On a Subordination Associated with a Certain Subclass of Analytic Functions Defined by Salagean Derivatives

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Abstract: In this paper we discuss some subordination results for a subclass of functions analytic in the unit disk $U$.

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1. Introduction

Let $A$ be the class of functions $f(z)$ analytic in the unit disk $U = \{z : |z| < 1\}$ and let $S$ denote a subclass of $A$ consisting of functions univalent in $U$ and normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

We denote the class of convex functions of order $\alpha$ by $K(\alpha)$, i.e.,

$$K(\alpha) = \{f \in S : \text{Re} \left( 1 + \frac{zf''}{f'} \right) > \alpha, z \in U\}$$

Definition 1.1 (Hadamard product or convolution). Given two functions $f(z)$ and $g(z)$, where $f(z)$ is defined as in (1) and $g(z)$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

The Hadamard product (or convolution) $f \ast g$ of $f(z)$ and $g(z)$ is defined by

$$(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g \ast f)(z)$$

Definition 1.2. Let $f(z)$ and $g(z)$ be analytic in the unit disk $U$. Then $f(z)$ is said to be subordinate to $g(z)$ in $U$ and we write $f(z) \prec g(z)$, $z \in U$.

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if there exists a Schwarz function $\omega(z)$, analytic in $U$ with $\omega(0) = 0, |\omega(z)| < 1$ such that

$$f(z) = g(\omega(z)), \quad z \in U$$

(3)

In particular, if the function $g(z)$ in univalent in $U$, then $f(z)$ is subordinate to $g(z)$ if

$$f(0) = g(0), f(U) \subseteq g(U)$$

(4)

**Definition 1.3.** A sequence $\{C_n\}_{n=1}^\infty$ of complex numbers is said to be a subordinating factor sequence of $f(z)$ if whenever $f(z)$ of the form (1) is analytic, univalent and convex in $U$, the subordination is given by $\sum_{n=1}^\infty a_n C_n z^n < f(z) \in U, a_1 = 1$.

We have the following theorem

**Theorem 1.4 ([1]).** The sequence $\{c_k\}_{k=1}^\infty$ is a subordinating factor sequence if and only if

$$Re\left\{ 1 + 2 \sum_{k=1}^\infty c_k z^k \right\} > 0 \quad (z \in U)$$

(5)

Let

$$S_n(\alpha) = \left\{ f \in A : Re\left( \frac{D^{n+1} f(z)}{D^n f(z)} \right) > \alpha, z \in U \right\}$$

(6)

Here $D^n f(z)$ is the Salagean derivatives, $n = 0, 1, 2, \ldots$. Such that

$$D^n f(z) = f(z)$$
$$D^1 f(z) = D f(z) = zf'(z)$$
$$D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]'$$

therefore,

$$D^n f(z) = z + \sum_{k=2}^\infty k^n a_k z^k$$

The class $S_n(\alpha)$ was studied by Salagean [2] and Kadioglu [3]. In [3] the following result was established

**Theorem 1.5 ([3]).** $f(z) \in S_n(\alpha)$ if and only if

$$\sum_{k=2}^\infty k^n (k - \alpha) |a_k| \leq 1 - \alpha$$

(7)

where $n \in \mathbb{N}, 0 \leq \alpha < 1$.

It is natural to consider the class $\bar{S}_n(\alpha)$ such that

$$\bar{S}_n(\alpha) = \left\{ f \in A : \sum_{k=2}^\infty k^n (k - \alpha) |a_k| \leq 1 - \alpha \right\}$$

(8)

$n = \mathbb{N} \cup [0], 0 \leq \alpha < 1$.

**Remark 1.6 ([4]).** If $n = 0$ and $\alpha = 0$ in $\bar{S}_n(\alpha)$ we have the class $S_0(0) = \left\{ f \in A : \sum_{k=2}^\infty k |a_k| \leq 1 \right\}$ which is the subclass of the class of starlike function.

**Remark 1.7 ([5]).** If $n = 0$ in $\bar{S}_n(\alpha)$ we have the class $S_0(\alpha) = \left\{ f \in A : \sum_{k=2}^\infty (k - \alpha) |a_k| \leq 1 - \alpha \right\}$ which is the subclass of class of starlike function of order $\alpha$.

**Remark 1.8 ([4]).** If $n = 1$ and $\alpha = 0$ in $\bar{S}_n(\alpha)$ we have the class $S_1(0) = \left\{ f \in A : \sum_{k=2}^\infty k^2 |a_k| \leq 1 \right\}$ which is the subclass of class of convex function.

