Generalized Doubt Fuzzy Structure of \( BG \)-algebra

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Abstract: In this paper, we introduced the concept of generalized \((\in, \in \lor \notin qk)\)-doubt fuzzy subalgebra and generalized \((\in, \in \lor \notin qk)\)-doubt fuzzy ideal in \( BG \)-algebra by using the combined notion of not quasi coincidence \((\notin q)\) of a fuzzy point to a fuzzy set and the notion doubt fuzzy ideals in \( BCK/BCI \)-algebras. Some characterizations of these generalized doubt fuzzy subalgebra and generalized doubt fuzzy ideal in \( BG \)-algebra are derived. We investigated characterizations of \((\in, \in \lor \notin qk)\)-doubt fuzzy subalgebra and \((\in, \in \lor \notin qk)\)-doubt fuzzy ideals by using level sets and \((\in \lor \notin qk)\)-level sets.

MSC: 06F35, 03E72, 03G25.

Keywords: \( BG \)-algebra, Fuzzy ideal, Doubt fuzzy ideal, \((\in, \in \lor \notin qk)\)-doubt fuzzy subalgebra, \((\in, \in \lor \notin qk)\)-doubt fuzzy ideal.

1. Introduction

The concept of fuzzy sets was first proposed by Zadeh ([23]) in 1965. Rosenfeld ([18]) was the first who consider the case of a groupoid in terms of fuzzy sets. Since then these ideas have been applied to other algebraic structures such as group, semigroup, ring, field, topology, vector spaces etc. Imai and Iseki ([9]) introduced \( BCK \)-algebra as a generalization of notion of the concept of set theoretic difference and propositional calculus and in the same year Iseki ([11]) introduced the notion of \( BCI \)-algebra which is a generalization of \( BCK \)-algebra. Xi Ougen ([20]) applied the concept of fuzzy set to \( BCK \)-algebra and discussed some properties. Since then \( B \)-algebras was introduced in [17] by Neggers and Kim and which is related to several classes of algebras such as \( BCI \)/\( BCK \)-algebras. In [12] Kim and Kim introduced the notion of \( B \)-algebra which is a generalization of \( B \)-algebra. Fuzzy subalgebras of \( B \)-algebras introduced in [1] by Ahn and Lee and the fuzzification of ideals of \( B \)-algebras were studied in [16] by R. Muthuraj et al. Huang [8] fuzzified \( BCI \)-algebras in little different ways. Jun et al. [7, 22] renamed Huang’s definition as doubt(anti) fuzzy ideals in \( BCK \)/\( BCI \)-algebras. Biswas [6] introduced the concept of anti fuzzy subgroup. The concept of doubt fuzzy BF-algebras was introduced by Saeid in [19] and the concept of doubt fuzzy ideal of BF-algebras was introduced by Barbhuiya [3].

Bhakat and Das [4, 5] used the relation of “belongs to” and “quasi coincident with” between fuzzy point and fuzzy set to introduce the concept of \((\in, \in \lor \notin q)\)-fuzzy subgroup, \((\in, \in \lor \notin q)\)-fuzzy subring and \((\in, \in \lor \notin q)\)-level subset. Jun [21] introduced \((\alpha, \beta)\)-fuzzy ideals of \( BCK \)/\( BCI \)-algebras. In fact, the \((\in, \in \lor \notin q)\)-fuzzy subgroup is an important generalization of Rosenfeld’s

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fuzzy subgroup. Further in [13] Larimi generalized $(\in, \in \lor q)$-fuzzy ideals to $(\in, \in \land q)$-fuzzy ideals. Reza Ameri et al [2] introduced the notion of $(\in, \in \land q)$-fuzzy subalgebras in $BCK/BCI$-algebras. In this paper, we combined the notion of not quasi coincidence $q$ of a fuzzy point to a fuzzy set and the notion doubt (anti) fuzzy ideals in $BCK/BCI$-algebras, we introduced the concept of generalized doubt fuzzy subalgebra and generalized doubt fuzzy ideal in $BG$-algebra. Some characterizations of these generalized doubt fuzzy subalgebra and generalized doubt fuzzy ideal in $BG$-algebra are derived. We investigated characterizations of $(\in, \in \lor q)$-doubt fuzzy subalgebra and $(\in, \in \land q)$-doubt fuzzy ideals by using level sets and $(\in \lor q)$-level sets.

2. Preliminaries

Definition 2.1 ([12]). A $BG$-algebra is a non-empty set $X$ with a constant $0$ and a binary operation $\ast$ satisfying the following axioms:

1. $x \ast x = 0$
2. $x \ast 0 = x$
3. $(x \ast y) \ast (0 \ast y) = x$ for all $x, y \in X$.

For brevity we also call $X$ a $BG$-algebra. A non empty subset $S$ of $BG$-algebra $X$ is said to be a subalgebra of $X$ if $x \ast y \in S$, $\forall x, y \in X$. A nonempty subset $I$ of a $BG$-algebra $X$ is called an ideal of $X$ if $(I_1)$ $0 \in I$ and $(I_2)$ $x \ast y \in I, y \in I \Rightarrow x \in I$ for all $x, y \in X$. A fuzzy subset $\mu$ of $X$ is called a doubt fuzzy ideal [22] of $X$ if it satisfies the following conditions:

$DF_1$ $\mu(0) \leq \mu(x)$ and $DF_2$ $\mu(x) \leq \max\{\mu(x \ast y), \mu(y)\}$ $\forall x, y \in X$.

Definition 2.2 ([4, 14]). A fuzzy set $\mu$ of the form

$$\mu(y) = \begin{cases} t, & \text{if } y = x, \ t \in (0, 1]; \\ 0, & \text{if } y \neq x. \end{cases}$$

is called a fuzzy point with support $x$ and value $t$ and it is denoted by $x_t$ [4, 14]. Let $\mu$ be a fuzzy set in $X$ and $x_t$ be a fuzzy point then

1. If $\mu(x) \geq t$ then we say $x_t$ belongs to $\mu$ and write $x_t \in \mu$.
2. If $\mu(x) + t > 1$ then we say $x_t$ quasi-coincident with $\mu$ and write $x_t \in q \mu$.
3. If $x_t \in \in \lor q \mu$ $\Leftrightarrow x_t \in \mu$ or $x_t \in q \mu$.
4. If $x_t \in \in \land q \mu$ $\Leftrightarrow x_t \in \mu$ and $x_t \in q \mu$.

The symbol $x_t \in q \mu$ means $x_t \in q \mu$ does not hold and $\in \land q$ means $\in \lor q$. For a fuzzy point $x_t$ and a fuzzy set $\mu$ in set $X$, Pu and Liu ([14]) gave meaning to the symbol $x_t \in q \mu$ where $\alpha \in \{\in, q, \in \lor q, \in \land q\}$.

Definition 2.3 ([2, 13]). Let $\mu$ be a fuzzy set in $X$ and $x_t$ be a fuzzy point then

1. If $\mu(x) < t$ then we say $x_t$ does not belongs to $\mu$ and write $x_t \in \mu$.
2. If $\mu(x) + t \leq 1$ then we say $x_t$ not quasi-coincident with $\mu$ and write $x_t \not\in q \mu$.
3. If $x_t \not\in \in \lor q \mu$ $\Leftrightarrow x_t \not\in \mu$ and $x_t \not\in q \mu$. 

