Properties of \((i,j)-\beta\)-compact Spaces

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Abstract: A kind of new \((i,j)-\beta\)-compactness axiom is introduced in \(L\)-bitopological spaces, where \(L\) is a fuzzy lattice. And its topological properties are systematically studied.

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1. Introduction

It is known that compactness and its stronger and weaker forms play very important roles in topology. Based on fuzzy topological spaces introduced by Chang [3], various kinds of fuzzy compactness [3, 6, 10] have been established. However, these concepts of fuzzy compactness rely on the structure of \(L\) and \(L\) is required to be completely distributive. In [9], for a complete De Morgan algebra \(L\), Shi introduced a new definition of fuzzy compactness in \(L\)-topological spaces using open \(L\)-sets and their inequality. This new definition does not depend on the structure of \(L\). In this paper, A kind of new \((i,j)\)-\(\beta\)-compactness axiom is introduced in \(L\)-bitopological spaces, where \(L\) is a fuzzy lattice. And its topological properties are systematically studied.

2. Preliminaries

Throughout this paper \(X\) and \(Y\) will be nonempty ordinary sets and \(L = L(\leq, \lor, \land')\) will denote a fuzzy lattice, that is, a completely distributive lattice with a smallest element 0 and largest element 1 (0 ≠ 1) and with an order reversion involution \(a \rightarrow a' (a \in L)\). We shall denote by \(L^X\) the lattice of all \(L\)-subsets of \(X\) and if \(A \in X\) by \(\chi_A\) the characteristic function of \(A\). An \(L\)-topological space is a pair \((X, \tau)\), where \(\tau\) is a subfamily of \(L^X\) which contains 0, 1 and is closed for any suprema and finite infima. \(\tau\) is called an \(L\)-topology on \(X\). Each member of \(\tau\) is called an open \(L\)-set and its quasi complementation is called a closed \(L\)-set. An \(L\)-bitopological space (or \(L\)-bts for short) is an ordered triple \((X, \tau_1, \tau_2)\), where \(\tau_1\) and \(\tau_2\) are subfamilies of \(L^X\) which contains 0, 1 and is closed for any suprema and finite infima. An \(L\)-bitopological space (or \(L\)-bts for short) is an ordered triple \((X, \tau_1, \tau_2)\), where \(\tau_1\) and \(\tau_2\) are subfamilies of \(L^X\) which contains 0, 1 and is closed for any suprema and finite infima. An element \(p\) of \(L\) is called prime if and only if \(p \neq 1\) and whenever \(a, b \in L\) with \(a \land b \leq p\) then \(a \leq p\) or \(b \leq p\) [5, 6]. The set of all prime elements of \(L\) will be denoted by \(pr(L)\). An element \(a\) of \(L\) is called union

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determined the prime element of the fuzzy lattice $L^X$. It is obvious that $p \in \text{pr}(L)$ if and only if $p' \in M(L)$. Warner [12] has determined the prime element of the fuzzy lattice $L^X$. We have $\text{pr}(L^X) = \{x_p : x \in X \text{ and } p \in \text{pr}(L)\}$, where for each $x \in X$ and each $p \in \text{pr}(L), x_p : X \to L$ is the L-subset defined by

$$x_p(y) = \begin{cases} p & \text{if } y=x, \\ 1 & \text{otherwise.} \end{cases}$$

These $x_p$ are called the L-points of $X$ and we say that $x_p$ is a member of an L-subset $f$ and write $x_p \in f$ if and only if $f(x) \notin p$. Thus, the union irreducible elements of $L^X$ are the function $x_\alpha : X \to L$ defined by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y=x, \\ 0 & \text{otherwise,} \end{cases}$$

where $x \in X$ and $\alpha \in M(L)$. Hence, we have $M(L^X) = \{x_\alpha : x \in X \text{ and } \alpha \in M(L)\}$. As these $x_\alpha$ are identified with the L-points $x_p$ of $X$, we shall refer to them as fuzzy points. When $x_\alpha \in M(L^X)$, we hall $x$ and $\alpha$ the support of $x_\alpha$ ($x = \text{Supp}_x$) and the height of $x_\alpha(\alpha = h(x_\alpha))$, respectively.

