Connectedness and Compactness via Semi-Star-Regular Open Sets

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Abstract: In this paper, we introduce new concepts namely, semi*-r-connectedness and semi*-r-compactness using semi*-regular open sets. We investigate their basic properties. We also discuss their relationships with already existing concepts of connectedness and compactness.

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1. Introduction

In 1974, Das [2] defined the concept of semi-connectedness in topological spaces and investigated its properties. Compactness is one of the most important, useful and fundamental concepts in topology. In 1981, Dorsett [5] introduced and studied the concept of semi-compact spaces. Since then, Hanna and Dorsett [6], Ganster [5] investigated the properties of semi-compact spaces. PasunkiliPandian.S [12] introduced semi*-pre-compact spaces and investigated their properties. Robert, A. and Pious Missier, S. recently introduced and studied semi*-connectedness and semi*-compactness [16] in topological spaces. The authors have defined semi*-regular open sets [13] and semi*-regular closed sets [13] and investigated their properties. In this paper, we introduce the concept of semi*-regular connected spaces. We investigate their basic properties. We also discuss their relationship with already existing concepts namely connectedness, semi-connectedness, semi-pre connectedness and semi*-\(\alpha\)-connectedness. Further we define semi*-regular compact spaces and investigate their properties. We also show the relationship of semi*-r-compactness with each of the concepts of compactness, semi-compactness semi*-compactness and semi*-pre compactness.

2. Preliminaries

Throughout this paper X will always denote a topological space. If A is a subset of the space X, \(Cl(A)\) and \(Int(A)\) denote the closure and the interior of A respectively.
Definition 2.1. Let $A$ be a subset of a topological space $(X, \tau)$ be called

1. generalized closed (briefly g-closed) \cite{7} if $\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

2. generalized open (briefly g-open) \cite{7} if $X \setminus A$ is g-closed in $X$.

Definition 2.2. A be a subset of $X$. The generalized closure \cite{4} of $A$ is defined as the intersection of all g-closed sets containing $A$ and is denoted by $\text{Cl}^*(A)$.

Definition 2.3. A subset $A$ of a topological space $(X, \tau)$ is called

1. semi-open \cite{8} (resp. $\alpha$-open \cite{9}, semi $\alpha$-open \cite{10}, semi*-open \cite{17}, semi*-preopen \cite{1}, semi*-regular open \cite{13}, semi*-preopen \cite{12}) if $A \subseteq \text{Cl}(\text{Int}(A))$ (resp. $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$, $A \subseteq \text{Cl}(\text{Int}(\text{Cl}(\text{Int}(A))))$, $A \subseteq \text{Cl}^*(\text{Int}(A))$, $A \subseteq \text{Cl}^*(\text{rInt}(A))$, $A \subseteq \text{Cl}^*(\text{pInt}(A)))$.

2. semi-closed \cite{2} (resp. semi $\alpha$-closed \cite{11}, semipreclosed \cite{1}, semi*-closed \cite{18}, semi*-regular closed \cite{13}, semi*-preclosed \cite{12}) if $\text{Int}(\text{Cl}(A)) \subseteq A$ (resp. $\text{Int}(\text{Cl}(\text{Int}(\text{Cl}(A)))) \subseteq A$, $\text{Int}(\text{Cl}(\text{Int}(A))) \subseteq A$, $\text{Int}^*(\text{Cl}(A)) \subseteq A$, $\text{Int}^*(\text{rCl}(A)) \subseteq A$, $\text{Int}^*(\text{pCl}(A)) \subseteq A$).

Definition 2.4. Let $A$ be a subset of $X$. Then the semi*-regular closure \cite{13} of $A$ is defined as the intersection of all semi*-regular closed sets in $X$ containing $A$ and is denoted by $s^*r\text{Cl}(A)$.

Definition 2.5 \cite{13}.

