Efficiently Dominating ($\gamma$, ed)-Number of Graphs

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Abstract: In this paper, we introduce the new concept connected and independent edge detour domination number of a graph and obtain the connected and independent edge detour domination number for some well known graphs.

Keywords: Edge detour domination, efficient domination, efficient($\gamma$,ed)-number of graphs.

1. Introduction

The concept of domination was introduced by Ore and Berge [8]. Let G be a finite, undirected connected graph with neither loops nor multiple edges. A subset D of V(G) is a dominating set of G if every vertex in V-D is adjacent to at least one vertex in D. The minimum cardinality among all dominating sets of G is called the domination number $\gamma(G)$ of G. We consider connected graphs with at least two vertices. For basic definitions and terminologies, we refer Harary [3]. For vertices u and v in a connected graph G, the detour distance $D(u,v)$ is the length of longest u-v path in G. A u-v path of length $D(u,v)$ is called a u - v detour. A subset S of V is called a detour set if every vertex in G lie on a detour joining a pair of vertices of S. The detour number $dn(G)$ of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called a detour basis of G. These concepts were studied by chartrand [4]. A subset S of V is called an edge detour set of G if every edge in G lie on a detour joining a pair of vertices of S. The edge detour number $dn_1(G)$ of G is the minimum order of its edge detour sets and any edge detour set of order $dn_1(G)$ is an edge detour basis. A graph G is called an edge detour graph if it has an edge detour set. Edge detour graphs were introduced and studied by Santhakumaran and Athisayanathan [10]. Let G be a connected graph with at least two vertices. An edge detour dominating set is a subset S of V(G) which is both a dominating and an edge detour set of G. An edge detour dominating set is said to be minimal edge detour dominating set of G if no proper subset of S is an edge detour dominating set of G. An edge detour dominating set S is said to be minimum edge detour dominating set of G if there exists no edge detour dominating set $S'$ such that $S'$. The smallest cardinality of an edge detour dominating set of G is called the edge detour domination number of G. It is denoted by ($\gamma_e$, eD). Any edge detour dominating set G of minimum cardinality is called $\gamma_e$-D-set of G.

The edge detour domination number of graphs were introduced and studied by A.Mahalakshmi, K.Palani and

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S.Somasundaram [7]. A subset $S$ of $V(G)$ is called an efficient dominating if for every $v \in V(G)$, $|N[v] \cap S| = 1$. A graph $G$ is efficient if $G$ has an efficient dominating set. Efficient dominating graphs were introduced and studied by D.W.Bange, A.E.Barkauskas and P.J.Slater[1]. Let $G$ be a connected graph and $S$ be an edge detour dominating set of $G$. Then, $S$ is an efficiently dominating $(\gamma, eD)$-set of $G$ if for every $v \in V(G)$, $|N[v] \cap S| = 1$. The minimum cardinality of $S$ is the efficiently dominating $(\gamma, eD)$-set of $G$ and is denoted by $e\gamma_{eD}(G)$. An efficiently dominating $(\gamma, eD)$-set of minimum cardinality is called a $e\gamma_{eD}$-set of $G$. A graph $G$ is said to be an efficiently dominating graph if it has an efficiently dominating $(\gamma, eD)$-set.

In our paper [7], we need the symbol $\gamma_{eD}$-set to represent any edge detour dominating set which we now changed to $(\gamma, eD)$-set.

Theorem 1.1 ([6]). The domination number of some standard graphs are given as follows.

1. $\gamma(P_p) = \lceil \frac{p}{2} \rceil$, $p \geq 3$.
2. $\gamma(C_p) = \lceil \frac{p}{2} \rceil$, $p \geq 3$.
3. $\gamma(K_p) = \gamma(W_p) = \gamma(K_{1,n}) = 1$.
4. $\gamma(K_m,n) = 2$ if $m,n \geq 2$.

The following results are from [10].

Theorem 1.2. Every end vertex of an edge detour graph $G$ belongs to every edge detour set of $G$. Also, if the set of all end vertices of $G$ is an edge detour set, then $S$ is the unique edge detour basis for $G$.