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Remark 3.2. A fuzzy subset \( \mu \) of a BG-algebra \( X \) is said to be \((\alpha, \beta)\)-fuzzy ideal of \( X \) if

\[
\alpha, \beta \in [0, 1], \\forall x, y \in X \quad \mu(x \star y) \leq M(\mu(x), \mu(y), 0.5)
\]

Definition 3.3. A fuzzy subset \( \mu \) of a BG-algebra \( X \) is an \((\varepsilon, \in \lor q_{k})\)-doubt fuzzy subalgebra of \( X \) if

\[
\mu(x \star y) \leq \max\{\mu(x), \mu(y), \frac{1-k}{2}\} \quad \text{for all} \quad x, y \in X.
\]

Remark 3.4. A fuzzy subset \( \mu \) of a BG-algebra \( X \) is an \((\varepsilon, \in \lor q)\)-doubt fuzzy subalgebra of \( X \) iff

\[
\mu(x \star y) \leq M(\mu(x), \mu(y), 0.5)
\]

3. Generalized Doubt Fuzzy Structure of BG-algebra

Definition 3.1. A fuzzy subset \( \mu \) of a BG-algebra \( X \) is an \((\in, \in \lor q)\)-doubt fuzzy subalgebra of \( X \) if

\[
\mu(x \star y) \leq \max\{\mu(x), \mu(y), \frac{1-k}{2}\} \quad \text{for all} \quad x, y \in X.
\]

Definition 3.2. A fuzzy subset \( \mu \) of a BG-algebra \( X \) is an \((\varepsilon, \in \lor q_{k})\)-doubt fuzzy subalgebra of \( X \) if

\[
\mu(x \star y) \leq M(\mu(x), \mu(y), 0.5)
\]

Theorem 3.5. A fuzzy subset \( \mu \) of a BG-algebra \( X \) is an \((\varepsilon, \in \lor q_{k})\)-doubt fuzzy ideal of \( X \) iff

\[
\forall x \in X, \quad x_{\mu} \Rightarrow 0_{\mu} \lor q_{k} \mu
\]

where \( M\{t, s\} = \max\{t, s\} \) and \( t, s \in (0, 1] \).
**Proof.** First let \( \mu \) be an \( (\varepsilon, \in \vee q_k) \)-doubt fuzzy ideal of \( X \). To prove conditions (1) and (2). Since \( \mu \) is an \( (\varepsilon, \in \wedge q_k) \)-doubt fuzzy ideal of \( X \).

\[
\begin{align*}
\mu(0) &\leq M\{\mu(x), \frac{1-k}{2}\} \\
\mu(x) &\leq M\{\mu(x \ast y), \mu(y), \frac{1-k}{2}\} \quad \text{for all } x, y \in X.
\end{align*}
\]

Let \( x \in X \) and \( t \in [0, 1] \) such that \( x \in \mu \) i.e., \( \mu(x) < t \). Now

\[
(1) \implies \mu(0) \leq M\{\mu(x), \frac{1-k}{2}\} \leq M\{t, \frac{1-k}{2}\} = \begin{cases} t & \text{if } t > \frac{1-k}{2} \\
\frac{1-k}{2} & \text{if } t \leq \frac{1-k}{2} \end{cases}
\]

\[
\implies \mu(0) < t \quad \text{or} \quad \mu(0) < \frac{1-k}{2}
\]

\[
\implies \mu(0) < t \quad \text{or} \quad \mu(0) + t < \frac{1-k}{2} + \frac{1-k}{2} = 1-k
\]

\[
\implies \mu(0) < t + k < 1
\]

\[
\implies x q_k \mu \quad \text{or} \quad 0 \in \wedge q_k \mu
\]

Therefore \( x _ q_k \mu \implies 0 \in \wedge q_k \mu \) which proves (1). Again let \( x, y \in X \) such that \( (x \ast y) q_k \mu \) and \( y q_k \mu \) where \( t, s \in (0, 1] \) i.e., \( \mu(x \ast y) < t \) and \( \mu(y) < s \).

\[
(2) \implies \mu(x) \leq M\{\mu(x \ast y), \mu(y), \frac{1-k}{2}\} \leq M\{t, s, \frac{1-k}{2}\} = \begin{cases} M(t, s) & \text{if } M(t, s) > \frac{1-k}{2} \\
\frac{1-k}{2} & \text{if } M(t, s) \leq \frac{1-k}{2} \end{cases}
\]

\[
\implies \mu(x) < M(t, s) \quad \text{or} \quad \mu(x) < \frac{1-k}{2}
\]

\[
\implies \mu(x) < M(t, s) \quad \text{or} \quad \mu(x) + M(t, s) < \frac{1-k}{2} + \frac{1-k}{2} = 1-k
\]

\[
\implies \mu(x) < M(t, s) + k < 1
\]

\[
\implies x q_{M(t,s)} \mu \quad \text{or} \quad x q_{M(t,s)} \mu
\]

Therefore \( (x \ast y) q_k \mu \) which proves (2).

Conversely, Suppose \( \mu \) satisfies conditions (1) and (2). To prove \( \mu \) is an \( (\varepsilon, \in \vee q_k) \)-doubt fuzzy ideal of \( X \). If possible \( \mu \) is not an \( (\varepsilon, \in \vee q_k) \)-doubt fuzzy ideal of \( X \). The at least one of \( \mu(0) > M\{\mu(x), \frac{1-k}{2}\} \) or \( \mu(x) > M\{\mu(x \ast y), \mu(y), \frac{1-k}{2}\} \) must hold for some \( x, y \in X \). Suppose \( \mu(0) > M\{\mu(x), \frac{1-k}{2}\} \) holds. Choose a real number \( t \) such that

\[
\mu(0) > t > M\{\mu(x), \frac{1-k}{2}\}
\]

\[
\implies \mu(x) < t \quad \text{or} \quad x q_k \mu \implies 0 \in \wedge q_k \mu \quad \text{[By condition (1)]} \quad \implies 0 \in \mu \quad \text{or} \quad 0 q_k \mu \implies \mu(0) < t \quad \text{or} \quad \mu(0) + t + k \leq 1 \quad \text{first part is not true by (3), therefore we have} \mu(0) + t + k \leq 1 \implies \mu(0) + t \leq 1 - k \implies 1 - k \geq \mu(0) + t \implies t + t = 2t \quad \text{[Since } \mu(0) > t \text{ by (3)]} \implies t \leq \frac{1-k}{2},
\]
which contradicts (3) again. Hence we must have \( \mu(0) \leq M \{ \mu(x), \frac{1-k}{2} \} \). Again if \( \mu(x) > M \{ \mu(x+y), \mu(y), \frac{1-k}{2} \} \) holds for some \( x, y \in X \). Then choose a real number \( t \) such that
\[
\mu(x) > t > M \left\{ \mu(x+y), \mu(y), \frac{1-k}{2} \right\}
\] (4)
\[
\Rightarrow \mu(x+y) < t \quad \text{and} \quad \mu(y) < t \Rightarrow (x+y) \in \mathcal{I} \quad \text{and} \quad (y) \in \mathcal{I} \Rightarrow (x)_{M(1,1)} \in \mathcal{I} \quad \text{[By condition (2)]} \Rightarrow (x)_{\mathcal{I}} \quad \text{or} \quad (x)_{\mathcal{I}} \Rightarrow \mu(x) < t \\
\text{or} \mu(x)+t+k \leq 1 \quad \text{first part is not true by (4), therefore we have} \mu(x)+t \leq 1-k \Rightarrow 1-k \geq \mu(x)+t > t+t = 2t \\
\text{[Since} \mu(x) > t \text{by (4)]} \Rightarrow t \leq \frac{1-k}{2} \text{which contradicts (4). Hence we must have} \mu(x) \leq M \{ \mu(x+y), \mu(y), \frac{1-k}{2} \}. \text{Hence} \mu \text{ is an} (\epsilon, \epsilon) \text{-doubt fuzzy ideal of} X.\]

**Theorem 3.6.** A fuzzy subset \( \mu \) of a BG-algebra \( X \) is an \((\epsilon, \epsilon)\)-doubt fuzzy subalgebra of \( X \) iff
\[
(x, y, \mu_{\mathcal{I}}) \Rightarrow (x+y)_{M(1,1)} \in \mathcal{I} \quad \text{for all} \quad x, y \in X
\]

where \( M(t, s) = \max\{t, s\} \) and \( t, s \in (0, 1] \).