**Definition 2.1** ([1]). Let $(X, \tau_1, \tau_2)$ be an L-cts, $A \in L^X$. Then $A$ is called an $(i, j)$-β-open set if $A \leq j \text{ Cl}(i \text{ Int}(j \text{ Cl}(A)))$.

The complement of an $(i, j)$-β-open set is called an $(i, j)$-β-closed set. Also, $(i, j)$-βO($L^X$) and $(i, j)$-βC($L^X$) will always denote the family of all $(i, j)$-β-open sets and $(i, j)$-β-closed sets respectively. Obviously, $A \in (i, j)$-βO($L^X$) if and only if $A' \in (i, j)$-βC($L^X$).

**Definition 2.2** ([1]). Let $(L^X, \tau_1, \tau_2)$ be an L-bitopological space, $A, B \in L^X$. Let $A, B \in (i, j)$-βO($L^X$) and $A \leq B \in (i, j)$-βC($L^X$). Then $(i, j)$-βInt($A$) and $(i, j)$-βCl($A$) are called the $(i, j)$-β-interior and $(i, j)$-β-closure of $A$ respectively.

**Definition 2.3** ([11]). Let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ be two L-bitopological spaces. A function $f(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $(i, j)$-β-continuous if and only if $f^{-1}(g)$ is $(i, j)$-β-open in $(X, \tau_1, \tau_2)$ for each $g \in \sigma_i$.

**Definition 2.4** ([2]). Let $\alpha \in M(L)$ and $g \in L^X$. A collection $\eta$ of L-subsets is said to form an α-level filter base in the L-subset $g$ if and only if for any finite subcollection $\{f_1, ..., f_n\}$ of $\eta$, there exists $x \in X$ with $g(x) \geq \alpha$ such that $(\bigwedge_{i=1}^{n} f_i)(x) \geq \alpha$. When $g$ is the whole space $X$, then $\eta$ is an α-level filter base if and only if for any finite subcollection $\{f_1, ..., f_n\}$ of $\eta$ there exists $x \in X$ such that $(\bigwedge_{i=1}^{n} f_i)(x) \geq \alpha$.

**Lemma 2.5** ([9]). Let $(X, \tau)$ be a topological space, $f$ be an L-subset in the L-ts $(X, \omega(\tau))$ and $p \in \text{pr}(L)$. Then we have

1. $(\text{Cl}(f))^{-1}(\{t \in L : t \notin p\}) \subseteq \text{Cl}(f^{-1}(\{t \in L : t \notin p\}))$.
2. $(\text{Int}(f))^{-1}(\{t \in L : t \notin p\}) \subseteq \text{Int}(f^{-1}(\{t \in L : t \notin p\}))$.

**Lemma 2.6** ([9]). Let $(X, \tau)$ be a topological space and $A \subseteq X$. Considering the L-ts $(X, \omega(\tau))$ and

$$f(x) = \begin{cases} e \in L & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

we have the following

$$\text{Cl}(f)(x) = \begin{cases} e & \text{if } x \in \text{Cl}(A), \\ 0 & \text{otherwise,} \end{cases}$$
and
\[
\text{Int}(f)(x) = \begin{cases} 
eq 0 & \text{if } x \in \text{Int}(A), \\
eq 0 & \text{otherwise}. \end{cases}
\]

**Definition 2.7** ([2]). Let \((X, \tau)\) be an L-ts and \(g \in L^X, r \in L\).

1. A collection \(\mu = \{f_i\}_{i \in I}\) of L-subsets is called an r-level cover of \(g\) if and only if \((\bigvee_{i \in I} f_i)(x) \not\in r\) for all \(x \in X\) with \(g(x) \geq r\). If each \(f_i\) is open then \(\mu\) is called an r-level open cover of \(g\). If \(g\) is the whole space \(X\), then \(\mu\) is called an r-level cover of \(X\) if and only if \((\bigvee_{i \in I} f_i)(x) \not\in r\) for all \(x \in X\).