1. Every Semi*-regular open set is Semi-$\alpha$-open.

2. Every Semi*-regular open set is Semi*-pre-open.

3. Every Semi*-regular open set is Semi*-open.

4. Every Semi* regular open set is Semi open.

5. Every Semi*regular open set is Semi $\alpha$-open.

6. Every Semi*regular open set is Semi pre-open.

7. Every Semi*regular open set is regular generalized open set.

8. Every Semi*regular open set is generalized pre regular open set.

Definition 2.6 \cite{15}. If $A$ is a subset of $X$, the semi*r-Frontier of $A$ is defined by $s^*r\text{Fr}(A) = s^*r\text{Cl}(A) \setminus s^*r\text{Int}(A)$.

Result 2.7 \cite{15}. If $A$ is a subset of $X$, then $s^*r\text{Fr}(A) = s^*r\text{Cl}(A) \setminus s^*r\text{Int}(A)$. Let $A$ be a subset of a space $X$. Then $A$ is semi*r-regular if and only if $s^*r\text{Fr}(A) = \emptyset$.

Theorem 2.8 \cite{13}. If $A$ is a subset of $X$, then

1. $s^*r\text{Cl}(X \setminus A) = X \setminus s^*r\text{Int}(A)$.

2. $s^*r\text{Int}(X \setminus A) = X \setminus s^*r\text{Cl}(A)$.

3. $A$ is semi*regular closed if and only if $s^*r\text{Cl}(A) = A$. 
Definition 2.9. A topological space $X$ is said to be connected [19] (resp. semi-connected [2], $\alpha$-connected, semi-$\alpha$-connected [16], semi*-pre-connected [12]) if $X$ cannot be expressed as the union of two disjoint nonempty open (resp. semi-open, $\alpha$-open, semi*-open, semi*-preopen) sets in $X$.

Definition 2.10 ([19]). A subset $A$ of a topological space $(X, \tau)$ is called clopen if it is both open and closed in $X$.

Theorem 2.11 ([19]). A topological space $X$ is connected if and only if the only clopen subsets of $X$ are $\emptyset$ and $X$.

Definition 2.12. A collection $B$ of open (resp. semi-open) sets in $X$ is called an open (resp. semi-open) cover of $A \subseteq X$ if $A \subseteq \bigcup\{U_\alpha : U_\alpha \in B\}$ holds.

Definition 2.13. A space $X$ is said to be compact [19] (resp. semi compact [3]) if every open (resp. semi-open) cover of $X$ has a finite subcover.

Definition 2.14 ([15]). A function $f : X \to Y$ is said to be

1. semi*-$r$-continuous if $f^{-1}(V)$ is semi*regular open in $X$ for every open set $V$ in $Y$.

2. semi*-$r$-irresolute if $f^{-1}(V)$ is semi*regular open in $X$ for every semi*regular open set $V$ in $Y$.

3. semi*regular open if $f(V)$ is semi*regular open in $Y$ for every open set $V$ in $X$.

4. semi*regular closed if $f(V)$ is semi*regular closed in $Y$ for every closed set $V$ in $X$.

5. pre-semi*regular open if $f(V)$ is semi*regular open in $Y$ for every semi*regular open set $V$ in $X$.

6. pre-semi*regular closed if $f(V)$ is semi*regular closed in $Y$ for every semi*regular closed set $V$ in $X$.

7. semi*-$r$-totally continuous if $f^{-1}(V)$ is clopen in $X$ for every semi*regular closed set $V$ in $Y$.

8. contra-semi*-$r$-continuous if $f^{-1}(V)$ is semi*regular closed in $X$ for every open set $V$ in $Y$.

9. contra-semi*-$r$-irresolute if $f^{-1}(V)$ is semi*regular closed in $X$ for every semi*regular open set $V$ in $Y$.

Theorem 2.15 ([15]). Let $f : X \to Y$ be a function. Then

1. $f$ is semi*-$r$-continuous if and only if $f^{-1}(F)$ is semi*regular-closed in $X$ for every closed set $F$ in $Y$.