**Theorem 3.7.** A fuzzy subset \( \mu \) of a BG-algebra \( X \) is a doubt fuzzy ideal if and only if \( \mu \) is an \((\epsilon, \epsilon)\)-doubt fuzzy ideal.

**Proof.** Let \( \mu \) be a doubt fuzzy ideal of \( X \), to prove that \( \mu \) is an \((\epsilon, \epsilon)\)-doubt fuzzy ideal. It is enough to show that

(i). \( x_{\mathcal{I}} \Rightarrow 0_{\mathcal{I}} \) for all \( x \in X \)

(ii). \( x+y_{\mathcal{I}} \Rightarrow x_{M(1,1)} \in \mathcal{I} \) for all \( x, y \in X \).

Where \( M(t, s) = \max\{t, s\} \) and \( t, s \in (0, 1] \). Let \( x \in X \), such that \( x_{\mathcal{I}} \Rightarrow 0_{\mathcal{I}} \) where \( t \in (0, 1) \), then \( \mu(x) < t \). Now \( \mu(0) \leq \mu(x) < t \) [Since \( \mu \) is a doubt fuzzy ideal] \( \Rightarrow 0_{\mathcal{I}}. \) Therefore \( x_{\mathcal{I}} \Rightarrow 0_{\mathcal{I}} \). Let \( x, y \in X \), such that \( x+y_{\mathcal{I}} \Rightarrow 0_{\mathcal{I}} \), then \( \mu(x+y) \Rightarrow 0_{\mathcal{I}} \). Now \( \mu(x) \leq \max\{\mu(x+y), \mu(y)\} < \max\{t, s\} = M(t, s) \) [Since \( \mu \) is a doubt fuzzy ideal] \( \Rightarrow x_{M(1,1)} \Rightarrow 0_{\mathcal{I}} \). Hence \( \mu \) is an \((\epsilon, \epsilon)\)-doubt fuzzy ideal of \( X \).

Conversely, let \( \mu \) be an \((\epsilon, \epsilon)\)-doubt fuzzy ideal of \( X \). Let \( x \in X \) and \( \mu(x) = t \), where \( t, s \in [0, 1] \). Then \( \mu(x) < t + \delta \) where \( \delta \) is arbitrary small positive number. Therefore \( (x+y)_{\mathcal{I}} \Rightarrow 0_{\mathcal{I}} \) [Since \( \mu \) is an \((\epsilon, \epsilon)\)-doubt fuzzy ideal of \( X \)] \( \Rightarrow 0_{\mathcal{I}} \Rightarrow 0_{\mathcal{I}} \Rightarrow 0_{\mathcal{I}} \). Again let \( x, y \in X \) and \( \mu(x+y) = t \), where \( t, s \in [0, 1] \) then \( \mu(x+y) < t+\delta, \mu(y) < s + \delta \) \( \text{where} \delta \text{is arbitrary small positive number. Therefore} x+y_{\mathcal{I}} \Rightarrow x_{M(1,1)} \Rightarrow 0_{\mathcal{I}} \) [Since \( \mu \) is an \((\epsilon, \epsilon)\)-doubt fuzzy ideal of \( X \)] \( \Rightarrow 0_{\mathcal{I}} \Rightarrow 0_{\mathcal{I}} \Rightarrow 0_{\mathcal{I}} \). Hence \( \mu \) is a doubt fuzzy ideal of \( X \).

**Theorem 3.8.** A fuzzy subset \( \mu \) of a BG-algebra \( X \) is a doubt fuzzy subalgebra if and only if \( \mu \) is an \((\epsilon, \epsilon)\)-doubt fuzzy subalgebra.

**Theorem 3.9.** If \( \mu \) is a \((q, q)\)-doubt fuzzy ideal of a BG-algebra \( X \), then it is also an \((\epsilon, \epsilon)\)-doubt fuzzy ideal of \( X \).

**Proof.** Let \( \mu \) be a \((q, q)\)-doubt fuzzy ideal of a BG-algebra \( X \). Let \( x, y \in X \) such that \( (x+y)_{\mathcal{I}} \Rightarrow (x+y)_{\mathcal{I}} \Rightarrow (x+y)_{\mathcal{I}} \). Then \( \mu(x+y) < t \) and \( \mu(y) < s \) \( \Rightarrow \mu(x+y) < t+1 \) and \( \mu(y) < s+1 \) \( \Rightarrow \mu(x+y) = \mu(y) < t+1 \) and \( \mu(y) < s+1 \) \( \Rightarrow (x+y)_{\mathcal{I}} \Rightarrow (x+y)_{\mathcal{I}} \). Since \( \mu \) is a \((q, q)\)-doubt fuzzy ideal \( X \). Therefore we have \( x_{M(1,1)} \Rightarrow x_{M(1,1)} \Rightarrow x_{M(1,1)} \Rightarrow x_{M(1,1)} \Rightarrow x_{M(1,1)} \Rightarrow x_{M(1,1)} \Rightarrow x_{M(1,1)} \Rightarrow x_{M(1,1)} \). Hence \( \mu \) is an \((\epsilon, \epsilon)\)-doubt fuzzy ideal of \( X \).

**Remark 3.10.** Converse of Theorem 3.9 is not true as seen from the following example.
**Example 3.11.** Consider $BG$-algebra $X = \{0, 1, 2, 3\}$ with the following Cayley table.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a map $\mu : X \to [0, 1]$ by $\mu(0) = \mu(1) = 0.37, \mu(2) = \mu(3) = 0.46$. Then it is easy to verify that $\mu$ is an $(\varepsilon, \varepsilon)$-doubt fuzzy ideal $X$, but not an $(q, q)$-doubt fuzzy ideal of $X$ because if $x = 2, y = 1, t = 0.4, s = 0.6$ then $(x * y)_\mu, y\mu, \mu$ but $\mu(x) + M(t, s) = \mu(2) + M(0.4, 0.6) = 0.46 + 0.6 = 1.06 > 1$.

**Theorem 3.12.** If $\mu$ is a $(q, q)$-doubt fuzzy subalgebra of a $BG$-algebra $X$, then it is also an $(\varepsilon, \varepsilon)$-doubt fuzzy subalgebra of $X$.

**Theorem 3.13.** Let $\mu$ be an $(\varepsilon, \varepsilon \lor q_k)$-doubt fuzzy ideal of $X$.

1. $\mu(0) > \frac{1-k}{2}$ for some $x \in X$, then $\mu$ is also an $(\varepsilon, \varepsilon)$-doubt fuzzy ideal of $X$.

2. $\mu(x) \leq \frac{1-k}{2}$ for some $x \in X$, then $\mu(0) \leq \frac{1-k}{2}$.

**Proof.**

1. Let $\mu$ be an $(\varepsilon, \varepsilon \lor q_k)$-doubt fuzzy ideal of $X$ and $\mu(x) > \frac{1-k}{2}$ for some $x \in X$. Let $x_i \varepsilon \mu \Rightarrow \mu(x) < t$. Therefore $\frac{1-k}{2} < \mu(x) < t$ also $\mu(0) > \frac{1-k}{2}$. Therefore $\mu(0) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1-k \Rightarrow \mu(0) + t + k > 1$ that is $0_q_k \mu$. Since $\mu$ is an $(\varepsilon, \varepsilon \lor q_k)$-doubt fuzzy ideal so we must have $0_q \mu$. Hence $x_i \varepsilon \mu \Rightarrow 0_q \mu$.

Again let $(x * y)_\mu, y\mu, \mu \Rightarrow \mu(x * y) < t$ and $\mu(y) < s$ Therefore $\frac{1-k}{2} < \mu(x * y) < t$ and $\frac{1-k}{2} < \mu(y) < s \Rightarrow M(t, s_k) > \frac{1-k}{2}$. Also $\mu(x) > \frac{1-k}{2}$. Therefore $\mu(x) + M(t, s_k) > \frac{1-k}{2} + \frac{1-k}{2} = 1-k \Rightarrow \mu(x) + M(t, s_k) + k > 1 \Rightarrow x_{M(t,s)}q_k \mu$. Since $\mu$ is an $(\varepsilon, \varepsilon \lor q_k)$-doubt fuzzy ideal so we must have $0_q \mu$. Hence $(x * y)_\mu, y\mu, \mu \Rightarrow x_{M(t,s)}q_k \mu$. Therefore $\mu$ is an $(\varepsilon, \varepsilon)$-doubt fuzzy ideal of $X$.