2. An r-level cover \(\mu = \{f_i\}_{i \in I}\) of \(g\) is said to have a finite r-level subcover if there exists a finite subset \(F\) of \(J\) such that \((\bigvee_{i \in F} f_i)(x) \not\in r\) for all \(x \in X\) with \(g(x) \geq r\).

**Definition 2.8.** Let \((X, \tau)\) be an L-ts and \(g \in L^X\). Then \(g\) is said to be compact [7] if and only if for every prime \(p \in L\) and every collection \(\{f_i\}_{i \in I}\) of open L-subsets with \((\bigvee_{i \in I} f_i)(x) \not\in p\) for all \(x \in X\) with \(g(x) \geq p\), there exists a finite subset \(F\) of \(J\) such that \((\bigvee_{i \in F} f_i)(x) \not\in p\) for all \(x \in X\) with \(g(x) \geq p\), that is, every p-level open cover of \(g\) has a finite p-level subcover, where \(p \in pr(L)\). If \(g\) is the whole space, then the L-ts \((X, \tau)\) is called compact.

### 3. \((i, j)\)-\(\beta\)-compactness and its Goodness

**Definition 3.1.** Let \((X, \tau_1, \tau_2)\) be an L-bts and \(g \in L^X\). The \(g\) is called \((i, j)\)-\(\beta\)-compact if and only if every p-level cover of \(g\) consisting of \((i, j)\)-\(\beta\)-open L-subsets has a finite p-level subcover, where \(p \in pr(L)\). If \(g\) is the whole space, then we say that the L-bts \((X, \tau_1, \tau_2)\) is \((i, j)\)-\(\beta\)-compact.

**Lemma 3.2.** Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(A \subset X\). If \(A\) is \((i, j)\)-\(\beta\)-open in \((X, \tau_1, \tau_2)\), then \(\chi_A\) is \((i, j)\)-\(\beta\)-open in the L-bts \((X, \omega(\tau_1), \omega(\tau_2))\).

**Theorem 3.3.** Let \((X, \tau_1, \tau_2)\) be a bitopological space. Then \((X, \tau_1, \tau_2)\) is \((i, j)\)-\(\beta\)-compact if and only if the L-bts \((X, \omega(\tau_1), \omega(\tau_2))\) is \((i, j)\)-\(\beta\)-compact.

**Proof.** Let \(p \in pr(L)\) and \(\{f_i\}_{i \in I}\) be a p-level \((i, j)\)-\(\beta\)-open cover of \((X, \omega(\tau_1), \omega(\tau_2))\). Then \((\bigvee_{i \in I} f_i)(x) \not\in p\) for all \(x \in X\). Hence for each \(x \in X\) there is \(i \in I\) such that \(f_i(x) \not\in p\), that is, \(x \in f_i^{-1}(\{t \in L : t \not\in p\})\).

So, \(X = \bigcup_{i \in I} f_i^{-1}(\{t \in L : t \not\in p\})\). Because \(f_i\) is \((i, j)\)-\(\beta\)-open in \((X, \omega(\tau_1), \omega(\tau_2))\), there is an \((i, j)\)-preopen L-subset \(g_i\) in \((X, \omega(\tau_1), \omega(\tau_2))\) such that \(g_i \subseteq f_i \subseteq \text{Cl}(g_i)\) for every \(i \in I\). Hence by Lemma 2.5, we get
\[
g_i^{-1}(\{t \in L : t \not\in p\}) \subset f_i^{-1}(\{t \in L : t \not\in p\}) \subset \text{Cl}(g_i)^{-1}(\{t \in L : t \not\in p\}) \subset \text{Cl}(g_i^{-1}(\{t \in L : t \not\in p\}))\] (i.e., \(X, \tau_1, \tau_2\). Since \((X, \tau_1, \tau_2)\) is \((i, j)\)-\(\beta\)-compact, there is a finite subset \(F\) of \(J\) such that \(X = \bigcup_{i \in F} f_i^{-1}(\{t \in L : t \not\in p\})\), that is, \((\bigvee_{i \in F} f_i)(x) \not\in p\) for all \(x \in X\). Hence, \((X, \omega(\tau_1), \omega(\tau_2))\) is \((i, j)\)-\(\beta\)-compact.

