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$\alpha-$Cubic and $\beta-$Cubic Functional Equations

Research Article

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Abstract: In this paper, we established the general solution and generalized Ulam - Hyers stability of $\alpha-$cubic functional equation $2[\alpha f(w-\alpha z)+f(\alpha w+z)]=\alpha(\alpha^2+1)[f(w+z)+f(w-z)]-2(\alpha^4-1)f(z)$, where $\alpha\neq 0,\pm 1$ and $\beta-$cubic functional equation $\beta f(w+\beta z)-f(\beta w+z)-[\beta f(w-\beta z)-f(\beta w-z)]=2(\beta^4-1)f(z)$, where $\beta\neq 0,\pm 1$ in Banach Space using direct and fixed point methods.

MSC: 39B52, 32B72, 32B82.

Keywords: Cubic functional equations, generalized Ulam - Hyers stability, Banach space, fixed point.

1. Introduction

The survey of stability problems for functional equations is connected to the eminent Ulam problem [32] (in 1940), with reference to the stability of group homomorphisms, which was first solved by D. H. Hyers [13], in 1941. This stability problem was also generalized by a number of authors [2, 12, 25, 28, 30]. We cite also other pertinent research works [1, 11, 14, 16, 19, 29]. The solution and stability of the following cubic functional equations

$$C(x+2y)+3C(x)=3C(x+y)+C(x-y)+6C(y), \tag{1}$$

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x), \tag{2}$$

$$f(x+y+2z)+f(x+y-2z)+f(2x)+f(2y)=2[f(x+y)+2f(x+z)+2f(y+z)+2f(x-z)+2f(y-z)], \tag{3}$$

$$3f(x+3y)-f(3x+y)=12[f(x+y)+f(x-y)]+80f(y)-48f(x), \tag{4}$$

$$g(2x-y)+g(x-2y)=6g(x-y)+3g(x)-3g(y), \tag{5}$$

$$f(2x\pm y\pm z)+f(\pm y\pm z)+2f(\pm y)+2f(\pm z)$$
$$=2f(x\pm y\pm z)+f(x\pm y)+f(x\pm z)+f(-x\pm y)+f(-x\pm z)+6f(x), \tag{6}$$

$$kf(x+ky)-f(kx+y)=\frac{k(k^2-1)}{2}[f(x+y)+f(x-y)]+(k^4-1)f(y)-2k(k^2-1)f(x), k\geq 2 \tag{7}$$

$$\frac{a+\sqrt{k}\,b}{2}f\left(ax+\sqrt{k}\,by\right)+\frac{a-\sqrt{k}\,b}{2}f\left(ax-\sqrt{k}\,by\right)+k(a^2-kb^2)b^2f(y)$$
$$=k(ab)^2f(x+y)+(a^2-kb^2)a^2f(x), a\neq\pm 1,0; b\neq\pm 1,0; k>0 \tag{8}$$

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were investigated by J.M. Rassias [26], K.W. Jun, H.M. Kim [15], Y.S. Jung, I.S. Chang [18], K. Ravi et. al., [31], M.Arunkumar [3, 4], M.J.Rassias et. al., [17], J.M.Rassias., et.al., [27]. Now, we will recall the fundamental results in fixed point theory.

Theorem 1.1 (Banach's contraction principle). *Let (X,d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

(A_1). *$d(Tx,Ty) \leq Ld(x,y)$, for some (Lipschitz constant) $L < 1$. Then,*

(1). The mapping T has one and only fixed point $x^* = T(x^*)$;

(2). The fixed point for each given element x^* is globally attractive, that is

(A_2). *$\lim_{n\to\infty} T^n x = x^*$, for any starting point $x \in X$;*

(3). One has the following estimation inequalities:

(A_3). *$d(T^n x, x^*) \leq \frac{1}{1-L} \ d(T^n x, T^{n+1}x), \forall \ \ n \geq 0, \forall \ \ x \in X$;*

(A_4). *$d(x,x^*) \leq \frac{1}{1-L} \ d(x,x^*), \forall \ \ x \in X$.*

Theorem 1.2 (The alternative of fixed point [20]). *Suppose that for a complete generalized metric space (X,d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*

(B_1). *$d(T^n x, T^{n+1}x) = \infty \quad \forall \ \ n \geq 0$, or*

(B_2). *there exists a natural number n_0 such that:*

(1). $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;

(2). The sequence $(T^n x)$ is convergent to a fixed point y^* of T

(3). y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;

(4). $d(y^*, y) \leq \frac{1}{1-L} \ d(y, Ty)$ for all $y \in Y$.

In this paper, we established the general solution and generalized Ulam - Hyers stability of $\alpha-$cubic functional equation

$$2[\alpha f(w-\alpha z) + f(\alpha w + z)] = \alpha(\alpha^2+1)[f(w+z)+f(w-z)] - 2(\alpha^4-1)f(z) \tag{9}$$

where $\alpha \neq 0, \pm 1$ and $\beta-$cubic functional equation

$$\beta f(w+\beta z) - f(\beta w + z) - [\beta f(w-\beta z) - f(\beta w - z)] = 2(\beta^4-1)f(z) \tag{10}$$

where $\beta \neq 0, \pm 1$ in Banach Space using direct and fixed point methods.

2. General Solution of (9) and (10)

In this section, we present the general solution of the $\alpha-$cubic and $\beta-$cubic functional equations. To prove the solution, let us take W and Z be real vector spaces.

Lemma 2.1. *If a mapping $f : W \to Z$ satisfies the functional equation (9), then the following properties hold*

(1). $f(0) = 0$,

(2). $f(aw) = a^3 f(w)$, *for all* $w \in W$.

(3). $f(-z) = -f(z)$, *for all* $z \in W$*; that is,* f *is an odd function.*

Proof.

(1). Replacing (w, z) by $(0, 0)$ in (9), we get

$$\begin{aligned}
&2[\alpha + 1]f(0) = 2\alpha(\alpha^2 + 1)f(0) - 2(\alpha^4 - 1)f(0) \\
&(-2\alpha^3 + 2\alpha^4)f(0) = 0 \\
&(-\alpha^3 + \alpha^4)f(0) = 0 \\
&f(0) = 0
\end{aligned}$$

since $\alpha \neq 0, \pm 1$.

(2). Setting z by 0 in (9), we obtain

$$\begin{aligned}
&2[\alpha f(w) + f(\alpha w)] = \alpha(\alpha^2 + 1)[f(w) + f(w)] \\
&[\alpha f(w) + f(\alpha w)] = \alpha(\alpha^2 + 1)f(w) \\
&[\alpha f(w) + f(\alpha w)] = [\alpha^3 + \alpha]f(w) \\
&f(\alpha w) = \alpha^3 f(w)
\end{aligned}$$

for all $w \in W$.