2. Let $\mu$ be an $(\varepsilon, \varepsilon \lor q_k)$-doubt fuzzy ideal of $X$ and $\mu(x) \leq \frac{1-k}{2}$ for some $x \in X$. Now $\mu(0) \leq M(\mu(x), \frac{1-k}{2}) = M\left(\frac{1-k}{2}, \frac{1-k}{2}\right) = \frac{1-k}{2}$.

**Corollary 3.14.** Let $\mu$ be an $(\varepsilon, \varepsilon \lor q)$-doubt fuzzy ideal of $X$.

1. $\mu(0) > 0.5$ for some $x \in X$, then $\mu$ is also an $(\varepsilon, \varepsilon)$-doubt fuzzy ideal of $X$.

2. $\mu(x) \leq 0.5$ for some $x \in X$, then $\mu(0) \leq 0.5$.

**Theorem 3.15.** Let $\mu$ be an $(\varepsilon, \varepsilon \lor q_k)$-doubt fuzzy subalgebra of $X$.

1. $\mu(0) > \frac{1-k}{2}$ for some $x \in X$, then $\mu$ is also an $(\varepsilon, \varepsilon)$-doubt fuzzy subalgebra of $X$.

2. $\mu(x) \leq \frac{1-k}{2}$ for some $x \in X$, then $\mu(0) \leq \frac{1-k}{2}$.

**Proof.**

1. Same as Theorem 3.13 (1).

2. Since $\mu(0) = \mu(x * x) \leq M\left(\mu(x), \mu(x), \frac{1-k}{2}\right)$ for some $x \in X$.

**Theorem 3.16.** A fuzzy set $\mu$ in $X$ is an $(\varepsilon, \varepsilon \lor q_k)$-doubt fuzzy ideal of $X$ if and only if the set $\overline{\mu} = \{x \in X | \mu(x) < t \}$ is an ideal of $X$ for all $t \in \left(\frac{1-k}{2}, 1\right]$.
Proof. Assume that $\mu$ be an $(\in,\in\lor qk)$-doubt fuzzy ideal of $X$. Let $t \in (\frac{1-k}{2}, 1]$ and $x \in \mathcal{P}_t$, therefore $\mu(x) < t$. It follows that

$$\mu(0) \leq M\left\{ \mu(x), \frac{1-k}{2} \right\} < M\left\{ t, \frac{1-k}{2} \right\} = t$$

Therefore $\mu(0) < t \Rightarrow 0 \in \mathcal{P}_t$, that is $x \in \mathcal{P}_t \Rightarrow 0 \in \mathcal{P}_t$. Again let $x \ast y, y \in \mathcal{P}_t$. Therefore $\mu(x \ast y) < t$ and $\mu(y) < t$. It follows that

$$\mu(x) \leq M\left\{ \mu(x \ast y), \mu(y), \frac{1-k}{2} \right\} < M\left\{ t, \frac{1-k}{2} \right\} = t$$

Which implies $x \in \mathcal{P}_t$. Therefore $x \ast y, y \in \mathcal{P}_t \Rightarrow y \in \mathcal{P}_t$. Hence $\mathcal{P}_t$ is an ideal of $X$.

Conversely, suppose that $\mathcal{P}_t$ is an ideal of $X$ for all $t \in (\frac{1-k}{2}, 1]$ and let

$$\mu(0) \leq M\left\{ \mu(x), \frac{1-k}{2} \right\}$$

is not valid, then there exists some $a \in X$ such that

$$\mu(0) > M\left\{ \mu(a), \frac{1-k}{2} \right\}$$

Hence we can take $t \in (\frac{1-k}{2}, 1]$ such that

$$\mu(0) \geq t > M\left\{ \mu(a), \frac{1-k}{2} \right\}$$

Which shows that $0 \notin \mathcal{P}_t$ which is a contradiction. Since $\mathcal{P}_t$ is an ideal of $X$, Therefore we must have

$$\mu(0) \leq M\left\{ \mu(x), \frac{1-k}{2} \right\}$$

Again let

$$\mu(x) \leq M\left\{ \mu(x \ast y), \mu(y), \frac{1-k}{2} \right\}$$

is not valid, then there exists some $a, b \in X$ such that

$$\mu(a) > M\left\{ \mu(a \ast b), \mu(b), \frac{1-k}{2} \right\}$$

hence we can take $t \in (\frac{1-k}{2}, 1]$ such that

$$\mu(a) \geq t > M\left\{ \mu(a \ast b), \mu(b), \frac{1-k}{2} \right\}$$

Which implies $a \ast b, b \in \mathcal{P}_t$. Since $\mathcal{P}_t$ is an ideal of $X$, it follows that $a \in \mathcal{P}_t$, so that $\mu(a) < t$. This is again a contradiction, therefore

$$\mu(x) \leq M\left\{ \mu(x \ast y), \mu(y), \frac{1-k}{2} \right\}$$

is valid. Consequently $\mu$ is an $(\in, \in\lor qk)$-doubt fuzzy ideal of $X$.

Corollary 3.17. A fuzzy set $\mu$ in $X$ is an $(\in, \in\lor qk)$-doubt fuzzy ideal of $X$ if and only if the set $\mathcal{P}_t = \{ x \in X | \mu(x) < t \}$ is an ideal of $X$ for all $t \in (0.5, 1]$.

Theorem 3.18. A fuzzy set $\mu$ in $X$ is an $(\in, \in\lor qk)$-doubt fuzzy subalgebra of $X$ if and only if the set $\mathcal{P}_t = \{ x \in X | \mu(x) < t \}$ is a subalgebra of $X$ for all $t \in (\frac{1-k}{2}, 1]$. 

$\square$
Theorem 3.19. Let A be a non empty subset of a BG-algebra X. Consider the fuzzy set $\mu_A$ in X defined by

$$
\mu_A(x) = \begin{cases} 
0 & \text{if } x \in A \\
1 & \text{otherwise}
\end{cases}
$$

Then A is an ideal of X iff $\mu_A$ is an $(\varepsilon, \in \mathcal{Q}_{k})$-doubt fuzzy ideal of X.

Proof. Let A be an ideal of X, then $(\mu_A)_t = \{ x \in X | \mu_A(x) < t \} \forall t \in (\frac{1-k}{2}, 1] = A$, which is an ideal. Hence by above theorem $\mu_A$ is an $(\varepsilon, \in \mathcal{Q}_{k})$-doubt fuzzy ideal of X.

Conversely, assume that $\mu_A$ is an $(\varepsilon, \in \mathcal{Q}_{k})$-doubt fuzzy ideal of X. Let $x \in A$, then

$$
\mu_A(0) \leq M \left\{ \mu_A(x), \frac{1-k}{2} \right\} = M \left\{ 0, \frac{1-k}{2} \right\} = \frac{1-k}{2} < 1 \forall k \in [0, 1)
$$

Therefore $\mu_A(0) < 1 \Rightarrow \mu_A(0) = 0 \Rightarrow 0 \in A$. Again let $x \ast y, y \in A$, then

$$
\mu_A(x) \leq M \left\{ \mu(x \ast y), \mu(y), \frac{1-k}{2} \right\} = M \left\{ 0, 0, \frac{1-k}{2} \right\} = \frac{1-k}{2} < 1 \forall k \in [0, 1)
$$

Therefore $\mu_A(x) < 1 \Rightarrow \mu_A(x) = 0 \Rightarrow x \in A$. Hence A is an ideal of X.