Conversely let \(\{A_i\}_{i \in I}\) be an \((i, j)\)-\(\beta\)-open cover of \((X, \tau_1, \tau_2)\). Then by Lemma 3.2 \(\chi_{A_i}\) is a family of \((i, j)\)-\(\beta\)-open L-subsets in \((X, \omega(\tau_1), \omega(\tau_2))\) such that \(1 = \bigvee_{i \in F} \chi_{A_i}(x) \not\in p\) for all \(x \in X\) and for all \(p \in pr(L)\), that is, \(\chi_{A_i}\) is a \(p\)-level \((i, j)\)-\(\beta\)-open cover of \((X, \omega(\tau_1), \omega(\tau_2))\). Since \((X, \omega(\tau_1), \omega(\tau_2))\) is \((i, j)\)-\(\beta\)-compact, there is a finite \(F\) of \(J\) such that \((\bigvee_{i \in F} \chi_{A_i})(x) \not\in p\) for all \(x \in X\). Hence \((\bigvee_{i \in F} \chi_{A_i}) = 1\) for all \(x \in X\), that is, \(X = \bigcup_{i \in F} A_i\) and therefore \((X, \tau_1, \tau_2)\) is \((i, j)\)-\(\beta\)-compact.\]
Theorem 3.4. Let $(X, \tau_1, \tau_2)$ be an L-bts. Then $g \in L^X$ is $(i, j)\beta$-compact if and only if for every $\alpha \in M(L)$ and every collection $\{h_i\}_{i \in J}$ of $(i, j)\beta$-closed L-subsets with $(\bigwedge_{i \in J} h_i)(x) \geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, there is a finite subset $F$ of $J$ such that $(\bigwedge_{i \in F} h_i)(x) \geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$.

Proof. This follows immediately from Definition 3.1. □

Theorem 3.5. Let $(X, \tau_1, \tau_2)$ be an L-bts. Then $g \in L^X$ is $(i, j)\beta$-compact if and only if for every $p \in pr(L)$ and every collection $\{f_i\}_{i \in J}$ of $(i, j)\beta$-open L-subsets with $(\bigvee_{i \in J} f_i \cup g')(x) \geq p$ for all $x \in X$, there is a finite subset $F$ of $J$ such that $(\bigvee_{i \in F} f_i \cup g')(x) \geq p$ for all $x \in X$.

Proof. Let $p \in pr(L)$ and $\{f_i\}_{i \in J}$ be a collection of $(i, j)\beta$-open L-subsets with $(\bigvee_{i \in J} f_i \cup g')(x) \geq p$ for all $x \in X$. Then $(\bigvee_{i \in J} f_i \cup g')(x) \geq p$ for all $x \in X$ with $g(x) \geq p'$. Since $g$ is $(i, j)\beta$-compact, there is a finite subset $F$ of $J$ such that $(\bigvee_{i \in F} f_i)(x) \geq p$ for all $x \in X$ with $g(x) \geq p'$. Take an arbitrary $x \in X$. If $g'(x) \leq p$, then $g'(x) \vee (\bigvee_{i \in F} f_i)(x) = (\bigvee_{i \in F} f_i \cup g')(x) \geq p$ because $(\bigvee_{i \in F} f_i)(x) \geq p$. If $g'(x) \geq p$, then we have $g'(x) \vee (\bigvee_{i \in F} f_i)(x) = (\bigvee_{i \in F} f_i \cup g')(x) \geq p$. Thus, we have $(\bigvee_{i \in F} f_i \cup g')(x) \geq p$ for all $x \in X$.