(3). Letting (w, z) by $(0, z)$ in (9), we arrive

$$\begin{aligned}
&2[\alpha f(-\alpha z) + f(z)] = \alpha(\alpha^2 + 1)[f(z) + f(-z)] - 2(\alpha^4 - 1)f(z) \\
&f(-z)[2\alpha^4 - \alpha(\alpha^2 + 1)] = f(z)[-2 - 2(\alpha^4 - 1) + \alpha(\alpha^2 + 1)] \\
&f(-z)[2\alpha^4 - \alpha^3 - \alpha] = f(z)[-2\alpha^4 + \alpha^3 + \alpha] \\
&f(-z) = -f(z)
\end{aligned}$$

holds for all $z \in W$, since $\alpha \neq 0, \pm 1$. Thus f is an odd function. □

Lemma 2.2. *If a mapping* $f : W \to Z$ *satisfies the functional equation (10), then the following properties hold*

(1). $f(0) = 0$,

(2). $f(-z) = -f(z)$, *for all* $z \in W$*; that is,* f *is an odd function.*

(3). $f(\beta z) = \beta^3 f(z)$, *for all* $w \in W$.

Proof.

(1). Replacing (w,z) by $(0,0)$ in (10), we get

$$\begin{aligned} &\beta f(0)-f(0)-[\beta f(0)-f(0)]=2(\beta^4-1)f(0)\\ &2(\beta^4-1)f(0)=0\\ &f(0)=0. \end{aligned}$$

since $\beta\neq 0,\pm 1$.

(2). Setting (w,z) by $(0,z)$ in (10), we obtain

$$\beta f(\beta z)-f(z)-[\beta f(-\beta z)-f(-z)]=2(\beta^4-1)f(z) \tag{11}$$

for all $z\in W$. Replacing z by $-z$ in (11), we have

$$\beta f(-\beta z)-f(-z)-[\beta f(\beta z)-f(z)]=2(\beta^4-1)f(-z) \tag{12}$$

for all $z\in W$. Adding (11) and (12), we reach

$$f(-z)=-f(z)$$

for all $z\in W$. Thus f is an odd function.

(3). Using (2) in (11), we arrive

$$\begin{aligned} &\beta f(\beta z)-f(z)+\beta f(\beta z)-f(z)=2(\beta^4-1)f(z)\\ &2(\beta f(\beta z)-f(z))=2(\beta^4-1)f(z)\\ &\beta f(\beta z)=\beta^4 f(z)\\ &f(\beta z)=\beta^3 f(z) \end{aligned}$$

holds for all $z\in W$, since $\beta\neq 0,\pm 1$. □

3. Stability of (9)

In this section, we present the generalized Ulam - Hyers - Rassias of the $\alpha-$cubic functional equation. Throughout this section, we assume $\mathcal{W}$ be a normed space and $\mathcal{Z}$ be a Banach space.

3.1. Banach Space: Direct Method

Theorem 3.1. *Let $a=\pm 1$ and $\Delta_\alpha:\mathcal{W}^2\longrightarrow[0,\infty)$ be a function such that*

$$\sum_{b=0}^{\infty}\frac{\Delta_\alpha\left(\alpha^{ba}w,\alpha^{ba}z\right)}{\alpha^{3a}}\quad \text{converges in } \mathbb{R} \quad \text{and} \quad \lim_{b\to\infty}\frac{\Delta_\alpha\left(\alpha^{ba}w,\alpha^{ba}z\right)}{\alpha^{3a}}=0 \tag{13}$$

for all $w,z\in\mathcal{W}$. Let $f:\mathcal{W}\longrightarrow\mathcal{Z}$ be a function fulfilling the inequality

$$\left\|2[\alpha f(w-\alpha z)+f(\alpha w+z)]-\alpha(\alpha^2+1)[f(w+z)+f(w-z)]+2(\alpha^4-1)f(z)\right\|\leq\Delta_\alpha(w,z) \tag{14}$$

for all $w, z \in \mathcal{W}$. Then there exists a unique cubic function $\mathcal{C}_\alpha : \mathcal{W} \longrightarrow \mathcal{Z}$ which satisfies (9) and

$$\|f(w) - \mathcal{C}_\alpha(w)\| \leq \frac{1}{2\alpha^3} \sum_{b=\frac{1-a}{2}}^{\infty} \frac{\Delta_\alpha\left(\alpha^{ba}w, 0\right)}{\alpha^{3ba}} \tag{15}$$

where $\mathcal{C}_\alpha(w)$ is defined by

$$\mathcal{C}_\alpha(w) = \lim_{b\to\infty} \frac{f(\alpha^{ba}w)}{\alpha^{3ba}} \tag{16}$$

for all $w \in \mathcal{W}$.

Proof. **Case (i):** Assume $a = 1$.

Replacing (w, z) by $(w, 0)$ in (14), we get

$$\left\|2f(\alpha w) - 2\alpha^3 f(w)\right\| \leq \Delta_\alpha(w, 0) \tag{17}$$

for all $w \in \mathcal{W}$. Rewriting (17), we have

$$\left\|\frac{f(\alpha w)}{\alpha^3} - f(w)\right\| \leq \frac{\Delta_\alpha(w, 0)}{2\alpha^3} \tag{18}$$

for all $w \in \mathcal{W}$. Now replacing w by αw and dividing by α^3 in (18), we have

$$\left\|\frac{f(\alpha^2 w)}{\alpha^6} - \frac{f(\alpha w)}{\alpha^3}\right\| \leq \frac{\Delta_\alpha(\alpha w, 0)}{2\alpha^6} \tag{19}$$

for all $w \in \mathcal{W}$. Combining (18), (19) and using triangle inequality, we obtain

$$\begin{aligned}\left\|\frac{f(\alpha^2 w)}{\alpha^6} - f(w)\right\| &\leq \left\|\frac{f(\alpha^2 w)}{\alpha^6} - \frac{f(\alpha w)}{\alpha^3}\right\| + \left\|\frac{f(\alpha w)}{\alpha^3} - f(w)\right\| \\ &\leq \frac{1}{2\alpha^3}\left[\Delta_\alpha(w, 0) + \frac{\Delta_\alpha(\alpha w, 0)}{\alpha^3}\right]\end{aligned} \tag{20}$$

for all $w \in \mathcal{W}$. Generalizing, for a positive integer c, we land

$$\left\|\frac{f(\alpha^c w)}{\alpha^{3c}} - f(w)\right\| \leq \frac{1}{2\alpha^3} \sum_{b=0}^{c-1} \frac{\Delta_\alpha(\alpha^b w, 0)}{\alpha^{3b}} \tag{21}$$