Theorem 3.20. Let A be a non empty subset of a BG-algebra X. Consider the fuzzy set $\mu_A$ in X defined by

$$
\mu_A(x) = \begin{cases} 
0 & \text{if } x \in A \\
1 & \text{otherwise}
\end{cases}
$$

Then A is a subalgebra of X iff $\mu_A$ is an $(\varepsilon, \in \mathcal{Q}_{k})$-doubt fuzzy subalgebra of X.

Theorem 3.21. Let A be an ideal of X, then for every $t \in (\frac{1-k}{2}, 1]$, there exists an $(\varepsilon, \in \mathcal{Q}_{k})$-doubt fuzzy ideal $\mu$ of X, such that $\overline{\mu} = A$.

Proof. Let $\mu$ be a fuzzy set in X defined by

$$
\mu(x) = \begin{cases} 
0 & \text{if } x \in A \\
t & \text{otherwise}
\end{cases}
$$

for all $x \in X$, where $t \in (\frac{1-k}{2}, 1]$, $(\mu)_t = \{ x \in X | \mu(x) < t \} = A$. Hence $(\mu)_t$ is an ideal. Now if $\mu$ is not an $(\varepsilon, \in \mathcal{Q}_{k})$-doubt fuzzy ideal of X then at least one of condition(1) or condition (2) in Theorem 3.5 may not hold, suppose condition (1) does not holds then there exists some $a \in X$ such that $\mu(0) > M \{ \mu(a), \frac{1-k}{2} \}$ choose $t = \mu(0) + M \{ \mu(a), \frac{1-k}{2} \}/2$ then $\mu(0) > t > M \{ \mu(a), \frac{1-k}{2} \}$. Since A is an ideal of X, therefore $0 \in A$. Hence $\mu(0) < t \forall t \in (0, 1)$ which is a contradiction. Therefore we must have $\mu(0) \leq M \{ \mu(x), \frac{1-k}{2} \}$ for all $x, y \in X$.

Again if condition (2) does not holds then there exists some $a, b \in X$ such that $\mu(a) > M \{ \mu(a \ast b), \mu(b), \frac{1-k}{2} \}$. Choose $t = \mu(a) + M \{ \mu(a \ast b), \mu(b), \frac{1-k}{2} \}/2$ then $\mu(a) > t > M \{ \mu(a \ast b), \mu(b), \frac{1-k}{2} \}$. Hence $\mu(a \ast b) < t, \mu(b) < t$ and so $a \ast b, b \in (\mu)_t = A$. Since A is an ideal of X, therefore $a \in A$ hence $\mu(a) = 0 < t \forall t \in (0, 1)$ which is again a contradiction. Therefore $\mu(x) \leq M \{ \mu(x \ast y), \mu(y), \frac{1-k}{2} \}$ for all $x, y \in X$. Hence $\mu$ is an $(\varepsilon, \in \mathcal{Q}_{k})$-doubt fuzzy ideal of X.

Corollary 3.22. Let A be an ideal of X, then for every $t \in (0.5, 1]$, there exists an $(\varepsilon, \in \mathcal{Q}_{k})$-doubt fuzzy ideal $\mu$ of X, such that $\overline{\mu} = A$. 


Theorem 3.23. Let $A$ be an ideal of $X$, then for every $t \in (\frac{1+k}{2}, 1]$, their exists an $(\varepsilon, \in \vee q_k)$-doubt fuzzy subalgebra $\mu$ of $X$, such that $\overline{\mu t} = A$.

Definition 3.24. Let $\mu$ be a fuzzy set in BG-algebra $X$ and $t \in (0, 1]$, let

$$\mu_t = \{x \in X | x \in \mu \} = \{x \in X | \mu(x) \geq t\}$$

For $\mu > t = \{x \in X | x \in \mu \text{ and } t > \mu(x) > 1\}$

$$[\mu]_t = \{x \in X | x \in \mu \text{ or } \mu(x) \geq t\}$$

$$\overline{\mu} = \{x \in X | x \in \mu \text{ and } \mu(x) \leq t\}$$

$$[\overline{\mu}]_t = \{x \in X | x \in \mu \text{ or } \mu(x) > t\}$$

Here $[\mu]_t$ is called $t$ level set of $\mu$, $\overline{\mu}$ level set of $\mu$ and $[\overline{\mu}]_t$ is called $(\in \vee q_k)$ level set of $\mu$. Clearly $[\mu]_t = \mu \cup \mu_t$ and $[\overline{\mu}]_t = \mu \cup \mu_t$.

Theorem 3.25. Let $\mu$ be a fuzzy set in BG-algebra $X$. Then $\mu$ is an $(\varepsilon, \in \vee q_k)$-doubt fuzzy ideal of $X$ iff $[\mu]_t$ is an ideal of $X$ for all $t \in (0, 1]$. We call $[\mu]_t$ as $(\in \vee q_k)$ level set ideal of $\mu$.

Proof. Assume that $\mu$ is an $(\varepsilon, \in \vee q_k)$-doubt fuzzy ideal of $X$, to prove $[\mu]_t$ is an ideal of $X$. Let $x \in [\mu]_t$ for $t \in (0, 1]$ then $x \in \vee q_k \mu$ then $\mu(x) < t$ or $\mu(x) + t + k \leq 1$. Since $\mu$ is an $(\varepsilon, \in \vee q_k)$-doubt fuzzy ideal of $X$, therefore $\mu(0) \leq M \{\mu(x), \frac{1-k}{2}\}$ for all $x, y \in X$.

Case I: $\mu(x) < t$

$$\mu(0) \leq M \{\mu(0), \frac{1-k}{2}\}$$

$$\leq M \{t, \frac{1-k}{2}\} = \begin{cases} t, & \text{if } t > \frac{1-k}{2} \\ \frac{1-k}{2}, & \text{if } t \leq \frac{1-k}{2}. \end{cases}$$

$$\Rightarrow \mu(0) < t \text{ or } \mu(0) < \frac{1-k}{2}$$

$$\Rightarrow \mu(0) < t \text{ or } \mu(0) + t + k \leq 1$$

$$\Rightarrow \mu(0) < t \text{ or } \mu(0) + t + k < 1$$

$$\Rightarrow x \in \vee q_k \mu \text{ or } 0 \in \vee q_k \mu$$

Therefore $0 \in \vee q_k \mu$ i.e., $0 \in [\mu]_t$.

Case II: $\mu(x) + t + k \leq 1$

$$\mu(0) \leq M \{\mu(0), \frac{1-k}{2}\}$$

$$\leq M \{1-t-k, \frac{1-k}{2}\} = \begin{cases} \frac{1-k}{2}, & \text{if } t > \frac{1-k}{2} \\ 1-t-k, & \text{if } t \leq \frac{1-k}{2}. \end{cases}$$

$$\Rightarrow \mu(0) < \frac{1-k}{2} < t \text{ or } \mu(0) < 1-t-k$$

$$\Rightarrow \mu(0) < t \text{ or } \mu(0) + t + k < 1$$

$$\Rightarrow x \in \vee q_k \mu \text{ or } 0 \in \vee q_k \mu$$
Therefore in both cases $0, \in \mathcal{Q}_{k} \mu$, i.e., $0 \in [\mu_{t}]$. Again let $x \ast y, y \in [\mu_{t}]$ for $t \in (0, 1]$ then $(x \ast y) \in \mathcal{Q}_{k} \mu$ and $(y) \in \mathcal{Q}_{k} \mu$. Then $\mu(x \ast y) < t$ or $\mu(x \ast y) + t \leq 1$ and $\mu(y) < t$ or $\mu(y) + t \leq 1$. Since $\mu$ is an $(\in, \in \cup \ast)$-doubt fuzzy ideal of $X$. Therefore $\mu(x) \leq M\{\mu(x \ast y), \mu(y), \frac{1-k}{2}\}$ for all $x, y \in X$. Therefore we have the following cases:

Case I: Let $\mu(x \ast y) < t$ and $\mu(y) < t$

$$\mu(x) \leq M\{\mu(x \ast y), \mu(y), \frac{1-k}{2}\}$$

$$\leq M\{t, t, \frac{1-k}{2}\} = \begin{cases} 
 t & \text{if } t > \frac{1-k}{2} \\
 \frac{1-k}{2} & \text{if } t \leq \frac{1-k}{2}
\end{cases}$$

$$\Rightarrow \mu(x) < t \text{ or } \mu(x) < \frac{1-k}{2}$$

$$\Rightarrow \mu(x) < t \text{ or } \mu(x) + t < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$$

$$\Rightarrow \mu(x) < t \text{ or } \mu(x) + t + k < 1$$

$$\Rightarrow x_{t} \in \mu \text{ or } x_{t} \in \mu$$

$$\Rightarrow x_{t} \in \mathcal{Q}_{k} \mu$$

Therefore $x \in \mathcal{Q}_{k} \mu$, i.e., $x \in [\mu_{t}]$.