Conversely, let $p \in pr(L)$ and $\{f_i\}_{i \in J}$ be a p-level $(i, j)\beta$-open cover of $g$. Then $(\bigvee_{i \in J} f_i)(x) \geq p$ for all $x \in X$ with $g(x) \geq p'$. Hence $(\bigvee_{i \in J} f_i \cup g')(x) \geq p$ for all $x \in X$. From the hypothesis, there is a finite subset $F$ of $J$ such that $(\bigvee_{i \in F} f_i \cup g')(x) \geq p$ for all $x \in X$. Then $(\bigvee_{i \in F} f_i)(x) \geq p$ for all $x \in X$ with $g'(x) \leq p$. Thus $g$ is $(i, j)\beta$-compact. □

Definition 3.6. Let $(X, \tau_1, \tau_2)$ be an L-bts, $x_\alpha$ be an L-point in $M(L^X)$ and $S = (S_m)_{m \in D}$ be a net. $x_\alpha$ is called $(i, j)\beta$-cluster point of $S$ if and only if for each $(i, j)\beta$-closed L-subset $f$ with $f(x) \geq \alpha$ and for all $n \in D$, there is $m \in D$ such that $m \geq n$ and $S_m \not\subseteq f$, that is, $h(S_m) \not\subseteq f$ (Supp$S_m$).

Theorem 3.7. Let $(X, \tau_1, \tau_2)$ be an L-bts. Then $g \in L^X$ is $(i, j)\beta$-compact if and only if for every constant $\alpha$-net in $g$, where $\alpha \in M(L)$, has an $(i, j)\beta$-cluster point in $g$ with height $\alpha$.

Proof. Let $\alpha \in M(L)$ and $S = (S_m)_{m \in D}$ be a constant $\alpha$-net in $g$ without any $(i, j)\beta$-cluster point with height $\alpha$ in $g$. Then for each $x \in X$ with $g(x) \geq \alpha$, $x_\alpha$ is not an $(i, j)\beta$-cluster point of $S$, that is, there are $n_x \in D$ and an $(i, j)\beta$-closed L-subset $f_x$ with $f_x(x) \not\geq \alpha$ and $S_m \subseteq f_x$ for each $m \geq n_x$. Let $x^1, \ldots, x^k$ be elements of $X$ with $g(x^i) \geq \alpha$ for each $i \in \{1, \ldots, k\}$. Then there are $n_{x^1}, \ldots, n_{x^k} \in D$ and $(i, j)\beta$-closed L-subset $f_{x_i}$ with $f_x(x^i) \not\geq \alpha$ and $S_m \subseteq f_{x_i}$ for each $m \geq n_{x_i}$ and for each $i \in \{1, \ldots, k\}$. Since $D$ is a directed set, there is $n_0 \in D$ such that $n_0 \geq n_i$ for each $i \in \{1, \ldots, k\}$ and $S_m \subseteq f_{x_i}$ for each $m \geq n_0$ and each $i \in \{1, \ldots, k\}$. Therefore, consider the family $\mu = \{f_x\}_{x \in X}$ with $g(x) \geq \alpha$. Then $(\bigwedge_{i=1}^{n_0} f_{x_i})(y) \not\geq \alpha$ for all $y \in X$ with $g(y) \geq \alpha$ because $f_{x_i}(y) \not\geq \alpha$. We also have that for any finite subfamily $\nu = \{f_{x_1}, \ldots, f_{x_k}\}$ of $\mu$, there is $y \in X$ with $g(y) \geq \alpha$ and $(\bigwedge_{i=1}^{n_0} f_{x_i})(y) \not\geq \alpha$ since $S_m \subseteq f_{x_i}$ for each $m \geq n_0$ because $S_m \subseteq f_{x_i}$ for each $i \in \{1, \ldots, k\}$ and for each $m \geq n_0$. Hence, by Theorem 3.5, $g$ is not $(i, j)\beta$-compact.

