminus A$ is $\tau_{12}$-sarps-open.

Proof. Let $A \subseteq X$ be $\tau_{12}$-sarps-closed and let $F$ be a $\tau_{1}$-arps-closed set such that $F \subseteq \tau_{2}$-$scl A \setminus A$. Then by Theorem 3.21, $F = \emptyset$ that implies $F \subseteq \tau_{2}$-sint $(\tau_{2}$-$scl A \setminus A)$. By using Theorem 4.21, $\tau_{2}$-$scl A \setminus A$ is $\tau_{12}$-sarps-open.

Theorem 4.24. Let $A \subseteq B \subseteq X$ and let $\tau_{2}$-$scl A \setminus A$ be $\tau_{12}$-sarps-open. Then $\tau_{2}$-$scl A \setminus B$ is also $\tau_{12}$-sarps-open.

Proof. Suppose that $\tau_{2}$-$scl A \setminus A$ is $\tau_{12}$-sarps-open. Let $F$ be a $\tau_{1}$-arps-closed subset of $X$ with $F \subseteq \tau_{2}$-$scl A \setminus B$. Then $F \subseteq \tau_{2}$-$scl A \setminus A$. Since $\tau_{2}$-$scl A \setminus A$ is $\tau_{12}$-sarps-open, by Theorem 4.21, $F \subseteq \tau_{2}$-sint $(\tau_{2}$-$scl A \setminus A)$. By Lemma 2.3, $F = \emptyset$. Therefore $F \subseteq \tau_{2}$-sint $(\tau_{2}$-$scl A \setminus B)$. Hence $\tau_{2}$-$scl A \setminus B$ is $\tau_{12}$-sarps-open.

Theorem 4.25. If a set $A$ is $\tau_{12}$-sarps-open in $X$ and if $U$ is $\tau_{1}$-arps-open in $X$ with $\tau_{2}$-sint$A \cup (X \setminus A) \subseteq U$, then $U = X$.

Proof. Let $U$ be $\tau_{1}$-arps-open in $X$ with $\tau_{2}$-sint$A \cup (X \setminus A) \subseteq U$. Now $X \setminus U \subseteq \tau_{2}$-scl $(X \setminus A) \cap A = \tau_{2}$-scl $(X \setminus A) \setminus (X \setminus A)$. Suppose that $A$ is $\tau_{12}$-sarps-open. Since $X \setminus U$ is $\tau_{1}$-arps-closed and since $X \setminus A$ is $\tau_{12}$-sarps-closed, by Theorem 3.21, $X \setminus U = \emptyset$ and hence $U = X$.

Theorem 4.26. Let $X$ be a topological space and $A, B \subseteq X$. If $B$ is $\tau_{12}$-sarps-open and $\tau_{2}$-sint$B \subseteq A$, then $A \cap B$ is $\tau_{12}$-sarps-open.

Proof. Since $\tau_{2}$-sint$B \subseteq A$, $\tau_{2}$-sint$B \subseteq A \cap B \subseteq B$. Since $B$ is $\tau_{12}$-sarps-open, by Theorem 4.22, $A \cap B$ is $\tau_{12}$-sarps-open.

References