for all $w \in \mathcal{W}$. To prove the convergence of the sequence

$$\left\{\frac{f(\alpha^c w)}{\alpha^{3c}}\right\},$$

replacing w by $\alpha^d w$ and dividing by α^{3d} in (21), for any $c, d > 0$, we get

$$\begin{aligned}\left\|\frac{f(\alpha^{c+d} w)}{\alpha^{3(c+d)}} - \frac{f(\alpha^d w)}{\alpha^{3d}}\right\| &= \frac{1}{\alpha^{3d}}\left\|\frac{f(\alpha^c \cdot \alpha^d w)}{\alpha^{3c}} - f(\alpha^d w)\right\| \\ &\leq \frac{1}{2\alpha^3} \sum_{b=0}^{c-1} \frac{\Delta_\alpha(\alpha^{b+d} w, 0)}{\alpha^{3(b+d)}} \\ &\leq \frac{1}{2\alpha^3} \sum_{b=0}^{\infty} \frac{\Delta_\alpha(\alpha^{b+d} w, 0)}{\alpha^{3(b+d)}} \\ &\to 0 \quad as \;\; d \to \infty\end{aligned}$$

for all $w \in \mathcal{W}$. Thus it follows that the sequence $\left\{\frac{f(\alpha^c w)}{\alpha^{3c}}\right\}$ is a Cauchy in $\mathcal{Z}$. Define a mapping $\mathcal{C}_\alpha(w) : \mathcal{W} \to \mathcal{Z}$ by

$$\mathcal{C}_\alpha(w) = \lim_{c\to\infty} \frac{f(\alpha^c w)}{\alpha^{3c}} \tag{22}$$

for all $w \in \mathcal{W}$. Letting c tends to ∞ in (21) and using (22), we see that (15) holds for all $w \in \mathcal{W}$. In order to show that $\mathcal{C}_\alpha$ satisfies (9), replacing (w, z) by $(\alpha^c w, \alpha^c z)$ and dividing by α^{3c} in (14), we have

$$\frac{1}{\alpha^{3c}} \left\|2[\alpha f(\alpha^c(w - \alpha z)) + f(\alpha^c(\alpha w + z))] - \alpha(\alpha^2 + 1)[f(\alpha^c(w + z)) + f(\alpha^c(w - z))] + 2(\alpha^4 - 1)f(\alpha^c z)\right\| \le \frac{1}{\alpha^{3c}}\Delta_\alpha\left(\alpha^c w, \alpha^c z\right)$$

for all $w, z \in \mathcal{W}$. Letting c tends to ∞ in the above inequality and using (22), we arrive

$$\left\|2[\alpha\mathcal{C}_\alpha(w - \alpha z) + \mathcal{C}_\alpha(\alpha w + z)] - \alpha(\alpha^2 + 1)[\mathcal{C}_\alpha(w + z) + \mathcal{C}_\alpha(w - z)] + 2(\alpha^4 - 1)\mathcal{C}_\alpha(z)\right\| = 0$$

for all $w, z \in \mathcal{W}$. Hence, $\mathcal{C}_\alpha$ satisfies (9), for all $w, z \in \mathcal{W}$.

To prove that $\mathcal{C}_\alpha$ is unique, we assume now that there is $\mathcal{C'}_\alpha$ as another cubic mapping satisfying (9) and the inequality (15). Then it is easily note that

$$\mathcal{C}_\alpha(\alpha^s x) = \alpha^{3s}\mathcal{C}_\alpha(x), \qquad \mathcal{C'}_\alpha(\alpha^s x) = \alpha^{3s}\mathcal{C'}_\alpha(x)$$

for all $w \in \mathcal{W}$ and all $s \in \mathbb{N}$. Thus

$$\begin{aligned}\left\|\mathcal{C}_\alpha(w) - \mathcal{C'}_\alpha(w)\right\| &= \frac{1}{\alpha^{3d}}\left\|\mathcal{C}_\alpha(\alpha^d w) - \mathcal{C'}_\alpha(\alpha^d w)\right\| \\ &\le \frac{1}{\alpha^{3d}}\left\{\left\|\mathcal{C}_\alpha(\alpha^d w) - f(\alpha^d w)\right\| + \left\|f(\alpha^d w) - \mathcal{C'}_\alpha(\alpha^d w)\right\|\right\} \\ &\le \frac{1}{\alpha^3}\sum_{b=0}^{\infty}\frac{\Delta_\alpha(\alpha^{b+d}x, 0)}{\alpha^{3(b+d)}}\end{aligned}$$

for all $w \in \mathcal{W}$. Therefore, as $d \to \infty$ in the above inequality, we arrive the uniqueness of $\mathcal{C}_\alpha$. Hence the theorem holds for $a = 1$.

Case (ii): Assume $a = -1$.

Now replacing w by $\frac{x}{w}$ in (17), we get

$$\left\|f(w) - \alpha^3 f\left(\frac{x}{w}\right)\right\| \le \frac{1}{2}\Delta_\alpha\left(\frac{x}{w}, 0\right) \tag{23}$$

for all $w \in \mathcal{W}$. The rest of the proof is similar to that of case $a = 1$. Thus for $a = -1$ also the theorem holds. hence the proof is complete. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the stabilities of (9).

Corollary 3.2. *Let $f : \mathcal{W} \longrightarrow \mathcal{Z}$ be a mapping. If there exist real numbers p and q such that*

$$\left\|2[\alpha f(w - \alpha z) + f(\alpha w + z)] - \alpha(\alpha^2 + 1)[f(w + z) + f(w - z)] + 2(\alpha^4 - 1)f(z)\right\| \le \begin{cases} p, \\ p\left\{||w||^q + ||z||^q\right\}, \\ p\left\{||w||^q||z||^q + \left\{||w||^{2q} + ||z||^{2q}\right\}\right\}, \end{cases} \tag{24}$$

for all $w, z \in \mathcal{W}$, then there exists a unique cubic function $\mathcal{C}_\alpha : \mathcal{W} \longrightarrow \mathcal{Z}$ such that

$$\|f(w) - \mathcal{C}_\alpha(w)\| \le \begin{cases} \dfrac{p}{2|\alpha^3 - 1|}, \\ \dfrac{p||w||^q}{2|\alpha^3 - \alpha^q|}, & q \ne 3, \\ \dfrac{p||w||^{2q}}{2|\alpha^3 - \alpha^{2q}|}, & 2q \ne 3, \end{cases} \tag{25}$$

for all $w \in \mathcal{W}$.