Conversely, suppose that $g$ is not $(i, j)\beta$-compact. Then by Theorem 3.5, there exist $\alpha \in M(L)$ and a collection $\mu = \{f_i\}_{i \in J}$ of $(i, j)\beta$-closed L-subsets with $(\bigwedge_{i \in J} f_i)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, but for any finite subfamily $\nu$ of $\mu$ there is $x \in X$ with $g(x) \geq \alpha$ and $(\bigwedge_{i \in \nu} f_i)(x) \not\geq \alpha$. Consider the family of all finite subsets of $\mu$, $2(\mu)$, with the order $v_1 \leq v_2$ if and only if $v_1 \subseteq v_2$. Then $2(\mu)$ is a directed set. So, writing $x_\alpha$ as $S_\alpha$ for every $\nu \in 2(\mu)$, $(X_{\nu})_{\nu \in 2(\mu)}(x_\alpha)$ is a constant $\alpha$-net in $g$ because the height of $S_\alpha$ for all $\nu \in 2(\mu)$ is $\alpha$ and $S_\alpha \subseteq g$ for all $\nu \in 2(\mu)$, that is, $g(x) \geq \alpha$. $(S_\nu)_{\nu \in 2(\mu)}$ also satisfies the condition that for each $(i, j)\beta$-closed L-subset $f_i$ in $\nu$ we have $x_\alpha = S_\alpha \subseteq f_i$. Let $y \in X$ with $g(y) \geq \alpha$. Then $(\bigwedge_{i \in J} f_i)(y) \not\geq \alpha$, that is, there
exists \( j \in J \) with \( f_j(y) \geq \alpha \). Let \( \nu_0 = \{ f_j \} \). So, for any \( v \geq \nu_0, S_v \leq \bigwedge_{f_j \in v} f_j \leq \bigwedge_{f_j \in \nu_0} f_j = f_j \). Thus, we get an \((i,j)\)-\( \beta \)-closed \( L \)-subset \( f_j \) with \( f_j(y) \geq \alpha \) and \( \nu_0 \in 2^J \) such that for any \( v \geq \nu_0, S_v \leq f_j \). That means that \( y_\alpha \in M(L^X) \) is not an 
(i,j)-\( \beta \)-cluster point \((X_\alpha\),\( \nu_2 \)) for all \( y \in X \) with \( g(y) \geq \alpha \). Hence, the constant \( \alpha \)-net \((S_\alpha),\nu_2\) has no 
(i,j)-\( \beta \)-cluster point in \( g \) with height \( \alpha \).

\[ \square \]

**Corollary 3.8.** An \( L \)-bts \((X, \tau_1, \tau_2)\) is \((i,j)\)-\( \beta \)-compact if and only if every constant \( \alpha \)-net in \((X, \tau_1, \tau_2)\) has an \((i,j)\)-\( \beta \)-cluster point with height \( \alpha \), where \( \alpha \in M(L) \).

**Definition 3.9.** Let \((X, \tau_1, \tau_2)\) be an \( L \)-bts and \( \eta \) an \( \alpha \)-level filter base, where \( \alpha \in M(L) \). An \( L \)-point \( x_r \in M(L^X) \) is called an \((i,j)\)-\( \beta \)-cluster point of \( \eta \) if \( \bigwedge_{f \in \eta} (i,j)\)-\( \beta \)Cl(f)(x) \( \geq r \).

**Theorem 3.10.** Let \((X, \tau_1, \tau_2)\) be an \( L \)-bts. Then \( g \in L^X \) is \((i,j)\)-\( \beta \)-compact if and only if every \( \alpha \)-filter base in \( g \), where \( \alpha \in M(L) \), has an \((i,j)\)-\( \beta \)-cluster point \( x_\alpha \) in \( g \) with height \( \alpha \).