**Remark 1.9 ([5]).** If $n = 1$ in $\bar{S}_n(\alpha)$ we have the class $S_1(\alpha) = \left\{ f \in A : \sum_{k=2}^\infty k(k - \alpha) |a_k| \leq 1 - \alpha \right\}$ which is the subclass of class of convex function of order $\alpha$.
2. Main Result

Our main result in this paper is the following theorem.

**Theorem 2.1.** Let \( f(z) \in \widetilde{S}_n(\alpha) \), then

\[
\frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]} (f*g)(z) < g(z)
\]

where \( n \in N \cup \{0\} \), \( 0 \leq \alpha < 1 \), \( g(z) \) is a convex function. and

\[
\text{Re}(f(z)) > -\frac{(1-\alpha)+2^n(2-\alpha)}{2^n(2-\alpha)}
\]

The constant factor \( \frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]} \) cannot be replaced by a larger one.

**Proof.** Let \( f(z) \in \widetilde{S}_n(\alpha) \) and suppose that \( g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in K(\alpha) \) i.e. \( g(z) \) is a convex function of order \( \alpha \). Then by definition,

\[
\frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]} (f*g)(z) = \frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]} (z + \sum_{k=2}^{\infty} a_k b_k z^k)
\]

\[
= \sum_{k=1}^{\infty} \frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]} a_k b_k z^k, \quad a_1 = 1
\]

Hence, by Definition 1.3, to show subordination (9) it is enough to prove that

\[
\left\{ \frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]} a_k \right\}_{k=1}^{\infty}
\]

is a subordinating factor sequence with \( a_1 = 1 \). Therefore by Theorem 1.1, it is sufficient to show that

\[
\text{Re} \left\{ 1 + 2 \sum_{k=1}^{\infty} \frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]} a_k z^k \right\} > 0, \quad (z \in U)
\]

(13)

Now,

\[
\text{Re} \left\{ 1 + 2 \sum_{k=1}^{\infty} \frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]} a_k z^k \right\} = \text{Re} \left\{ 1 + \frac{2^n(2-\alpha)z}{1-\alpha+2^n(2-\alpha)} + \frac{2n(2-\alpha)z}{1-\alpha+2^n(2-\alpha)} \sum_{k=2}^{\infty} a_k z^k \right\}
\]

\[
> \text{Re} \left\{ 1 - \frac{2^n(2-\alpha)r}{1-\alpha+2^n(2-\alpha)} - \frac{1}{1-\alpha+2^n(2-\alpha)} \sum_{k=2}^{\infty} k^n(k-\alpha)|a_k| r \right\}
\]

\[
> \text{Re} \left\{ 1 - \frac{2^n(2-\alpha)r}{1-\alpha+2^n(2-\alpha)} - \frac{(1-\alpha)r}{1-\alpha+2^n(2-\alpha)} \right\}
\]

\[
= 1 - r > 0
\]

(14)

Since \(|z| = r < 1\). Therefore, we obtain

\[
\text{Re} \left\{ 1 + 2 \sum_{k=1}^{\infty} \frac{2^n(2-\alpha)}{2[(1-\alpha)+2^n(2-\alpha)]} a_k z^k \right\} > 0, \quad (z \in U)
\]

which is (13) that we are to established. We now show that

\[
\text{Re}(f(z)) > -\frac{2(1-\alpha)+2^n(2-\alpha)}{2^n(2-\alpha)}
\]
Taking \( g(z) = \frac{z}{1-z} \) which is a convex function (9) becomes

\[
\frac{2^n(2-\alpha)}{2[(1-\alpha) + 2^n(2-\alpha)]} f(z) \ast \frac{z}{1-z} \quad z = \frac{z}{1-z}
\]

and note that \( f(z) \ast \frac{z}{1-z} \). Since

\[
\text{Re} \left( \frac{z}{1-z} \right) > -\frac{1}{2}, \quad |z| = r
\]

which implies that

\[
\text{Re} \left\{ \frac{2^n(2-\alpha)}{2[(1-\alpha) + 2^n(2-\alpha)]} f(z) \ast \frac{z}{1-z} \right\} > -\frac{1}{2}
\]

Hence, we have

\[
\text{Re}(f(z)) > \frac{(1-\alpha) + 2^n(2-\alpha)}{2^n(2-\alpha)}
\]

which is the (10). To show the sharpness of the constant factor \( \frac{2^n(2-\alpha)}{2[(1-\alpha) + 2^n(2-\alpha)]} \) we consider the function:

\[
f_1(z) = \frac{z(2^n(2-\alpha)) + (1-\alpha)z^2}{2^n(2-\alpha)}
\]