2. $f$ is semi*-$r$-irresolute if and only if $f^{-1}(F)$ is semi*regular closed in $X$ for every semi*regular closed set $F$ in $Y$.

3. $f$ is contra-semi*-$r$-continuous if and only if $f^{-1}(F)$ is semi*regular open in $X$ for every closed set $F$ in $Y$.

4. $f$ is contra-semi*-$r$-irresolute if and only if $f^{-1}(F)$ is semi*regular open in $X$ for every semi*regular closed set $F$ in $Y$.

5. $f$ is semi*-$r$-totally continuous if $f^{-1}(F)$ is clopen in $X$ for every semi*regular closed set $F$ in $Y$.

3. Semi*regular Connected Spaces

In this section we introduce semi*regular connected spaces. We give characterizations for semi*regular connected spaces and also investigate their basic properties.

Definition 3.1. A topological space $X$ is said to be semi*regular connected if $X$ cannot be expressed as the union of two disjoint nonempty semi*regular open sets in $X$. 
Example 3.2. Let \( X = \{a, b, c, d\} \) and the topology \( \tau = \{X, \emptyset, \{a\}, \{d\}, \{a, d\}, \{a, c, d\}\} \). \( S^*\)\( RO(X) = \{\emptyset, \{a, b, d\}\} \). Then the space \( X \) is semi*-regular connected.

Theorem 3.3.

(1). Every semi*-connected space is semi*-regular connected.

(2). Every semi*pre-connected space is semi*-regular connected.

(3). Every semi*-\( \alpha \)-connected space is semi*-regular connected.

(4). Every semi connected space is semi*-regular connected.

(5). Every semi \( \alpha \)-connected space is semi*-regular connected.

(6). Every semi pre connected space is semi*-regular connected.

Proof. (1). Let \( X \) be semi*-connected space. Suppose \( X \) is not semi*-regular connected. Then there exists a proper non empty subset \( B \) of \( X \) which is both semi*-regular open and semi*-regular closed in \( X \). Since every semi*-regular closed (open)set is semi*-closed(open)set then \( X \) is not semi*-connected. This proves (1). In the similar manner we can prove (2), (3), (4), (5) and (6).

Remark 3.4. It can be seen from the following example that the converse of each of the statements in Theorem 3.3 is not true.

Example 3.5. Let \( X = \{a, b, c, d\} \) and the topology \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\} \)

\[
S^*O(X) &= \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}\} \\
S^*\alpha O(X) &= \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, d\}, \{a, c, d\}\} \\
S^* PO(X) &= \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, c\}, \{a, d\}, \{a, c, d\}\} \\
S^* RO(X) &= \{X, \emptyset, \{b\}, \{a, c\}\}.
\]

Then \( X \) is semi*-regular connected space but not semi*-connected , not semi*\( \alpha \)-connected and not semi*pre connected spaces.

Definition 3.6. The sets \( A \) and \( B \) in a topological space \( X \) are said to be semi*-regular separated if \( A \cap s^*r\text{Cl}(B) = s^*\text{rCl}(A) \cap B = \emptyset \).

Theorem 3.7. For a topological space \( X \), the following statements are equivalent:

(1). \( X \) is semi*-regular connected.

(2). \( X \) cannot be expressed as the union of two disjoint nonempty semi*-regular closed sets in \( X \).

(3). The only semi*\( r \)-regular subsets of \( X \) are \( \emptyset \) and \( X \) itself.

(4). Every semi*-regular continuous function of \( X \) into a discrete space \( Y \) is constant.

(5). Every nonempty proper subset of \( X \) has nonempty semi*\( r \)-frontier.

(6). \( X \) cannot be expressed as the union of the two non-empty semi*-regular separated sets.
Proof.  (1) ⇒ (2) Let X be a semi*regular connected space. Suppose $X = A \cup B$ where A and B are disjoint nonempty semi*regular closed sets. Then $A = X \cap B$ and $B = X \cap A$ are disjoint non-empty semi*regular open sets in X. This is a contradiction to the fact that X is semi*regular connected. This proves (2).