**Proof.** Assume that \( \eta \) is an \( \alpha \)-level filter base in \( g \) with no \((i,j)\)-\( \beta \)-cluster point in \( g \) with height \( \alpha \), where \( \alpha \in M(L) \). Then for each \( x \in X \) with \( g(x) \geq \alpha \), \( x \alpha \) is not an \((i,j)\)-\( \beta \)-cluster point of \( \eta \), that is, there is \( f_\alpha \in \eta \) with \((i,j)\)-\( \beta \)Cl(f)(x) \( \geq \alpha \). Hence \((i,j)\)-\( \beta \)Cl(f_\alpha)(x) \( \leq \alpha' = p \in pr(L) \). This means that the collection \( \{ (i,j)\)-\( \beta \)Cl(f_\alpha(x) \} \in X \) with \( g(x) \geq \alpha \) is a \( \beta \)-level \((i,j)\)-\( \beta \)-open cover of \( g \). Since \( g \) is \((i,j)\)-\( \beta \)-compact, there are \((i,j)\)-\( \beta \)Cl(f_\alpha(\ldots),\ldots, (i,j)\)-\( \beta \)Cl(f_k(x)) such that \( \bigwedge_{i=1}^n (i,j)\)-\( \beta \)Cl(f_\alpha(x) \} \\leq p \) for all \( x \in X \) with \( g(x) \geq p = \alpha \). Hence \( \bigwedge_{i=1}^n (i,j)\)-\( \beta \)Cl(f_\alpha(x) \} \\geq \alpha \) for all \( x \in X \) with \( g(x) \geq \alpha \) which implies that \( \bigwedge_{i=1}^n f_\alpha(x) \} \\geq \alpha \) for all \( x \in X \) with \( g(x) \geq \alpha \). This is a contradiction.

Conversely, suppose that \( g \) is not \((i,j)\)-\( \beta \)-compact. Then there is a \( \beta \)-level \((i,j)\)-\( \beta \)-open cover \( \mu \) of \( g \) with no finite \( \beta \)-level subcover, where \( p \in pr(L) \). Hence for each finite subcollection \( \{ h_1, \ldots, h_n \} \) of \( \mu \), there exists \( x \in X \) with \( g(x) \geq p \) such that \( \bigvee_{i=1}^n h_i(x) \geq p \). Thus, \( \eta = \{ h : h \in \mu \} \) forms an \( \alpha \)-level filter base in \( g \). By the hypothesis, \( \mu \) has an \((i,j)\)-\( \beta \)-cluster point \( y_\alpha \in M(L^X) \) in \( g \) with height \( \alpha \), that is, \( g(y) \geq \alpha \) and \( \bigwedge_{h \in \mu} (i,j)\)-\( \beta \)Cl(h)(y) \( \geq \alpha \) for all \( h \in \mu \), which yields a contradiction.

\[ \square \]

**Corollary 3.11.** An \( L \)-bts \((X, \tau_1, \tau_2)\) is \((i,j)\)-\( \beta \)-compact if and only if every \( \alpha \)-filter base has an \( (i,j)\)-\( \beta \)-cluster point with height \( \alpha \), where \( \alpha \in M(L) \).

**Theorem 3.12.** Let \((X, \tau_1, \tau_2)\) be an \( L \)-bts and \( g, h \in L^X \). If \( g \) and \( h \) are \((i,j)\)-\( \beta \)-compact, then \( g \lor h \) is \((i,j)\)-\( \beta \)-compact.

**Proof.** Let \( p \in pr(L) \) and \( \{ f_i \}_{i \in J} \) be a collection of \((i,j)\)-\( \beta \)-open \( L \)-subsets with \( \bigvee_{i \in J} f_i(x) \) \( \leq p \) for all \( x \in X \) with \( (g \lor h)(x) \geq p \) \. Since \( p \) is prime, we have \( (g \lor h)(x) \geq p \) if and only if \( g(x) \geq p \) or \( h(x) \geq p \). So, by the \((i,j)\)-\( \beta \)-compactness of \( g \) and \( h \), there are finite subsets \( E, F \) of \( J \) such that \( \bigvee_{i \in E} f_i(x) \) \( \leq p \) for all \( x \in X \) with \( g(x) \geq p \) and \( \bigvee_{i \in F} f_i(x) \) \( \leq p \) for all \( x \in X \) with \( h(x) \geq p \). Then \( \bigvee_{i \in E \cup F} f_i(x) \) \( \leq p \) for all \( x \in X \) with \( (g \lor h)(x) \geq p \). Thus, \( g \lor h \) is \((i,j)\)-\( \beta \)-compact.