Proof. If we substitute

$$\Delta_\alpha(w,z) = \begin{cases} p, \\ p\left\{\|w\|^q + \|z\|^q\right\}, \\ p\left\{\|w\|^q\|z\|^q + \left\{\|w\|^{2q} + \|z\|^{2q}\right\}\right\}, \end{cases}$$

in (17) of Theorem 3.1, we reach (25) as desired. □

3.2. Banach Space: Fixed Point Method

Theorem 3.3. *Let $f : \mathcal{W} \longrightarrow \mathcal{Z}$ be a mapping for which there exists a function $\Delta_\alpha : \mathcal{W}^2 \longrightarrow [0,\infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\ell_i^{3n}} \Delta_\alpha(\ell_i^n w, \ell_i^n z) = 0 \tag{26}$$

where

$$\ell_i = \begin{cases} \alpha & if \quad i = 0, \\ \frac{1}{\alpha} & if \quad i = 1 \end{cases} \tag{27}$$

such that the functional inequality

$$\left\|2[\alpha f(w-\alpha z) + f(\alpha w + z)] - \alpha(\alpha^2+1)[f(w+z) + f(w-z)] + 2(\alpha^4-1)f(z)\right\| \leq \Delta_\alpha(w,z) \tag{28}$$

holds for all $w,z \in \mathcal{W}$. Assume that there exists $L = L(i)$ such that the function

$$\Delta_\alpha(w,0) = \frac{1}{2}\Delta_\alpha\left(\frac{w}{\alpha},0\right)$$

with the property

$$\frac{1}{\ell_i^3}\Delta_\alpha(\ell_i w,0) = L\ \Delta_\alpha(w,0) \tag{29}$$

for all $w \in \mathcal{W}$. Then there exists a unique cubic mapping $\mathcal{C}_\alpha : \mathcal{W} \longrightarrow \mathcal{Z}$ satisfying the functional equation (9) and

$$\|f(w) - \mathcal{C}_\alpha(w)\| \,\|\leq \left(\frac{L^{1-i}}{1-L}\right)\Delta_\alpha(w,0) \tag{30}$$

for all $w \in \mathcal{W}$.

Proof. Consider the set

$$\mathcal{S} = \{f_a/f_a : \mathcal{W} \longrightarrow \mathcal{Z},\ f_a(0) = 0\}$$

and introduce the generalized metric $d : \mathcal{S}\times\mathcal{S} \to [0,\infty]$ as follows:

$$d(f,f_a) = \inf\{\omega \in (0,\infty) :\| f(w) - f_a(w) \|\leq \omega\ \Delta_\alpha(w,0), w \in \mathcal{W}\}. \tag{31}$$

It is easy to show that $(\mathcal{S}, d)$ is complete with respect to the defined metric. Let us define the linear mapping $J : \mathcal{S} \longrightarrow \mathcal{S}$ by

$$Jf_a(x) = \frac{1}{\ell_i^3} f_a(\ell_i x),$$

for all $w \in \mathcal{W}$. For given $f, f_a \in \mathcal{S}$ let $\omega \in [0,1)$ be an arbitrary constant with $d(f,f_a) \in \omega$ that is

$$\| f(w) - f_a(w) \|\leq \omega\ \Delta_\alpha(w,0), w \in \mathcal{W}.$$

So, we have

$$\begin{aligned}\| f(w)-f_a(w) \| &= \left\| \frac{1}{\ell_i^3} f(\ell_i w) - \frac{1}{\ell_i^3} f_a(\ell_i w) \right\| \\ &\le \frac{\omega}{\ell_i^3} \Delta_\alpha(\ell_i w, 0) \\ &= L\omega \Delta_\alpha(w,0)\end{aligned}$$

for all $w \in \mathcal{W}$, that is,

$$d(Jf, Jf_a) \le Ld(f, f_a), \qquad \forall \;\; f, f_a \in \mathcal{S}.$$

This implies J is a strictly contractive mapping on $\mathcal{S}$ with Lipschitz constant L. It follows from (31),(17) and (29) for the case $i=0$, we reach

$$\left\|2f\left(\alpha w\right) - 2\alpha^3 f(w)\right\| \le \Delta_\alpha(w,0), w \in \mathcal{W} \tag{32}$$

and

$$\left\| \frac{f\left(\alpha w\right)}{\alpha^3} - f(w) \right\| \le \frac{1}{2\alpha^3} \Delta_\alpha(w,0), w \in \mathcal{W}. \tag{33}$$

So, we obtain

$$\|Jf(w) - f(w)\| \le L \; \Delta_\alpha(w,0), w \in \mathcal{W}. \tag{34}$$

Hence,

$$d(Jf, f) \le L^{1-0}, f \in \mathcal{S} \tag{35}$$

Replacing $w = \frac{x}{\alpha}$ in (32) and (29) for the case $i=1$, we get

$$\left\|2f\left(w\right) - 2\alpha^3 \left(\frac{w}{\alpha}\right)\right\| \le \Delta_\alpha \left(\frac{w}{\alpha}, 0\right), w \in \mathcal{W} \tag{36}$$

Then,

$$\|f(w) - Jf(w)\| \le \frac{1}{2} \left(\frac{w}{\alpha}, 0\right), w \in \mathcal{W} \tag{37}$$

and

$$\|f\left(w\right) - Jf(w)\| \le L^{1-1} \Delta_\alpha(w,0), w \in \mathcal{W} \tag{38}$$

Thus, we obtain

$$d(f, Jf) \le L^{1-1}, f \in \mathcal{S} \tag{39}$$

Hence, from (35) and (39), we arrive

$$d(Jf, f) \le L^{1-i}, f \in \mathcal{S} \tag{40}$$

where $i = 0, 1$. Hence property (FP1) holds. It follows from property (FP2) that there exists a fixed point $\mathcal{C}_\alpha$ of J in $\mathcal{S}$ such that

$$\mathcal{C}_\alpha(w) = \lim_{n\to\infty} \frac{1}{\ell_i^{3n}} f(\ell_i^n w) \tag{41}$$

for all $w \in \mathcal{W}$. In order to show that $\mathcal{C}_\alpha$ satisfies (9), replacing (w, z) by $(\ell_i^n w, \ell_i^n z)$ and dividing by ℓ_i^{3n} in (28), we have

$$\frac{1}{\ell_i^{3n}} \left\|2[\alpha f(\ell_i^n(w-\alpha z)) + f(\ell_i^n(\alpha w + z))] - \alpha(\alpha^2+1)[f(\ell_i^n(w+z)) + f(\ell_i^n(w-z))] + 2(\alpha^4-1)f(\ell_i^n z)\right\| \leq \frac{1}{\ell_i^{3n}} \Delta_\alpha(\ell_i^n w, \ell_i^n z)$$

for all $w, z \in \mathcal{W}$, and so the mapping $\mathcal{C}_\alpha$ is cubic. i.e., $\mathcal{C}_\alpha$ satisfies the functional equation (9). By property (FP3), $\mathcal{C}_\alpha$ is the unique fixed point of J in the set

$$\Delta = \{\mathcal{C}_\alpha \in \mathcal{S} : d(f, \mathcal{C}_\alpha) < \infty\},$$

such that

$$\|f(w) - \mathcal{C}_\alpha(w)\| \leq \omega \Delta_\alpha(w, 0), w \in \mathcal{W}.$$