(2)⇒(1) Assume that X cannot be expressed as the union of two disjoint nonempty semi*regular closed sets in X. Suppose $X = A \cup B$ where A and B are disjoint nonempty semi*regular open sets. Then $A = X \cap B$ and $B = X \cap A$ are disjoint non-empty semi*regular closed sets in X. This is a contradiction to (2).

(1)⇒(3) Suppose X is a semi*regular connected space. Let A be non-empty proper subset of X that is A is semi*r-regular set. Then $X \cap A$ is a non-empty semi*regular open(or semi*regular closed) and $X = A \cup (X \cap A)$. This is a contradiction to $X$ is semi*regular connected.

(3)⇒(1) Suppose $X = A \cup B$ where A and B are disjoint non-empty semi*regular open sets. Then $A = X \cap B$ is semi*regular closed. Thus A is a non-empty proper subset that is semi*r-regular. This is a contradiction to (3).

(3)⇒(4) Let f be a semi*regular continuous function of the semi*regular connected space X into the discrete space Y. Then for each $y \in Y$, $f^{-1}\{y\}$ is a semi*r-regular set of X. Since X is semi*regular connected, $f^{-1}\{y\} = \emptyset$ or X. If $f^{-1}\{y\} = \emptyset$ for all $y \in Y$, then f fails to be a function. Therefore $f^{-1}\{y_0\} = X$ for a unique $y_0 \in Y$. This implies $f(X) = \{y_0\}$ and hence f is a constant function.

(4)⇒(3) Let U be a semi*r-regular set in X. Suppose $U \neq \emptyset$. We claim that $s^*rFr(U) \neq \emptyset$. If possible, let $s^*rFr(U) = \emptyset$. Then by the Result 2.7, A is semi*r-regular. This is a contradiction.

(3)⇒(5) Suppose that every non-empty proper subset of X has a non-empty semi*regular frontier. The only semi*r-regular subsets of X are $\emptyset$ and X itself. On the contrary, suppose that X has a non-empty proper subset A which is semi*r-regular. By the Result 2.7, $s^*rFr(A) = \emptyset$. This contradiction proves (3).

(1)⇒(6) Suppose $X = A \cup B$ where A and B are disjoint non-empty semi*r-separated sets in X. Since $A \cap s^*rCl(B) = \emptyset$, $s^*rCl(B) \subseteq X \cap A = B$ and hence $s^*rCl(B) = B$ and so by Theorem 2.8 (3), B is semi*regular closed. Therefore A is
semi\-regular open. Similarly, B is semi\-regular open. Hence X is not semi\-r-connected. This is contradiction to (1).

(6)⇒(1) Suppose X is not semi\-r-connected. Then X can be written as $X = A \cup B$ where A and B are disjoint non-empty semi\-regular open sets. Now $A = X \cap B$ is semi\-regular closed and hence by Theorem 2.8 (3), $s^rCl(A) = A$ and so $s^rCl(A) \cap B = \emptyset$. Similarly $A \cap s^rCl(B) = \emptyset$. Thus A and B are nonempty semi\-regular separated sets. This is a contradiction to (6).

**Theorem 3.8.** Let $f : X \to Y$ be a semi\-regular continuous bijection and X be semi\-regular connected. Then Y is connected.

**Proof.** Let $f : X \to Y$ be semi\-regular continuous surjection and X be semi\-regular connected. Let $V$ be a clopen subset of Y. By Definition 2.14 (1) $f^{-1}(V)$ is semi\-regular open and by Theorem 2.15 (1), $f^{-1}(V)$ is semi\-regular closed and hence $f^{-1}(V)$ is semi\-r-regular in X. Since X is semi\-regular connected, by Theorem 3.7 $f^{-1}(V) = \emptyset$ or X. Hence $V = \emptyset$ or Y. This proves that Y is connected.