Applying (10) with \( g(z) = \frac{z}{1-z} \) and \( f(z) = f_1(z) \) we have

\[
\frac{z(2^n(2-\alpha)) + (1-\alpha)z^2}{2[(1-\alpha) + 2^n(2-\alpha)]} < \frac{z}{1-z}
\]

By using the fact that

\[
|\text{Re}(z)| \leq |z|
\]

We show that

\[
\min_{z \in U} \left\{ \frac{\text{Re} \left\{ \frac{z(2^n(2-\alpha)) + (1-\alpha)z^2}{2[(1-\alpha) + 2^n(2-\alpha)]} \right\}}{2[(1-\alpha) + 2^n(2-\alpha)]} \right\} = -\frac{1}{2}
\]

We have that

\[
\left| \text{Re} \left\{ \frac{z(2^n(2-\alpha)) + (1-\alpha)z^2}{2[(1-\alpha) + 2^n(2-\alpha)]} \right\} \right| \leq \left| \frac{z(2^n(2-\alpha)) + (1-\alpha)z^2}{2[(1-\alpha) + 2^n(2-\alpha)]} \right| = \frac{|z(2^n(2-\alpha)) + (1-\alpha)z|}{2[(1-\alpha) + 2^n(2-\alpha)]} \leq \frac{2^n(2-\alpha) - (1-\alpha)}{2[(1-\alpha) + 2^n(2-\alpha)]} = \frac{1}{2}
\]

This implies that

\[
\left| \text{Re} \left\{ \frac{z(2^n(2-\alpha)) - (1-\alpha)z^2}{2[(1-\alpha) + 2^n(2-\alpha)]} \right\} \right| \leq \frac{1}{2}
\]

i.e.,

\[
-\frac{1}{2} \leq \left| \text{Re} \left\{ \frac{z(2^n(2-\alpha)) - (1-\alpha)z^2}{2[(1-\alpha) + 2^n(2-\alpha)]} \right\} \right| \leq \frac{1}{2}
\]

Hence, we have

\[
\min_{z \in U} \left\{ \text{Re} \left\{ \frac{z(2^n(2-\alpha)) - (1-\alpha)z^2}{2[(1-\alpha) + 2^n(2-\alpha)]} \right\} \right\} \geq -\frac{1}{2}
\]

i.e.,

\[
\min_{z \in U} \left\{ \text{Re} \left\{ \frac{2^n(2-\alpha)}{2[(1-\alpha) + 2^n(2-\alpha)]} f_1 \ast \frac{z}{1-z} \right\} \right\} \geq -\frac{1}{2}
\]

which completes the proof of Theorem 1.3.
3. Some Applications

Taking \( n = 0 \) in Theorem 2.1, we obtain the following:

**Corollary 3.1.** If the function \( f(z) \) defined by (1) is in \( \tilde{S}_n(\alpha) \) then

\[
\frac{2^\alpha}{2-\alpha} (f * g)(z) \prec g(z), \quad (z \in U; \ g \in K(\alpha)) \quad \text{and} \quad \Re(f(z)) > -\frac{3 - 2\alpha}{2 - \alpha}
\]

which is a result of [6].

Taking \( n = 0 \) and \( \alpha = 0 \) in Theorem 2.1, we obtain the following:

**Corollary 3.2.** If the function \( f(z) \) defined by (1) in \( \tilde{S}_n(\alpha) \) then

\[
\frac{3}{2}(f * g)(z) \prec g(z), \quad (z \in U; \ g \in K(\alpha)) \quad \text{and} \quad \Re(f(z)) > -\frac{3}{2}
\]

which is a result of [7].

Taking \( n = 1 \) in Theorem 2.1, we obtain the following:

**Corollary 3.3.** If the function \( f(z) \) defined by (1) in \( \tilde{S}_n(\alpha) \) then

\[
\frac{5 - 3\alpha}{4 - 2\alpha} (f * g)(z) \prec g(z), \quad (z \in U; \ g \in K(\alpha)) \quad \text{and} \quad \Re(f(z)) > -\frac{5 - 3\alpha}{4 - 2\alpha}
\]

which is the result generalized by [7].

Taking \( n = 1 \) and \( \alpha = 0 \) in Theorem 2.1, we obtain the following:

**Corollary 3.4.** If the function \( f(z) \) defined by (1.1) in \( \tilde{S}_n(\alpha) \) then

\[
\frac{5}{4} (f * g)(z) \prec g(z), \quad (z \in U; \ g \in K(\alpha)) \quad \text{and} \quad \Re(f(z)) > -\frac{5}{4}
\]

which is the result generalized by [4].

References