Case II: $\mu(x \ast y) < t$ and $\mu(y) + t + k \leq 1$

$$\mu(x) \leq M\{\mu(x \ast y), \mu(y), \frac{1-k}{2}\}$$

$$\leq M\{t, 1-t-k, \frac{1-k}{2}\} = \begin{cases} 
 t & \text{if } t > \frac{1-k}{2} \\
 1-t-k & \text{if } t \leq \frac{1-k}{2}
\end{cases}$$

$$\Rightarrow \mu(x) < t \text{ or } \mu(x) < 1-t-k$$

$$\Rightarrow \mu(x) < t \text{ or } \mu(x) + t + k < 1$$

$$\Rightarrow x_{t} \in \mu \text{ or } x_{t} \in \mu$$

$$\Rightarrow x_{t} \in \mathcal{Q}_{k} \mu$$

Therefore $x \in \mathcal{Q}_{k} \mu$, i.e., $x \in [\mu_{t}]$.

Case III: $\mu(x \ast y) + t + k \leq 1$ and $\mu(y) < t$. This is similar to Case II

Case IV: $\mu(x \ast y) + t + k \leq 1$ and $\mu(y) + t + k \leq 1$

$$\mu(x) \leq M\{\mu(x \ast y), \mu(y), \frac{1-k}{2}\}$$

$$\leq M\{1-t-k, 1-t-k, \frac{1-k}{2}\} = \begin{cases} 
 \frac{1-k}{2} & \text{if } t > \frac{1-k}{2} \\
 1-t-k & \text{if } t \leq \frac{1-k}{2}
\end{cases}$$

$$\Rightarrow \mu(x) < \frac{1-k}{2} \text{ or } \mu(x) < 1-t-k$$

$$\Rightarrow \mu(x) < t \text{ or } \mu(x) + t + k < 1$$

$$\Rightarrow x_{t} \in \mu \text{ or } x_{t} \in \mu$$

$$\Rightarrow x_{t} \in \mathcal{Q}_{k} \mu$$

Therefore for all cases $x \in \mathcal{Q}_{k} \mu$, i.e., $x \in [\mu_{t}]$. Hence $x \ast y, y \in [\mu_{t}] \Rightarrow x \in [\mu_{t}]$. That is $[\mu_{t}]$ is an ideal of $X$.

Conversely, let $\mu$ be a fuzzy set in $X$ and $t \in (0, 1]$ such that $[\mu_{t}]$ is an ideal of $X$. To prove $\mu$ is an $(\in, \in \cup \ast)$-doubt fuzzy
Hence ideal of $X$. If $\mu$ is not an $(\varepsilon, \in \cup_k)$-doubt fuzzy ideal of $X$, then at least one of the conditions of Definition 3.3 may not be hold, suppose condition (i) is not true, then there exists some $a \in X$ such that $\mu(0) > M\{\mu(a), \frac{1}{t-2}\}$ holds. Choose $t \in (0, 1]$ such that $\mu(0) > t > M\{\mu(a), \frac{1}{t-2}\} \Rightarrow \mu(0) > t \Rightarrow 0 \not\in \overline{\mu}_t \subseteq [\mu]_t$, which is a contradiction since $[\mu]_t$ is an ideal. Thus $\mu(0) \leq M\{\mu(x), \frac{1}{t-2}\}$ for all $x, y \in X$. Again if condition (ii) is not true, there exists some $a, b \in X$ such that $\mu(a) > M\{\mu(a \ast b), \mu(b), \frac{1}{t-2}\}$ holds. Choose $t \in (0, 1]$ such that $\mu(a) > t > M\{\mu(a \ast b), \mu(b), \frac{1}{t-2}\}$ then $a \ast b, b \in [\mu]_t$, which implies $a \not\in [\mu]_t$. Hence $\mu(a) < t$ or $\mu(a \ast b) + t + k \leq 1$ a contradiction. Thus $\mu(x) \leq M\{\mu(x \ast y), \mu(y), \frac{1}{t-2}\}$ for all $x, y \in X$. Hence $\mu$ is an $(\varepsilon, \in \cup_k)$-doubt fuzzy ideal of $X$.

Theorem 3.26. Let $\mu$ be a fuzzy set in $BG$-algebra $X$. Then $\mu$ is an $(\varepsilon, \in \cup_k)$-doubt fuzzy subalgebra of $X$ iff $[\mu]_t^k$ is a subalgebra of $X$ for all $t \in (0, 1]$. We call $[\mu]_t^k$ an $(\overline{\mu} \cup k)$ level subalgebra of $\mu$.

Theorem 3.27. Every an $(\varepsilon, \in)$-doubt fuzzy ideal of $X$ is an $(\varepsilon, \in \cup_k)$-doubt fuzzy ideal of $X$.

Theorem 3.28. Every an $(\varepsilon, \in)$-doubt fuzzy subalgebra of $X$ is an $(\varepsilon, \in \cup_k)$-doubt fuzzy subalgebra of $X$.

Remark 3.29. The converse of above Theorem 3.27 is not true as seen from following example.

Example 3.30. Consider $BG$-algebra $X = \{0, 1, 2, 3\}$ with the following Cayley table.

<table>
<thead>
<tr>
<th>$*$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a map $\mu : X \rightarrow [0, 1]$ by $\mu(0) = \mu(1) = 0.2, \mu(2) = 0.4, \mu(3) = 0.48$ then by Definition 3.3 it is easy to verify that $\mu$ is an $(\varepsilon, \in \cup_{0.5})$-doubt fuzzy ideal $X$. But not an $(\varepsilon, \in)$-doubt fuzzy ideal of $X$ because if $x = 3, y = 1 \mu(x \ast y) = \mu(3 \ast 1) = \mu(2) = 0.4 \Rightarrow (3 \ast 1)_{0.45} = 2_{0.45} \subseteq \mu$, also $1_{0.45} \subseteq \mu$ But $3_{0.45} \not\subseteq \mu$. Therefore $\mu$ is not an $(\varepsilon, \in)$-doubt fuzzy ideal of $X$.

Theorem 3.31. Every doubt fuzzy ideal is an $(\varepsilon, \in \cup_k)$-doubt fuzzy ideal of $X$.

Theorem 3.32. Every doubt subalgebra is a $(\varepsilon, \in \cup_k)$-doubt fuzzy subalgebra of $X$.

Remark 3.33. The converse of above Theorem 3.31 is not true which can be easily seen from Theorem 3.7 and Example 3.30.

Remark 3.34. The converse of above Theorem 3.32 is also not true.

Theorem 3.35. If $\mu$ is an $(\varepsilon, \in \cup_k)$-doubt fuzzy ideal of $X$. Then $\overline{\mu} > 1$ is an ideal of $X$, for all $t \in [0, \frac{1}{t-2}]$.