**Theorem 3.9.** Let $f : X \to Y$ be a semi\-r-irresolute bijection. If X is semi\-regular connected, so is Y.

**Proof.** Let $f : X \to Y$ be a semi\-r-irresolute surjection and let X be semi\-regular connected. Let $V$ be a subset of Y that is semi\-r-regular in Y. By Definition 2.14 (2) and by Theorem 2.15 (2), $f^{-1}(V)$ is semi\-r-regular in X. Since X is semi\-regular connected, $f^{-1}(V) = \emptyset$ or X. Hence $V = \emptyset$ or Y. This proves that Y is semi\-regular connected.

**Theorem 3.10.** Let $f : X \to Y$ be a pre-semi\-regular open and pre-semi\-regular closed bijection. If Y is semi\-regular connected, so is X.

**Proof.** Let A be subset of X that is semi\-r-regular in X. Since f is both pre-semi\-regular open and pre-semi\-regular closed, $f(A)$ is semi\-r-regular in Y. Since Y is semi\-regular connected, $f(A) = \emptyset$ or Y. Hence $A = \emptyset$ or X. Therefore by Theorem 3.7, X is semi\-regular connected.

**Theorem 3.11.** If $f : X \to Y$ is a semi\-regular open and semi\-regular closed bijection and Y is semi\-regular connected, then X is connected.

**Proof.** Let A be a clopen subset of X. Since f is semi\-regular open, $f(A)$ is semi\-regular open in Y. Since f is a semi\-regular closed map, $f(A)$ is semi\-regular closed in Y. Hence $f(A)$ is semi\-r-regular in Y. Since Y is semi\-regular connected, by Theorem 3.7, $f(A) = \emptyset$ or Y. Hence $A = \emptyset$ or X. By Theorem 2.11, X is connected.

**Theorem 3.12.** If there is a semi\-r-totally continuous function from a connected space X onto Y, then the only semi\-regular open sets in Y are $\emptyset$ and Y.

**Proof.** Let f be a semi\-r-totally continuous function from a connected space X onto Y. Let $V$ be any open set in Y. Then by Theorem 2.5 (2), $V$ is semi\-regular open in Y. Since f is semi\-r-totally continuous, $f^{-1}(V)$ is clopen in X. Since X is connected, by Theorem 2.11, $f^{-1}(V) = \emptyset$ or X. This implies $V = \emptyset$ or Y.

**Theorem 3.13.** If $f : X \to Y$ is a strongly semi\-r-continuous bijection and Y is a space with at least two points, then X is not semi\-r-regular connected.

**Proof.** Let $y \in Y$. Then $f^{-1}(\{y\})$ is a non-empty proper subset that is semi\-r-regular in X. Hence by Theorem 3.7, X is not semi\-regular connected.

**Theorem 3.14.** Let $f : X \to Y$ be a contra-semi\-regular continuous surjection and X be semi\-regular connected. Then Y is connected.
Proof. Let \( f : X \to Y \) be a contra-semi*r-regular continuous surjection and \( X \) be semi*r-regular connected. Let \( V \) be a clopen subset of \( Y \). By Definition 2.14 (8) and by Theorem 2.15 (3), \( f^{-1}(V) \) is semi*r-regular in \( X \). Since \( X \) is semi*r-regular connected, \( f^{-1}(V) = \emptyset \) or \( X \). Hence \( V = \emptyset \) or \( Y \). This proves that \( Y \) is connected.

Theorem 3.15. Let \( f : X \to Y \) be a semi*r-irresolute bijection. If \( X \) is semi*r-regular connected, so is \( Y \).