Theorem 1.3. If $G$ is an edge detour graph of order $p \geq 3$ such that $\{u,v\}$ is an edge detour basis of $G$, then $u$ and $v$ are not adjacent.

Theorem 1.4. If $T$ is a tree with $k$ end vertices, then $dn_1(T) = k$.

The following theorems are by A.Mahalakshmi, K.Palani, S.Somasundaram [7].

Theorem 1.5. $K_p$ is an edge detour dominating graph and for $p \geq 3$; $\gamma_{eD}(K_p) = 3$.

Theorem 1.6. $\gamma_{eD}(K_{1,n}) = n$.

Theorem 1.7.

$$\gamma_{eD}(P_n) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

Definition 1.8. A subset $S$ of $V$ is called an independent set of $G$ if no two vertices of $S$ are adjacent.

Definition 1.9. A subdivision of an edge $e = uv$ of a graph $G$ is the replacement of the edge $e$ by a path $\{u,v,w\}$. If every edge of $G$ is subdivided exactly once, then the resulting graph is called the subdivision graph $S(G)$.

Definition 1.10. Let $G_1$ and $G_2$ be two graphs with disjoint vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$ respectively. Then their union $G = G_1 \cup G_2$ is a graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. Clearly, $G_1 \cup G_2$ has $p_1 + p_2$ vertices and $q_1 + q_2$ edges.

Definition 1.11. If $G_1$ and $G_2$ are disjoint graphs, then the join of $G_1$ and $G_2$ is denoted by $G_1 + G_2$ and is defined as $V(G_1 + G_2) = V_1 \cup V_2$ and $E(G_1 + G_2) = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$.

In the next section, we introduce the definition of efficiently dominating $(\gamma, eD)$-graph and find its values for different graphs.
2. Efficiently Dominating \((\gamma, eD)\)-graph

Definition 2.1. Let \(G\) be a connected graph. An efficiently dominating \((\gamma, eD)\)-set of \(G\) is an edge detour dominating set of \(G\) such that for every \(v \in V(G)\), \(|N[v] \cap S| = 1\). The minimum cardinality of among all efficiently dominating \((\gamma, eD)\)-sets of \(G\) is the efficiently dominating \((\gamma, eD)\)-number of \(G\) and is denoted by \(e_{\gamma eD}(G)\). An efficiently dominating \((\gamma, eD)\)-set of minimum cardinality \(e_{\gamma eD}(G)\) is called a \(e_{\gamma eD}\)-set of \(G\). A graph \(G\) is said to be an efficiently dominating \((\gamma, eD)\)-graph if it has an efficiently dominating \((\gamma, eD)\)-set.

Example 2.2. Consider, the graph \(G\) in Figure 2.1 (a). Here, \(S = \{v_1, v_4\}\) is the minimum edge detour dominating set of \(G\) and \(|N[v_i] \cap S| = 1\) for all \(v_i \in V(G), 1 \leq i \leq 5\). Therefore, \(S\) is the minimum efficiently dominating \((\gamma, eD)\)-set of \(G\) and \(e_{\gamma eD}(G) = 2\). Therefore, the graph \(G\) is an efficiently dominating \((\gamma, eD)\)-graph.

\[\text{Figure 2.1}\]

For the graph \(G\) in Figure 2.1 (b), \(S = \{v_1, v_4, v_7, v_{10}\}\) is the minimum edge detour dominating set of \(G\) and \(|N[v_i] \cap S| = 1\) for all \(v_i \in V(G), 1 \leq i \leq 10\). Therefore, \(S\) is the minimum efficiently dominating \((\gamma, eD)\)-set of \(G\) and \(e_{\gamma eD}(G) = 4\). Therefore, the graph \(G\) is an efficiently dominating \((\gamma, eD)\)-graph.

Observation 2.3.

1. Any efficiently dominating \((\gamma, eD)\)-set contains only isolated vertices.
2. Every edge detour dominating set need not be an efficiently dominating \((\gamma, eD)\)-set.