Finally by property (FP4), we obtain

$$\|f(w) - \mathcal{C}_\alpha(w)\| \leq \|f(w) - Jf(w)\|.$$

This implies

$$\|f(w) - \mathcal{C}_\alpha(w)\| \leq \frac{L^{1-i}}{1-L}$$

which yields

$$\|f(w) - \mathcal{C}_\alpha(w)\| \leq \left(\frac{L^{1-i}}{1-L}\right) \Delta_\alpha(w, 0), w \in \mathcal{W}$$

So, the proof is completed. □

Using Theorem 3.3, we prove the following corollary concerning the stabilities of (9).

Corollary 3.4. *Let $f : \mathcal{W} \longrightarrow \mathcal{Z}$ be a mapping. If there exist real numbers p and q such that*

$$\left\|2[\alpha f(w-\alpha z) + f(\alpha w + z)] - \alpha(\alpha^2+1)[f(w+z) + f(w-z)] + 2(\alpha^4-1)f(z)\right\| \leq \begin{cases} p, \\ p\left\{||w||^q + ||z||^q\right\}, \\ p\left\{||w||^q||z||^q + \left\{||w||^{2q} + ||z||^{2q}\right\}\right\}, \end{cases} \tag{42}$$

for all $w, z \in \mathcal{W}$, then there exists a unique cubic function $\mathcal{C}_\alpha : \mathcal{W} \to \mathcal{Z}$ such that

$$\|f(w) - \mathcal{C}_\alpha(w)\| \leq \begin{cases} \dfrac{p}{2\alpha|\alpha^3-1|}, \\ \dfrac{p||w||^q}{2\alpha|\alpha^3-\alpha^q|}, & q \neq 3, \\ \dfrac{p||w||^{2q}}{2\alpha|\alpha^3-\alpha^{2q}|}, & 2q \neq 3, \end{cases} \tag{43}$$

for all $w \in \mathcal{W}$.

Proof. Let

$$\Delta_\alpha(w,z) = \begin{cases} p, \\ p\left\{||w||^q + ||z||^q\right\}, \\ p\left\{||w||^q||z||^q + \left\{||w||^{2q} + ||z||^{2q}\right\}\right\}, \end{cases}$$

for all $w, z \in \mathcal{W}$. Now

$$\frac{1}{\ell_i^{3n}}\Delta_\alpha(\ell_i^n w, \ell_i^n z) = \begin{cases} \frac{p}{\ell_i^{3n}}, \\ \frac{p}{\ell_i^{3n}}\{||\ell_i^n w||^q + ||\ell_i^n z||^q\}, \\ \frac{p}{\ell_i^{3n}}\left\{||\ell_i^n w||^q \; ||\ell_i^n z||^q + \left\{||\ell_i^n w||^{2q} + ||\ell_i^n z||^{2q}\right\}\right\} \end{cases} = \begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty. \end{cases}$$

Thus, (26) holds. But, we have

$$\Delta_\alpha(w,0) = \frac{1}{2}\Delta_\alpha\left(\frac{w}{\alpha}, 0\right)$$

has the property

$$\frac{1}{\ell_i^3}\Delta_\alpha(\ell_i w, 0) = L\ \Delta_\alpha(w,0)$$

for all $w \in \mathcal{W}$. Hence,

$$\Delta_\alpha(w,0) = \frac{1}{2}\Delta_\alpha\left(\frac{w}{\alpha}, 0\right) = \begin{cases} \frac{p}{2\alpha}, \\ \frac{p}{2\alpha \cdot \alpha^q}||w||^q, \\ \frac{p}{2\alpha \cdot \alpha^{2q}}||w||^{2q} \end{cases} \tag{44}$$

for all $w \in \mathcal{W}$. It follows from (44),

$$\frac{1}{\ell_i^3}\Delta_\alpha(\ell_i w, 0) = \begin{cases} \ell_i^{-3}\ \frac{p}{2\alpha}, \\ \ell_i^{q-3}\frac{p}{2\alpha}||w||^q \\ \ell_i^{2q-3}\frac{p}{2\alpha}||w||^{2q}. \end{cases}$$

Hence, the inequality (30) holds for

(i). $L = \ell_i^{-3}$ if $i = 0$ and $L = \frac{1}{\ell_i^{-3}}$ if $i = 1$;

(ii). $L = \ell_i^{q-3}$ for $q < 3$ if $i = 0$ and $L = \frac{1}{\ell_i^{q-3}}$ for $q > 3$ if $i = 1$;

(iii). $L = \ell_i^{2q-3}$ for $2q > 3$ if $i = 0$ and $L = \frac{1}{\ell_i^{2q-3}}$ for $2q > 3$ if $i = 1$.

Now, from (30), we prove the following cases for condition (i).

$$L = \ell_i^{-3}, i = 0 \qquad\qquad L = \frac{1}{\ell_i^{-3}}, i = 1$$

$$L = \alpha^{-3}, i = 0 \qquad\qquad L = \frac{1}{\alpha^{-3}}, i = 1$$

$$L = \alpha^{-3}, i = 0 \qquad\qquad L = \alpha^3, i = 1$$

$$\begin{aligned} \|\, f(w) - \mathcal{C}_\alpha(w)\,\| &\le \left(\frac{L^{1-i}}{1-L}\right)\Delta_\alpha(w,0) \\ &= \left(\frac{(\alpha^{-3})^{1-0}}{1-a^{-3}}\right)\cdot\frac{p}{2\alpha} \\ &= \left(\frac{\alpha^{-3}}{1-\alpha^{-3}}\right)\cdot\frac{p}{2\alpha} \\ &= \left(\frac{p}{2\alpha(\alpha^3-1)}\right) \end{aligned} \qquad \begin{aligned} \|\, f(w) - \mathcal{C}_\alpha(w)\,\| &\le \left(\frac{L^{1-i}}{1-L}\right)\Delta_\alpha(w,0) \\ &= \left(\frac{(\alpha^3)^{1-1}}{1-a^3}\right)\cdot\frac{p}{2\alpha} \\ &= \left(\frac{1}{1-\alpha^3}\right)\cdot\frac{p}{2\alpha} \\ &= \left(\frac{p}{2\alpha(1-\alpha^3)}\right) \end{aligned}$$