Proof. Let \( f : X \to Y \) be a semi*r-irresolute bijection and let \( X \) be semi*r regular connected. Let \( V \) be a subset of \( Y \) that is semi*r-regular in \( Y \). By Definition 2.14 (2) and by Theorem 2.15 (2), \( f^{-1}(V) \) is semi*r-regular in \( X \). Since \( X \) is semi*r-regular connected, \( f^{-1}(V) = \emptyset \) or \( X \). Hence \( V = \emptyset \) or \( Y \). This proves that \( Y \) is semi*r-regular connected.

Theorem 3.16. Every contra-semi*r-continuous function from a semi*r-regular connected space into a \( T_1 \) space is necessarily constant.

Proof. Let \( f : X \to Y \) be a contra-semi*r-continuous function and \( X \) be semi*r-regular connected and \( Y \) be \( T_1 \). Since \( Y \) is \( T_1 \), for each \( y \in Y \), \( \{y\} \) is closed in \( Y \). Since \( f \) is contra-semi*r-continuous, by Theorem 2.15 (3), \( f^{-1}(\{y\}) \) is semi*r-regular open in \( X \). Therefore \( \{f^{-1}(\{y\}) : y \in Y\} \) is a collection of pair wise disjoint semi*r-regular open sets in \( X \). Since \( X \) is semi*r-regular connected, \( f^{-1}(\{y_0\}) = X \) for some fixed \( y_0 \in Y \). Hence \( f(X) = y_0 \). Thus \( f \) is a constant function.

Theorem 3.17. Every contra-semi*r-irresolute function from a semi*r-regular connected space into a semi*r-\( T_1 \) space is necessarily constant.

Proof. Let \( f : X \to Y \) be a contra-semi*r-irresolute function and \( X \) be semi*r-regular connected and \( Y \) be semi*r-\( T_1 \). Since \( Y \) is semi*r-\( T_1 \), for each \( y \in Y \), \( \{y\} \) is semi*r-regular closed in \( Y \). Since \( f \) is contra-semi*r-continuous, \( f^{-1}(\{y\}) \) is semi*r-regular open in \( X \). Therefore \( \{f^{-1}(\{y\}) : y \in Y\} \) is a collection of pair wise disjoint semi*r-regular open sets in \( X \). Since \( X \) is semi*r-regular connected, \( f^{-1}(\{y_0\}) = X \) for some fixed \( y_0 \in Y \). Hence \( f(X) = y_0 \). Thus \( f \) is constant.

4. Semi*r-Compact Spaces

In this section we introduce semi*r-compact spaces and study their properties. We also give characterizations for these spaces.

Definition 4.1. A collection \( C \) of semi*r-regular open sets in \( X \) is called a semi*r-regular open cover of a subset \( B \) of \( X \) if \( B \subseteq \bigcup \{U_\alpha : U_\alpha \in C\} \) holds.

Definition 4.2. A space \( X \) is said to be semi*r-compact if every semi*r-regular open cover of \( X \) has a finite subcover.

Definition 4.3. A subset \( B \) of \( X \) is said to be semi*r-compact relative to \( X \) if for every semi*r-regular open cover \( C \) of \( B \), there is a finite subcollection of \( C \) that covers \( B \).

Remark 4.4. Every finite topological space is semi*r-regular compact.

Theorem 4.5.

1. Every semi-compact space is semi*r-regular compact space.
2. Every semi-pre-compact space is semi*r-regular compact space.
3. Every semi \( \alpha \)-compact space is semi*r-regular compact space.
4. Every semi*-compact space is semi*r-regular compact space.
5. Every semi-*pre-compact space is semi-*regular compact space.

6. Every semi-*a-compact space is semi-*regular compact space.

**Theorem 4.6.** Every semi-*regular closed subset of a semi-*regular compact space $X$ is semi-*regular compact relative to $X$.

**Definition 4.7.** Let $A$ be a semi-*regular closed subset of a semi-*regular compact space $X$. Let $B$ be a semi-*regular open cover of $A$. Then $B \cup \{XnA\}$ is a semi-*regular open cover of $X$. Since $X$ is semi-*regular compact, this cover contains a finite subcover of $X$ and hence contains a finite subcollection of $B$ that covers $A$. This shows that $A$ is semi-*regular compact relative to $X$.