Theorem 2.4. The path \(P_n, n \equiv 1 \pmod{3}\) is an efficiently dominating \((\gamma, eD)\)-graph and \(e_{\gamma eD}(P_n) = \gamma_{eD}(P_n) = \lceil \frac{n-1}{3} \rceil + 2\).

Proof. Let \(P_n = (v_1, v_2, v_3, ..., v_{3k+1}), k > 0\). \(S = \{v_1, v_4, v_7, ..., v_{3k+1}\}\) is a unique \(\gamma_{eD}\)-set of \(P_n\). Further, \(|N[v] \cap S| = 1\)
for all \( v \in P_n \). Hence, \( S \) is the unique efficiently dominating \((\gamma, eD)\)-set of \( P_n \). Therefore, \( e_{\gamma,D}(P_n) = \gamma_{eD}(P_n) \). By Theorem 1.8, \( \gamma_{eD}(P_n) = \lceil \frac{n-4}{3} \rceil + 2 \).

**Theorem 2.5.** The graph \( G = C_{3n} \), \( n > 1 \) is an efficiently dominating \((\gamma, eD)\)-graph and \( e_{\gamma,D}(C_{3n}) = n \).

**Proof.** Let \( V(C_{3n}) = \{v_1, v_2, ..., v_{3n}\} \). The only \( \gamma_{eD} \)-sets of \( C_{3n} \) are \( S_1 = \{v_1, v_4, ..., v_{3(n-1)+1}\} \), \( S_2 = \{v_2, v_5, ..., v_{3(n-1)+2}\} \) and \( S_3 = \{v_3, v_6, ..., v_{3n}\} \). Further, \( |N[v] \cap S_i| = 1 \) for all \( v \in C_{3n} \) and \( i = 1, 2, 3 \). Hence, \( C_{3n} \) is an efficiently dominating \((\gamma, eD)\) graph and \( S_i \), \( i = 1, 2, 3 \) are efficiently dominating \((\gamma, eD)\)-sets of \( C_{3n} \). Therefore, \( e_{\gamma,D}(C_{3n}) = \gamma_{eD}(C_{3n}) \). Therefore, by Theorem 1.9, \( e_{\gamma,D}(C_{3n}) = \gamma_{eD}(C_{3n}) = \lceil \frac{2n}{3} \rceil = n \).

**Theorem 2.6.** Every efficiently dominating \((\gamma, eD)\)-set of a graph \( G \) is independent. (i.e), No two vertices of an efficiently dominating \((\gamma, eD)\)-set are adjacent.

**Proof.** Let \( G \) be a graph and \( S \) be an \( e_{\gamma,D} \)-set of \( G \). Suppose \( u, v \in S \) such that \( u \) and \( v \) are adjacent. Then, \( \{u, v\} \subseteq N[u] \cap S \). Therefore, \( |N[u] \cap S| \neq 1 \). So, \( S \) is not an efficiently dominating \( e_{\gamma,D} \)-set of \( G \), which is a contradiction. Therefore, \( u \) and \( v \) are not adjacent. Since \( u \) and \( v \) are arbitrary, no two vertices are adjacent. Hence, every efficiently dominating \((\gamma, eD)\)-set is independent.

**Proposition 2.7.** Complete graph \( K_n \), \( n > 2 \) are not efficiently dominating \((\gamma, eD)\)-graphs.

**Proof.** By Observation 2.3 (ii), any efficiently dominating \((\gamma, eD)\)-set is also a \((\gamma, eD)\)-set. Further, any \((\gamma, eD)\)-set of \( K_n \) contains at least two vertices. As being vertices of \( K_n \), they are adjacent. Therefore, by Observation 2.3(ii), \( K_n \) has no efficiently dominating \((\gamma, eD)\)-graph.