Proof. Assume $\mu$ is an $(\varepsilon, \in \cup_k)$-doubt fuzzy ideal of $X$, and let $t \in (0, \frac{1}{t-2})$. Let $x \in X$ such that $x \in \overline{\mu} > 1 \Rightarrow x\overline{\mu} \Rightarrow \mu(x) + t + k \leq 1$. Now $\mu(0) \leq M\{\mu(x), \frac{1}{t-2}\}$ [Since $\mu$ is an $(\varepsilon, \in \cup_k)$-doubt fuzzy ideal of $X$] $\leq M\{1 - t - k, \frac{1}{t-2}\}$ [Since $t < \frac{1}{t-2}$] $= 1 - t - k \Rightarrow 0 \not\in \overline{\mu} \subseteq \mu$ for all $t \in (0, \frac{1}{t-2})$. Let $x, y \in X$ such that $x \ast y, y \in \overline{\mu} \Rightarrow (x \ast y) \overline{\mu} \Rightarrow x\overline{\mu} \Rightarrow x \in \overline{\mu}$. Hence $\overline{\mu} > 1$ is an ideal of $X$.

Corollary 3.36. If $\mu$ is an $(\varepsilon, \in \cup_k)$-doubt fuzzy ideal of $X$. Then $\overline{\mu} > 1$ is an ideal of $X$, for all $t \in [0, 0.5)$

Theorem 3.37. If $\mu$ is an $(\varepsilon, \in \cup_k)$-doubt fuzzy subalgebra of $X$. Then $\overline{\mu} > 1$ is an subalgebra of $X$, for all $t \in [0, \frac{1}{t-2})$. 

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Proposition 3.38. If \( k_1, k_2 \in (0, 1] \) such that \( k_1 < k_2 \), then every \((\varepsilon, \in \vee q_{k_1})\)-doubt fuzzy ideal of \( X \) is an \((\varepsilon, \in \vee q_{k_2})\)-doubt fuzzy ideal.

Proof. Here \( k_1, k_2 \in (0, 1] \) such that \( k_1 < k_2 \) and let \( \mu \) be an \((\varepsilon, \in \vee q_{k_1})\)-doubt fuzzy ideal of \( X \). Therefore

\[
\mu(0) \leq M \left\{ \mu(x), \frac{1-k_2}{2} \right\} \quad \text{for all } x, y \in X.
\]

\[
\leq M \left\{ \mu(x), \frac{1-k_1}{2} \right\} \quad \text{[Since } k_1 < k_2 \Rightarrow \frac{1-k_2}{2} \leq \frac{1-k_1}{2} \text{]}
\]

also \( \mu(x) \leq M\{\mu(x * y), \mu(y), \frac{1-k_2}{2} \} \) for all \( x, y \in X \)

\[
\leq M \left\{ \mu(x * y), \mu(y), \frac{1-k_1}{2} \right\}
\]

Hence by Definition 3.3 \( \mu \) is an \((\varepsilon, \in \vee q_{k_1})\)-doubt fuzzy ideal of \( X \).

Proposition 3.39. If \( k_1, k_2 \in (0, 1] \) such that \( k_1 < k_2 \), then every \((\varepsilon, \in \vee q_{k_2})\)-doubt fuzzy subalgebra of \( X \) is an \((\varepsilon, \in \vee q_{k_1})\)-doubt fuzzy ideal.

Remark 3.40. The converse of above Proposition 3.38 is not true as seen from following example.

Example 3.41. Let \( X \) and \( \mu \) as in Example 3.30. If \( k_1 = 0.1, k_2 = 0.4 \), then \( \mu \) is an \((\varepsilon, \in \vee q_{0.1})\)-doubt fuzzy ideal of \( X \) by Definition 3.3, but \( \mu \) is not an \((\varepsilon, \in \vee q_{0.4})\)-doubt fuzzy ideal of \( X \). Since \( (3 * 1)_{0.45} = 2_{0.45} \vee \mu, 1_{0.45} \vee \mu \) but \( 3_{0.45} \vee q_0.4 \mu \).

Corollary 3.42. If \( k_1, k_2 \in (0, 1] \) such that \( k_1 < k_2 \). If \( \mu \) be an \((\varepsilon, \in \vee q_{k_2})\)-doubt fuzzy ideal of \( X \), then \( \mu >_{k_1} \) is an ideal of \( X \) for all \( t \in (0, \frac{1-k_2}{2}) \).

Corollary 3.43. If \( k_1, k_2 \in (0, 1] \) such that \( k_1 < k_2 \). If \( \mu \) be an \((\varepsilon, \in \vee q_{k_2})\)-doubt fuzzy subalgebra of \( X \), then \( \mu >_{k_1} \) is a subalgebra of \( X \) for all \( t \in (0, \frac{1-k_2}{2}) \).

Theorem 3.44. Every \((\varepsilon, q_{k_1})\)-doubt fuzzy ideal of \( X \) is an \((\varepsilon, \in \vee q_{k_2})\)-doubt fuzzy ideal of \( X \).

Remark 3.45. The converse of above Theorem 3.44 is not true as seen from following example.

Example 3.46. Let \( BG \)-algebra as in Example 3.30 if we take \( \mu(0) = \mu(1) = \mu(2) = 0.3, \mu(3) = 0.48, t = 0.5, s = 0.6, k = 0.01 \) then it is easy to verify that \( \mu \) is an \((\varepsilon, \in \vee q_{k_1})\)-doubt fuzzy ideal \( X \). But not an \((\varepsilon, q_{k_2})\)-doubt fuzzy ideal of \( X \). Since \( \mu(3 * 1) = \mu(2) < 0.5 = t \) and \( \mu(1) < 0.6 = s \). But \( \mu(3) + M(t, s) + k = 0.48 + M(0.5, 0.6) + 0.01 = 0.48 + 0.6 + 0.01 = 1.09 > 1 \).

Theorem 3.47. Let \( \lambda \) and \( \mu \) be two \((\varepsilon, \in \vee q_{k_1})\)-doubt fuzzy ideals of \( X \) then \( \lambda \cup \mu \) is an \((\varepsilon, \in \vee q_{k_2})\)-doubt fuzzy ideal of \( X \).

Proof. Here \( \lambda \) and \( \mu \) both are \((\varepsilon, \in \vee q_{k_1})\)-fuzzy ideals of \( X \). Therefore

\[
\lambda(0) \leq M \left\{ \lambda(x), \frac{1-k}{2} \right\}
\]

\[
\mu(0) \leq M \left\{ \mu(x), \frac{1-k}{2} \right\} \quad \text{for all } x \in X
\]

\[
\lambda(x) \leq M \left\{ \lambda(x * y), \lambda(y), \frac{1-k}{2} \right\}
\]

\[
\mu(x) \leq M \left\{ \mu(x * y), \mu(y), \frac{1-k}{2} \right\} \quad \text{for all } x, y \in X
\]

Now, \( (\lambda \cup \mu)(0) = M(\lambda(0), \mu(0)) \)

\[
\leq M \left\{ M \left\{ \lambda(x), \frac{1-k}{2} \right\}, M \left\{ \mu(x), \frac{1-k}{2} \right\} \right\}
\]

\[
= M \left\{ M(\lambda(x), \mu(x)), \frac{1-k}{2} \right\}
\]
Proof. Let $X$ and $X$ be two $BG$-algebras. Then their Cartesian Product $X \times Y = \{(x,y) | x \in X, y \in Y\}$ is also a $BG$-algebra under the binary operation $*$ defined in $X \times Y$ by $(x,y) * (p,q) = (x*p, y*q)$ for all $(x,y), (p,q) \in X \times Y$.

Theorem 3.50. Let $\mu_1$ and $\mu_2$ be two $(\epsilon, \in \mathcal{V}_k)$-doubt fuzzy ideals of a $BG$-algebra $X$. Then their Cartesian product $\mu_1 \otimes \mu_2$ is defined by $(\mu_1 \otimes \mu_2)(x,y) = \max\{\mu_1(x), \mu_2(y), \frac{1}{2}\}$. Where $(\mu_1 \times \mu_2) : X \times X \rightarrow [0,1]$ \hspace{1cm} $\forall x, y \in X$.