\[ \square \]

**Theorem 3.13.** Let \((X, \tau_1, \tau_2)\) be an \( L \)-bts and \( g, h \in L^X \). If \( f \) is \((i,j)\)-\( \beta \)-compact and \( h \) is \((i,j)\)-\( \beta \)-closed, then \( g \land h \) is \((i,j)\)-\( \beta \)-compact.

**Proof.** Let \( p \in pr(L) \) and \( \{ f_i \}_{i \in J} \) be a collection of \((i,j)\)-\( \beta \)-open \( L \)-subsets with \( \bigvee_{i \in J} f_i(x) \) \( \leq p \) for all \( x \in X \) with \( g(x) \geq p \). Thus \( \mu = \{ f_i \}_{i \in J} \cup \{ h' \} \) is a family of \((i,j)\)-\( \beta \)-open \( L \)-subsets with \( \bigvee_{k \in \mu} k(x) \) \( \leq p \) for all \( x \in X \) with \( g(x) \geq p \). In fact, for each \( x \in X \) with \( g(x) \geq p \), if \( h(x) \geq p \), then \( (g \land h)(x) \geq p \) which implies that \( \bigvee_{i \in J} f_i(x) \) \( \leq p \), thus \( \bigvee_{k \in \mu} k(x) \) \( \leq p \). If \( h(x) \not\geq p \), then \( h'(x) \) \( \leq p \) which implies \( \bigvee_{k \in \mu} k(x) \) \( \leq p \). From the \((i,j)\)-\( \beta \)-compactness of \( g \) there is a
Properties of \((i,j)\)-\(\beta\)-compact Spaces

Let \(\mu = \{f_1, \ldots, f_n, h'\}\) with \((\bigvee_{k \in \upsilon} f_k)(x) \not\subseteq p\) for all \(x \in X\) with \((g \wedge h)(x) \geq p'\). Then \((\bigvee_{i=1}^{n} f_i)(x) \not\subseteq p\) for all \(x \in X\) with \((g \wedge h)(x) \geq p'\). Hence \(g \wedge h\) is \((i,j)\)-\(\beta\)-compact.

Corollary 3.14. Let \((X, \tau_1, \tau_2)\) be an \((i,j)\)-\(\beta\)-compact space and \(g\) be an \((i,j)\)-\(\beta\)-closed \(L\)-subset. Then \(g\) is \((i,j)\)-\(\beta\)-compact.

Theorem 3.15. Let \((X, \tau_1, \tau_2)\) be an \(L\)-bts where \(X\) is a finite set. Then \((X, \tau_1, \tau_2)\) is \((i,j)\)-\(\beta\)-compact.

Proof. Let \(\{f_i\}_{i \in J}\) be a \(p\)-level \((i,j)\)-\(\beta\)-open cover of \((X, \tau_1, \tau_2)\), where \(p \in \text{pr}(L)\). Then \((\bigvee_{i \in J} f_i)(x) \not\subseteq p\) for all \(x \in X\). Hence, for each \(x \in X\) there is \(i \in J\) such that \(x \in f_i^{-1}(\{t \in T : t \not\subseteq p\})\). Since \(X\) is finite subset \(F\) of \(J\) such that \(X = \bigcup_{i \in F} f_i^{-1}(\{t \in T : t \not\subseteq p\})\), that is, \((\bigvee_{i \in F} f_i)(x) \not\subseteq p\) for each \(x \in X\). Hence \((X, \tau_1, \tau_2)\) is \((i,j)\)-\(\beta\)-compact.

References