**Corollary 2.8.** Complete bipartite graphs are not efficiently dominating \((\gamma, eD)\)-graphs.

**Proof.** Let \( V_1, V_2 \) be the bipartition of \( V(K_{m,n}) \). Any, \( \gamma_{eD} \)-set of \( K_{m,n} \) contains at least one vertex from both \( V_1 \) and \( V_2 \). Obviously, they are adjacent. Therefore, by Theorem 2.6, \( K_{m,n} \) are not efficiently dominating \((\gamma, eD)\)-graphs.

**Observation 2.9.**

1. A graph \( G \) has no efficiently dominating \((\gamma, eD)\)-set if it has a vertex with more than one pendant edge.

2. For any graph \( e_{\gamma,D}(G) \leq \gamma_{eD}(G) \).

3. If \( G_1 \) and \( G_2 \) are efficiently dominating \((\gamma, eD)\)-graphs then, \( G_1 \cup G_2 \) is also an efficiently dominating \((\gamma, eD)\)-graph and \( e_{\gamma,D}(G_1 \cup G_2) = \gamma_{eD}(G_1) + \gamma_{eD}(G_2) \).

**Proposition 2.10.** An edge detour dominating set \( S \) is an efficiently dominating \((\gamma, eD)\)-set if and only if \( d(u, v) \geq 3 \).

**Proof.** Let \( G \) be a graph. \( S \) is an efficiently dominating \((\gamma, eD)\)-set of \( G \). Suppose, for all \( u, v \in S \) such that \( d(u, v) < 3 \).

**Case 1:** \( d(u, v) = 1 \). \( \{u, v\} \subseteq N[u] \cap S \). Therefore, \( N[u] \cap S \geq 1 \).

**Case 2:** \( d(u, v) = 2 \). Let \( uwv \) be a smallest \( u-v \) path in \( G \). Then, \( N[u] \cap S \geq 2 \). In both the cases, \( N[x] \cap S \neq 1 \) for at least one \( x \in S \). Therefore, \( S \) is not an efficiently dominating \((\gamma, eD)\)-set of \( G \), which is a contradiction. Therefore, \( d(u, v) \geq 3 \).

Conversely, \( d(u, v) \geq 3 \). Let \( S \) be an edge detour dominating set of \( G \). To prove that \( S \) is efficiently dominating \((\gamma, eD)\)-set of \( G \). Suppose not. Then, \( S \cap N[x] \neq 1 \) for at least one \( x \in S \). Therefore, there exist one vertex say \( u \) in \( S \) such that \( x \) is adjacent to \( u \). Then, obviously \( d(u, x) = 1 < 3 \). Which is a contradiction. Therefore, \( S \) is an efficiently dominating \((\gamma, eD)\)-set of \( G \).
Lemma 2.11. Wheel graph $W_{1,n}$ is an edge detour dominating graph, $\gamma_{eD}(W_{1,n}) = \begin{cases} 2fn = 6 \\ 3 \text{ otherwise} \end{cases}$

*Proof.* Let $G$ be a wheel graph with central vertex $v$.

**Case 1:** When $n = 6$, $S_i = \{v, v_{i+1}\}$, $i = 1, 2, 3$. Forms an edge detour dominating set of $G$. Therefore, $\gamma_{eD}(G) = 2$.

**Case 2:** Let $n > 6$. Now $v$ along with any two of the rim vertices forms an edge detour dominating set. Therefore, $\gamma_{eD}(G) \leq 3$.

**Claim:** $\gamma_{eD}(G) \neq 2$.

Let $S$ be any two element subset of $V(G)$. Suppose $S = \{v, v_i\}$, then the edge $vv_i$ does not lie on any detour joining $v$ and $v_i$. If $S = \{v, v_j\}$ two cases arise.

**Sub Case 2a:** If $v_i$ and $v_j$ are adjacent. Here, the edge $v_iv_j$ does not lie on any detour joining $v_i$ and $v_j$.