Also, from (30), we prove the following cases for condition (ii).

$$L = \ell_i^{q-3}, q < 3, i = 0 \qquad L = \frac{1}{\ell_i^{q-3}}, q > 3, i = 1$$

$$L = \alpha^{q-3}, q < 3, i = 0 \qquad L = \frac{1}{\alpha^{q-3}}, q < 3, i = 1$$

$$L = \alpha^{q-3}, q < 3, i = 0 \qquad L = \alpha^{3-q}, q > 3, i = 1$$

$$\begin{aligned} \| f(w) - \mathcal{C}_\alpha(w) \| &\leq \left(\frac{L^{1-i}}{1-L}\right)\Delta_\alpha(w,0) \\ &= \left(\frac{(\alpha^{q-3})^{1-0}}{1-\alpha^{q-3}}\right)\cdot\frac{p}{2\alpha\cdot\alpha^q} \\ &= \left(\frac{\alpha^{q-3}}{1-\alpha^{q-3}}\right)\cdot\frac{p}{2\alpha\cdot\alpha^q} \\ &= \left(\frac{\alpha^{q}}{\alpha^3-\alpha^{q}}\right)\cdot\frac{p}{2\alpha\cdot\alpha^q} \end{aligned} \qquad \begin{aligned} \| f(w) - \mathcal{C}_\alpha(w) \| &\leq \left(\frac{L^{1-i}}{1-L}\right)\Delta_\alpha(w,0) \\ &= \left(\frac{(\alpha^{3-q})^{1-1}}{1-\alpha^{3-q}}\right)\cdot\frac{p}{2\alpha\cdot\alpha^q} \\ &= \left(\frac{1}{1-\alpha^{3-q}}\right)\cdot\frac{p}{2\alpha\cdot\alpha^q} \\ &= \left(\frac{\alpha^{q}}{\alpha^q-\alpha^{3}}\right)\cdot\frac{p}{2\alpha\cdot\alpha^q} \end{aligned}$$

Finally, the proof of (30) for condition (iii) is similar to that of condition (ii). Hence the proof is complete. □

4. Stability of (10)

In this section, we present the generalized Ulam - Hyers - Rassias of the $\beta-$cubic functional equation. Throughout this section, we assume $\mathcal{W}$ be a normed space and $\mathcal{Z}$ be a Banach space.

4.1. Banach Space: Direct Method

Theorem 4.1. *Let $a = \pm 1$ and $\Delta_\beta : \mathcal{W}^2 \longrightarrow [0,\infty)$ be a function such that*

$$\sum_{b=0}^{\infty} \frac{\Delta_\beta\left(\beta^{ba}w, \beta^{ba}z\right)}{\beta^{3a}} \quad converges \quad in \quad \mathbb{R} \quad and \quad \lim_{b\to\infty} \frac{\Delta_\beta\left(\beta^{ba}w, \beta^{ba}z\right)}{\beta^{3a}} = 0 \tag{45}$$

for all $w, z \in \mathcal{W}$. Let $f : \mathcal{W} \longrightarrow \mathcal{Z}$ be a function fulfilling the inequality

$$\left\|\beta f(w+\beta z) - f(\beta w + z) - [\beta f(w-\beta z) - f(\beta w - z)] - 2(\beta^4-1)f(z)\right\| \leq \Delta_\beta(w,z) \tag{46}$$

for all $w, z \in \mathcal{W}$. Then there exists a unique cubic function $\mathcal{C}_\beta : \mathcal{W} \longrightarrow \mathcal{Z}$ which satisfies (10) and

$$\|f(w) - \mathcal{C}_\beta(w)\| \leq \frac{1}{2\beta^3} \sum_{b=\frac{1-a}{2}}^{\infty} \frac{\Delta_\beta\left(0, \beta^{ba}z\right)}{\beta^{3ba}} \tag{47}$$

where $\mathcal{C}_\beta(w)$ is defined by

$$\mathcal{C}_\beta(z) = \lim_{b\to\infty} \frac{f(\beta^{ba}z)}{\beta^{3ba}} \tag{48}$$

for all $z \in \mathcal{W}$.

Proof. **Case (i):** Assume $a = 1$.

Replacing (w, z) by $(0, z)$ in (46) and using oddness of f, we get

$$\left\|2\beta f(\beta z) - 2\beta^4 f(z)\right\| \leq \Delta_\beta(0,z) \tag{49}$$

for all $z \in \mathcal{W}$. Rewriting (49), we have

$$\left\|\frac{f(\beta z)}{\beta^3} - f(z)\right\| \leq \frac{\Delta_\beta(0,z)}{2\beta^3} \tag{50}$$

for all $w \in \mathcal{W}$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 4.1 concerning the stabilities of (10).

Corollary 4.2. *Let $f : \mathcal{W} \longrightarrow \mathcal{Z}$ be a mapping. If there exist real numbers p and q such that*

$$\left\|\beta f(w+\beta z) - f(\beta w+z) - [\beta f(w-\beta z) - f(\beta w - z)] - 2(\beta^4-1)f(z)\right\| \leq \begin{cases} p, \\ p\left\{||w||^q + ||z||^q\right\}, \\ p\left\{||w||^q||z||^q + \left\{||w||^{2q} + ||z||^{2q}\right\}\right\}, \end{cases} \tag{51}$$

for all $w, z \in \mathcal{W}$, then there exists a unique cubic function $\mathcal{C}_\beta : \mathcal{W} \longrightarrow \mathcal{Z}$ such that

$$\|f(w) - \mathcal{C}_\beta(w)\| \leq \begin{cases} \dfrac{p}{2|\beta^3-1|}, \\ \dfrac{p||w||^q}{2|\beta^3-\beta^q|}, & q \neq 3, \\ \dfrac{p||w||^{2q}}{2|\beta^3-\beta^{2q}|}, & 2q \neq 3, \end{cases} \tag{52}$$

for all $w \in \mathcal{W}$.