Theorem 3.51. Let $\mu_1$ and $\mu_2$ be two $(\epsilon, \in \mathcal{V}_k)$-doubt fuzzy ideals of a $BG$-algebra $X$. Then $\mu_1 \otimes \mu_2$ is also an $(\epsilon, \in \mathcal{V}_k)$-doubt fuzzy ideal of $X \times X$.

Definition 3.52. Let $X$ and $X'$ be two $BG$-algebras. Then a mapping $f : X \rightarrow X'$ is said to be homomorphism if $f(x * y) = f(x) * f(y) \hspace{1cm} \forall x, y \in X$.

Theorem 3.53. Let $X$ and $X'$ be two $BG$-algebras and $f : X \rightarrow X'$ be a homomorphism. Then $f(0) = 0'$.

Proof. Let $x \in X$ therefore $f(x) \in X'$. Now $f(0) = f(x * x) = f(x) * f(x) = 0'$.

Theorem 3.54. Let $X$ and $X'$ be two $BG$-algebras and $f : X \rightarrow X'$ be homomorphism. If $\mu$ be an $(\epsilon, \in \mathcal{V}_k)$-doubt fuzzy ideal of $X'$, then $f^{-1}(\mu)$ is an $(\epsilon, \in \mathcal{V}_k)$-doubt fuzzy ideal of $X$.

Proof. $f^{-1}(\mu)$ is defined as $f^{-1}(\mu)(x) = \{\mu(f(x)) | \forall x \in X$. Let $\mu$ be an $(\epsilon, \in \mathcal{V}_k)$-doubt fuzzy ideal of $X'$. Let $x \in X$ such that $x_{\epsilon} f^{-1}(\mu) \text{ then } f^{-1}(\mu)(x) < t \Rightarrow \mu(f(x)) < t \Rightarrow (f(x))_{\epsilon} f_{\mu} \Rightarrow ((f(0))_{\epsilon} f_{\mu} = \mu(0) + t \leq 1 \Rightarrow f^{-1}(\mu)(0) \leq t \Rightarrow f^{-1}(\mu)(0) < t \Rightarrow f^{-1}(\mu)(0) + t < 1 \Rightarrow 0_{\epsilon} f^{-1}(\mu)$ or $0_{\epsilon} f^{-1}(\mu) \Rightarrow 0_{\epsilon} \leq \mu_{\epsilon} f^{-1}(\mu)$. Therefore $x_{\epsilon} f^{-1}(\mu) = 0_{\epsilon} \leq \mu_{\epsilon} f^{-1}(\mu)$.

Again let $x, y \in X$ such that $(x*y), y_{\epsilon} f^{-1}(\mu)$ then $f^{-1}(\mu)(x * y) < t$ and $f^{-1}(\mu)(y) < s$. $\mu(f(x * y)) < t$ and $\mu(f(y)) < s \Rightarrow (f(x*y))_{\epsilon} f_{\mu} \Rightarrow (f(x))_{\epsilon} f_{\mu} \Rightarrow (f(x) * f(y))_{\epsilon} f_{\mu}$ and $f(x), f(y)$, $\mu$ since $f$ is a homomorphism $\Rightarrow ((f(x))_{\epsilon} f_{\mu})_{\epsilon} \leq f^{-1}(\mu)_{\epsilon}$ $\Rightarrow f^{-1}(\mu)(x) + M(t, s) + k \leq 1 \Rightarrow f^{-1}(\mu)(x) \leq f^{-1}(\mu)(x) = \mu_{\epsilon} f^{-1}(\mu)_{\epsilon} f^{-1}(\mu)_{\epsilon} f^{-1}(\mu)_{\epsilon}$. Therefore $(x * y), y_{\epsilon} f^{-1}(\mu) \Rightarrow x_{\epsilon} f^{-1}(\mu) = \mu_{\epsilon} f^{-1}(\mu)$. Hence $f^{-1}(\mu)$ is an $(\epsilon, \epsilon \mathcal{V}_k)$-doubt fuzzy ideal of $X$.

Theorem 3.55. Let $X$ and $X'$ be two $BG$-algebras and $f : X \rightarrow X'$ be an onto homomorphism. If $\mu$ be a fuzzy subset of $X'$ such that $f^{-1}(\mu)$ is an $(\epsilon, \in \mathcal{V}_k)$-doubt fuzzy ideal of $X$, then $\mu$ is also an $(\epsilon, \in \mathcal{V}_k)$-doubt fuzzy ideal of $X'$. "

Proof. Let $x', y' \in X'$ since $f$ is onto so there exists $x, y \in X$, such that $f(x) = x'$, $f(y) = y'$ also $f$ is homomorphism so $f(x * y) = f(x) * f(y) = x' * y'$ such that $x_{\epsilon} f_{\mu}$ where $t, s \in [0,1]$ then $\mu(x') < t \Rightarrow \mu(f(x)) < t \Rightarrow f^{-1}(\mu)(x) < t \Rightarrow x_{\epsilon} f^{-1}(\mu) = 0_{\epsilon} \leq \mu_{\epsilon} f^{-1}(\mu)$ (Since $f^{-1}(\mu)$ is an $(\epsilon, \in \mathcal{V}_k)$-doubt fuzzy ideal of $X'$) $\Rightarrow f^{-1}(\mu)(0) < t$ or $f^{-1}(\mu)(0) + 103
t + k ≤ 1 \implies \mu(f(0)) < t \text{ or } \mu(f(0)) + t + k ≤ 1 \implies \mu(0') < t \text{ or } \mu(0') + t + k ≤ 1 \implies 0'_{\in \neg \neg \neg \mu} \text{ or } 0'_{\in \neg \neg \neg \mu} \implies 0'_{\in \neg \neg \neg \mu} \text{ or } 0'_{\in \neg \neg \neg \mu}. \text{ Therefore } x'_t \in \mu \implies 0'_{\in \neg \neg \neg \mu}.

Again let \((x' \ast y'), y'_{\in \mu} \text{ where } t, s \in [0] \text{ then } \mu(x' \ast y') < t \text{ and } \mu(y') < s. \text{ Therefore } \mu(f(x \ast y)) < t \text{ and } \mu(f(y)) < s \implies f^{-1}(\mu)(x \ast y) < t \text{ and } f^{-1}(\mu)(y) < s \implies (x \ast y)_{\in \neg \neg \neg \inf}f^{-1}(\mu) \text{ and } (y)_{\in \neg \neg \neg \inf}f^{-1}(\mu) \implies (x)_{\in \neg \neg \neg \inf}f^{-1}(\mu) \in \neg \neg \neg \inf f^{-1}(\mu) \text{ [since } f^{-1}(\mu) \text{ is an } (\in, \leq, \in) \text{-fuzzy ideal of } X. \] \implies f^{-1}(\mu)(x) < M(t, s) \text{ or } f^{-1}(\mu)(y) + M(t, s) + k ≤ 1 \implies \mu(f(x)) \text{ or } \mu(f(y)) \text{ or } (x')_{\in \neg \neg \neg \inf}M(t, s)_{\in \neg \neg \neg \inf} \text{ or } x'_{M(t, s)_{\in \neg \neg \neg \inf}} \text{ or } x'_{M(t, s)_{\in \neg \neg \neg \inf}} \text{ or } x'_{M(t, s)_{\in \neg \neg \neg \inf}} \text{ or } x'_{M(t, s)_{\in \neg \neg \neg \inf}}. \text{ Therefore } (x' \ast y'), y'_{\in \mu} \implies x'_{M(t, s)_{\in \neg \neg \neg \inf}} \text{ or } x'_{M(t, s)_{\in \neg \neg \neg \inf}} \text{ or } x'_{M(t, s)_{\in \neg \neg \neg \inf}}. \text{ Hence } \mu \text{ is an } (\in, \leq, \in) \text{-fuzzy ideal of } X'. \]

References