**Sub Case 2b:** If $v_i$ and $v_j$ are non-adjacent. Since $n > 6$, some vertices of G are either not dominated by $v_i$ and $v_j$ or they do not lie on any detour joining $v_i$ and $v_j$. Therefore, $S$ cannot be a detour joining set of $G$. Since, $S$ is arbitrary, $\gamma_{eD}(G) \neq 2$. Hence, $\gamma_{eD}(G) = 3$. \[\square\]

**Corollary 2.12.** Wheel graph is a non - efficiently dominating $(\gamma, eD)$-graph.

*Proof.* Any edge detour dominating set of $W_n$ contains at least two of the rim vertices. Obviously, $d(u, v) = 2 < 3$, for any two rim vertices $u$ and $v$.

Therefore, by Proposition 2.10, any edge detour dominating set is not an efficiently dominating $(\gamma, eD)$-set. Therefore, wheel is not an efficiently dominating $(\gamma, eD)$-graph. \[\square\]

3. Efficiently Dominating $(\gamma, eD)$-Number of Subdivision Graphs

**Theorem 3.1.** The graph $S(P_n)$, subdivision of path $P_n$, $n \equiv 1 \pmod 3$ is an efficiently dominating $(\gamma, eD)$-graph.

*Proof.* The graph $S(P_n)$, $n \equiv 1 \pmod 3$ is again a path $P_m$ with $m \equiv 1 \pmod 3$. Therefore by Theorem 2.4, $S(P_n)$ where $n \equiv 1 \pmod 3$ is an efficiently dominating $(\gamma, eD)$-graph. \[\square\]

**Theorem 3.2.** The graph $S(C_{3n})$, subdivision of the cycle $C_{3n}$ where $n \geq 1$ is an efficiently dominating $(\gamma, eD)$-graph and $e\gamma_{eD}(S(C_{3n})) = 2n$.

*Proof.* The graph $S(C_{3n})$ where $n \geq 1$, is again a cycle $C_{2(3n)}$. Therefore, by Theorem 2.5, $S(C_{3n})$ is an efficiently dominating $(\gamma, eD)$-graph and also $e\gamma_{eD}(S(C_{3n})) = e\gamma_{eD}(C_{3(2n)}) = 2n$. \[\square\]

**Theorem 3.3.** The subdivision graph $S(K_n)$ is an efficiently dominating $(\gamma, eD)$-graph if and only if $n = 3$.

*Proof.* Let $n = 3$. $S(K_n) = C_6$. Hence, by Theorem 2.5, $\gamma(S(K_3)) = 2$. Let $n \geq 4$. Any $(\gamma, eD)$-set should contain at least three points. Therefore, it must contain at least two points such that $d(u, v) = 2$. Hence, the middle vertex in any shortest $u - v$ path is dominated by both $u$ and $v$. Therefore, any $(\gamma, eD)$-set cannot be an efficiently dominating $(\gamma, eD)$-set. Hence, $S(K_n)$ has no efficiently dominating $(\gamma, eD)$-set. \[\square\]

**Corollary 3.4.** The subdivision of a star graph $S(K_{1,n})$ is not an efficiently dominating $(\gamma, eD)$-graph.

*Proof.* Any efficiently dominating $(\gamma, eD)$-set $S$ of $S(K_{1,n})$ contain at least $n + 1$ vertices. Therefore, it must be at least two vertices $u$ and $v$ in $S$ such that $d(u, v) \leq 2$.

**Case 1:** $d(u, v) = 1$, then $|N(u) \cap S| = |N(v) \cap S| = 2$. Then, $S$ is not an efficiently dominating $(\gamma, eD)$-set.

**Case 2:** $d(u, v) = 2$, Hence, the middle vertex in any shortest $u - v$ path is dominated by both $u$ and $v$. Therefore, any $(\gamma, eD)$-set cannot be an efficiently dominating $(\gamma, eD)$-set. Hence, $S(K_{1,n})$ has no efficiently dominating $(\gamma, eD)$-set. \[\square\]
Remark 3.5. Let $G$ be a connected graph with $p \geq 2$ vertices. If every $(\gamma, eD)$-set of $G$ should contain at least two vertices of shortest length $\leq 2$, then $G$ contains no efficiently $(\gamma, eD)$-set.

References