**Theorem 4.8.** A space $X$ is semi-*regular compact if and only if for every family of semi-*regular closed sets in $X$ which has empty intersection has a finite subfamily with empty intersection.

**Proof.** Suppose $X$ is semi-*regular compact and $\{Fa : \alpha \in \Delta\}$ is a family of semi-*regular closed sets in $X$ such that
$$\cap\{Fa : \alpha \in \Delta\} = \emptyset.$$ Then $\cup\{XnFa : \alpha \in \Delta\}$ is a semi-*regular open cover for $X$. Since $X$ is semi-*regular compact, this cover has a finite subcover $\{XnF\alpha_1, XnF\alpha_2, \ldots, XnF\alpha_n\}$. That is, $X = \cup\{XnF\alpha_i : i = 1, 2, \ldots, n\}$. On taking the complements on both sides we get $\cap_{i=1}^n F\alpha_i = \emptyset$.

Conversely, suppose that every family of semi-*regular closed sets in $X$ which has empty intersection has a finite subfamily with empty intersection. Let $\{U\alpha : \alpha \in \Delta\}$ be a semi-*regular open cover for $X$. Then $\cup\{U\alpha : \alpha \in \Delta\} = X$. Taking the complements, we get $\cap\{XnU\alpha : \alpha \in \Delta\} = \emptyset$. Since $XnU\alpha$ is semi-*regular closed for each $\alpha \in \Delta$, by the assumption, there is a finite subfamily, $\{XnU\alpha_1, XnU\alpha_2, \ldots, XnU\alpha_n\}$ with empty intersection. That is $\cap_{i=1}^n (X\setminus U\alpha_i) = \emptyset$. Taking the complements on both sides, we get $\cup_{i=1}^n (X\setminus U\alpha_i) = X$ Hence $X$ is semi-*regular compact.

**Theorem 4.9.** Let $f : X \to Y$ be a semi-*r*-irresolute bijection. If $X$ is semi-*r*-compact, then so is $Y$.

**Proof.** Let $f : X \to Y$ be a semi-*r*-irresolute bijection and $X$ be semi-*r*-compact. Let $\{V\alpha\}$ be a semi-*regular open cover for $Y$. Then $\{f^{-1}(V\alpha)\}$ is a cover of $X$ by semi-*regular open sets. Since $X$ is semi-*r*-compact, $\{f^{-1}(V\alpha)\}$ contains a finite subcover, namely $\{f^{-1}(V\alpha_1), f^{-1}(V\alpha_2), \ldots, f^{-1}(V\alpha_n)\}$. Then $\{V\alpha_1, V\alpha_2, \ldots, V\alpha_n\}$ is a finite subcover for $Y$. Thus $Y$ is semi-*r*-compact.

**Theorem 4.10.** Let $f : X \to Y$ be a semi-*r*-continuous bijection and $X$ be semi-*r*-compact. Then $Y$ is compact.

**Proof.** Let $f : X \to Y$ be a semi-*r*-continuous bijection and $X$ be semi-*a-compact. Let $\{V\alpha\}$ be an open cover for $Y$. Then $\{f^{-1}(V\alpha)\}$ is a cover of $X$ by semi-*regular open sets. Since $X$ is semi-*r*-compact, $\{f^{-1}(V\alpha)\}$ contains a finite sub cover, namely $\{f^{-1}(V\alpha_1), f^{-1}(V\alpha_2), \ldots, f^{-1}(V\alpha_n)\}$. Then $\{V\alpha_1, V\alpha_2, \ldots, V\alpha_n\}$ is a cover for $Y$. Thus $Y$ is compact.