4.2. Banach Space: Fixed Point Method

Theorem 4.3. *Let $f : \mathcal{W} \longrightarrow \mathcal{Z}$ be a mapping for which there exists a function $\Delta_\beta : \mathcal{W}^2 \longrightarrow [0,\infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\ell_i^{3n}} \Delta_\beta(\ell_i^n w, \ell_i^n z) = 0 \tag{53}$$

where

$$\ell_i = \begin{cases} \beta \;\; if \;\; i = 0, \\ \frac{1}{\beta} \;\; if \;\; i = 1 \end{cases} \tag{54}$$

such that the functional inequality

$$\left\|\beta f(w+\beta z) - f(\beta w+z) - [\beta f(w-\beta z) - f(\beta w-z)] - 2(\beta^4-1)f(z)\right\| \leq \Delta_\beta(w,z) \tag{55}$$

holds for all $w, z \in \mathcal{W}$. Assume that there exists $L = L(i)$ such that the function

$$\Delta_\beta(0,z) = \frac{1}{2}\Delta_\beta\left(0, \frac{z}{\beta}\right)$$

with the property

$$\frac{1}{\ell_i^3}\Delta_\beta(\ell_i w, 0) = L\ \Delta_\beta(w,0) \tag{56}$$

for all $z \in \mathcal{W}$. Then there exists a unique cubic mapping $\mathcal{C}_\beta : \mathcal{W} \longrightarrow \mathcal{Z}$ satisfying the functional equation (10) and

$$\|f(z) - \mathcal{C}_\beta(z)\| \; \| \leq \left(\frac{L^{1-i}}{1-L}\right)\Delta_\beta(0,z) \tag{57}$$

for all $w \in \mathcal{W}$.

Proof. Consider the set

$$\mathcal{S} = \{f_b / f_b : \mathcal{W} \longrightarrow \mathcal{Z},\ f_b(0) = 0\}$$

and introduce the generalized metric $d : \mathcal{S} \times \mathcal{S} \to [0,\infty]$ as follows:

$$d(f, f_a) = \inf\{\omega \in (0,\infty) : \| f(z) - f_b(z) \| \leq \omega\ \Delta_\beta(0,z), z \in \mathcal{W}\}. \tag{58}$$

It is easy to show that $(\mathcal{S}, d)$ is complete with respect to the defined metric. Let us define the linear mapping $J : \mathcal{S} \longrightarrow \mathcal{S}$ by

$$Jf_b(x) = \frac{1}{\ell_i^3} f_b(\ell_i x),$$

for all $w \in \mathcal{W}$. □

Using Theorem 4.3, we prove the following corollary concerning the stabilities of (10).

Corollary 4.4. *Let $f : \mathcal{W} \longrightarrow \mathcal{Z}$ be a mapping. If there exists real numbers p and q such that*

$$\left\|\beta f(w+\beta z) - f(\beta w + z) - [\beta f(w-\beta z) - f(\beta w - z)] - 2(\beta^4-1)f(z)\right\| \leq \begin{cases} p, \\ p\left\{||w||^q + ||z||^q\right\}, \\ p\left\{||w||^q||z||^q + \left\{||w||^{2q} + ||z||^{2q}\right\}\right\}, \end{cases} \tag{59}$$

for all $w, z \in \mathcal{W}$, then there exists a unique cubic function $\mathcal{C}_\beta : \mathcal{W} \to \mathcal{Z}$ such that

$$\|f(z) - \mathcal{C}_\beta(z)\| \leq \begin{cases} \dfrac{p}{2\beta|\beta^3-1|}, \\ \dfrac{p||z||^q}{2\beta|\beta^3-\beta^q|}, & q \neq 3, \\ \dfrac{p||z||^{2q}}{2\beta|\beta^3-\beta^{2q}|}, & 2q \neq 3, \end{cases} \tag{60}$$

for all $w \in \mathcal{W}$.

4.3. Banach Space: Direct Method: Another Way

Theorem 4.5. *Let $a = \pm 1$ and $\Delta_\beta : \mathcal{W}^2 \longrightarrow [0, \infty)$ and $f : \mathcal{W} \longrightarrow \mathcal{Z}$ be functions satisfying (45) and (46) for all $w, z \in \mathcal{W}$. Then there exists a unique cubic function $\mathcal{C}_\beta : \mathcal{W} \longrightarrow \mathcal{Z}$ which satisfies (10) and*

$$\|f(z) - \mathcal{C}_\beta(z)\| \leq \frac{1}{\beta^3} \sum_{b=\frac{1-a}{2}}^{\infty} \frac{\Delta_\beta^G\left(\beta^{ba}z\right)}{\beta^{3ba}} \tag{61}$$

where $\Delta_\beta^G\left(\beta^{ba}z\right)$ and $\mathcal{C}_\beta(w)$ are defined by

$$\begin{aligned}\Delta_\beta^G\left(\beta^{ba}z\right) = \frac{1}{2\beta}\Big(\Delta_\beta\left(0, \beta^{ba}z\right) &+ \frac{1}{2(\beta^4-1)}\left[\Delta_\beta\left(0, \beta^{ba}z\right) + \Delta_\beta\left(0, -\beta^{ba}z\right)\right] \\ &+ \frac{\beta}{2(\beta^4-1)}\left[\Delta_\beta\left(0, \beta^{ba}\cdot\beta z\right) + \Delta_\beta\left(0, -\beta^{ba}\cdot\beta z\right)\right]\Big)\end{aligned} \tag{62}$$

and

$$\mathcal{C}_\beta(z) = \lim_{b\to\infty} \frac{f(\beta^{ba}z)}{\beta^{3ba}} \tag{63}$$

for all $z \in \mathcal{W}$.

Proof. **Case (i):** Assume $a = 1$. Setting (w, z) by $(0, z)$ in (46), we get

$$||\beta f(\beta z) - f(z) - [\beta f(-\beta z) - f(-z)] - 2(\beta^4-1)f(z)|| \leq \Delta_\beta(0, z) \tag{64}$$

for all $z \in \mathcal{W}$. Replacing z by $-z$ in (64), we have

$$||\beta f(-\beta z) - f(-z) - [\beta f(\beta z) - f(z)] - 2(\beta^4-1)f(-z)||\Delta_\beta(0, -z) \tag{65}$$

for all $z \in \mathcal{W}$. From (64) and (65), we arrive

$$\begin{aligned}||2(\beta^4-1)f(z)+2(\beta^4-1)f(-z)|| &\leq ||\beta f(\beta z)-f(z)-[\beta f(-\beta z)-f(-z)]-2(\beta^4-1)f(z)|| \\ &\quad + ||\beta f(-\beta z)-f(-z)-[\beta f(\beta z)-f(z)]-2(\beta^4-1)f(-z)|| \\ &\leq \Delta_\beta(0,z)+\Delta_\beta(0,-z) \end{aligned} \tag{66}$$

for all $z \in \mathcal{W}$. Rewriting (66), we arrive

$$||f(z)+f(-z)|| \leq \frac{1}{2(\beta^4-1)}\left[\Delta_\beta(0,z)+\Delta_\beta(0,-z)\right] \tag{67}$$

for all $z \in \mathcal{W}$. Replacing z by βz and multiplying both sides by β on (67), we land