**Theorem 4.11.** Let $f : X \to Y$ be a pre-semi-*r*-open injection. If $Y$ is semi-*r*-compact, then so is $X$.

**Proof.** Let $\{V\alpha\}$ be a semi-*regular open cover for $X$. Then $\{f(V\alpha)\}$ is a cover of $Y$ by semi-*regular open sets. Since $Y$ is semi-*r*-compact, $\{f(V\alpha)\}$ contains a finite subcover, namely $\{f(V\alpha_1), f(V\alpha_2), \ldots, f(V\alpha_n)\}$. Since $f$ is semi-*regular open injection, $\{V\alpha_1, V\alpha_2, \ldots, V\alpha_n\}$ is a finite subcover for $X$. Therefore $X$ is semi-*r*-compact.

**Theorem 4.12.** If $f : X \to Y$ is a semi-*regular open injection and $Y$ is semi-*r*-compact, then $X$ is compact.

**Proof.** Let $\{V\alpha\}$ be an open cover for $X$. Then $\{f(V\alpha)\}$ is a cover of $Y$ by semi-*regular open sets. Since $Y$ is semi-*r*-compact, $\{f(V\alpha)\}$ contains a finite subcover, namely $\{f(V\alpha_1), f(V\alpha_2), \ldots, f(V\alpha_n)\}$. Since $f$ is semi-*regular open injection, $\{V\alpha_1, V\alpha_2, \ldots, V\alpha_n\}$ is a finite sub cover for $X$. Thus $X$ is compact.
**Theorem 4.13.** Let $f : X \rightarrow Y$ be a contra-semi*r-continuous function and $Y$ be $T_1$. If $X$ is semi*r-compact, then the range of $f$ is finite. Further if $Y$ is infinite, $f$ cannot be onto.

**Proof.** Since $Y$ is $T_1$, for each $y \in Y$, $\{y\}$ is closed in $Y$. Since $f$ is contra-semi*r-continuous, by Theorem 2.15 (3), $f^{-1}(\{y\})$ is semi*regular open in $X$. Therefore $\{f^{-1}(\{y\}) : y \in Y\}$ is a semi*regular open cover for $X$. Since $X$ is semi*r-compact, there are $y_1, y_2, ..., y_n$ in $Y$ such that $\{f^{-1}(\{y_i\}) : i = 1, 2, ..., n\}$ is a cover of $X$ by semi*regular open sets. Therefore $\bigcup \{f^{-1}(\{y_i\}) : i = 1, 2, ..., n\} = X$. That is, $f^{-1}(\{y_1, y_2, ..., y_n\}) = X$. This implies $f(X) = \{y_1, y_2, ..., y_n\}$. Thus the range of $f$ is finite. If $Y$ is infinite, $f(X) \neq Y$. Hence $f$ cannot be onto.

**Theorem 4.14.** Let $f : X \rightarrow Y$ be a contra-semi*r-irresolute function and $Y$ be semi*r-$T_1$. If $X$ is semi*r-compact, then the range of $f$ is finite. Further if $Y$ is infinite, $f$ cannot be onto.

**Proof.** Since $Y$ is semi*r-$T_1$, for each $y \in Y$, $\{y\}$ is semi*regular closed in $Y$. Since $f$ is contra-semi*r-continuous, by Theorem 2.15 (4), $f^{-1}(\{y\})$ is semi*regular open in $X$. Therefore $\{f^{-1}(\{y\}) : y \in Y\}$ is a semi*regular open cover for $X$. Since $X$ is semi*r-compact, there are $y_1, y_2, ..., y_n$ in $Y$ such that $\{f^{-1}(\{y_i\}) : i = 1, 2, ..., n\}$ is a cover of $X$ by semi*regular open sets. Therefore $\bigcup \{f^{-1}(\{y_i\}) : i = 1, 2, ..., n\} = X$. That is, $f^{-1}(\{y_1, y_2, ..., y_n\}) = X$. This implies $f(X) = \{y_1, y_2, ..., y_n\}$. Thus the range of $f$ is finite. If $Y$ is infinite, $f(X) \neq Y$. Hence $f$ cannot be onto.

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**References**