$$\beta||f(\beta z)+f(-\beta z)|| \leq \frac{\beta}{2(\beta^4-1)}\left[\Delta_\beta(0,\beta z)+\Delta_\beta(0,-\beta z)\right] \tag{68}$$

for all $z \in \mathcal{W}$. With the help of (64), (67) and (68) we obtain

$$\begin{aligned}||2\beta f(\beta z)-2\beta^4 f(z)]|| &= ||\beta f(\beta z)+\beta f(\beta z)+\beta f(-\beta z)-\beta f(-\beta z)-f(z)-f(-z)-f(z)+f(-z)-2\beta^4 f(z)+2f(z)|| \\ &\leq ||\beta f(\beta z)-f(z)-[\beta f(-\beta z)-f(-z)]-2(\beta^4-1)f(z)|| \\ &\quad + ||-f(-z)-f(z)||+||\beta f(\beta z)+\beta f(-\beta z)|| \\ &\leq \Delta_\beta(0,z)+\frac{1}{2(\beta^4-1)}\left[\Delta_\beta(0,z)+\Delta_\beta(0,-z)\right]+\frac{\beta}{2(\beta^4-1)}\left[\Delta_\beta(0,\beta z)+\Delta_\beta(0,-\beta z)\right] \end{aligned} \tag{69}$$

for all $z \in \mathcal{W}$. It follows from (69), we get

$$\left\|\frac{f(\beta z)}{\beta^3}-f(z)\right\| \leq \frac{1}{2\beta^4}\left(\Delta_\beta(0,z)+\frac{1}{2(\beta^4-1)}\left[\Delta_\beta(0,z)+\Delta_\beta(0,-z)\right]+\frac{\beta}{2(\beta^4-1)}\left[\Delta_\beta(0,\beta z)+\Delta_\beta(0,-\beta z)\right]\right) \tag{70}$$

for all $z \in \mathcal{W}$. Define

$$\Delta_\beta^G(z)=\frac{1}{2\beta}\left(\Delta_\beta(0,z)+\frac{1}{2(\beta^4-1)}\left[\Delta_\beta(0,z)+\Delta_\beta(0,-z)\right]+\frac{\beta}{2(\beta^4-1)}\left[\Delta_\beta(0,\beta z)+\Delta_\beta(0,-\beta z)\right]\right) \tag{71}$$

for all $z \in \mathcal{W}$. Using (74) in (73), we arrive

$$\left\|\frac{f(\beta z)}{\beta^3}-f(z)\right\| \leq \frac{\Delta_\beta^G(z)}{\beta^3} \tag{72}$$

for all $z \in \mathcal{W}$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 4.5 concerning the stabilities of (10).

Corollary 4.6. *Let $f : \mathcal{W} \longrightarrow \mathcal{Z}$ be a mapping. If there exist real numbers p and q such that*

$$\left\|\beta f(w+\beta z)-f(\beta w+z)-[\beta f(w-\beta z)-f(\beta w-z)]-2(\beta^4-1)f(z)\right\| \leq \begin{cases} p, \\ p\left\{||w||^q+||z||^q\right\}, \\ p\left\{||w||^q||z||^q+\left\{||w||^{2q}+||z||^{2q}\right\}\right\}, \end{cases} \tag{73}$$

for all $w, z \in \mathcal{W}$, then there exists a unique cubic function $\mathcal{C}_\beta : \mathcal{W} \longrightarrow \mathcal{Z}$ such that

$$\|f(w)-\mathcal{C}_\beta(w)\| \leq \begin{cases} \dfrac{p(\beta^3+1)}{2(\beta^4-1)|\beta^3-1|}, \\ \dfrac{p(\beta^3+\beta^q)||z||^q}{2(\beta^4-1)|\beta^3-\beta^q|}, & q \neq 3, \\ \dfrac{p(\beta^3+\beta^{2q})||z||^{2q}}{2(\beta^4-1)|\beta^3-\beta^{2q}|}, & 2q \neq 3, \end{cases} \tag{74}$$

for all $w \in \mathcal{W}$.

4.4. Banach Space: Fixed Point Method: Another Way

Theorem 4.7. *Let $f : \mathcal{W} \longrightarrow \mathcal{Z}$ be a mapping for which there exists a function $\Delta_\beta : \mathcal{W}^2 \longrightarrow [0,\infty)$ with the condition (53) where ℓ_i is defined in (54) such that the functional inequality (55) holds for all $w, z \in \mathcal{W}$. Assume that there exists $L = L(i)$ such that the function*

$$\Delta_\beta^G(z) = \frac{1}{2}\Delta_\beta^G\left(\frac{z}{\beta}\right)$$

with the property

$$\frac{1}{\ell_i^3}\Delta_\beta^G(\ell_i z) = L\ \Delta_\beta^G(z) \tag{75}$$

for all $z \in \mathcal{W}$. Then there exists a unique cubic mapping $\mathcal{C}_\beta : \mathcal{W} \longrightarrow \mathcal{Z}$ satisfying the functional equation (10) and

$$\|f(z) - \mathcal{C}_\beta(z)\|\ \| \leq \left(\frac{L^{1-i}}{1-L}\right)\Delta_\beta^G(z) \tag{76}$$

for all $z \in \mathcal{W}$.

Proof. Consider the set $\mathcal{S} = \{f_c/f_c : \mathcal{W} \longrightarrow \mathcal{Z},\ f_c(0) = 0\}$ and introduce the generalized metric $d : \mathcal{S} \times \mathcal{S} \to [0,\infty]$ as follows:

$$d(f, f_c) = \inf\{\omega \in (0,\infty) : \| f(z) - f_c(z) \| \leq \omega\ \Delta_\beta^G(z), z \in \mathcal{W}\}. \tag{77}$$

It is easy to show that $(\mathcal{S}, d)$ is complete with respect to the defined metric. Let us define the linear mapping $J : \mathcal{S} \longrightarrow \mathcal{S}$ by $Jf_c(z) = \frac{1}{\ell_i^3} f_c(\ell_i z)$, for all $z \in \mathcal{W}$. □

Using Theorem 4.7, we prove the following corollary concerning the stabilities of (10).

Corollary 4.8. *Let $f : \mathcal{W} \longrightarrow \mathcal{Z}$ be a mapping. If there exist real numbers p and q such that*

$$\left\|\beta f(w+\beta z) - f(\beta w + z) - [\beta f(w-\beta z) - f(\beta w - z)] - 2(\beta^4-1)f(z)\right\| \leq \begin{cases} p, \\ p\left\{||w||^q + ||z||^q\right\}, \\ p\left\{||w||^q||z||^q + \left\{||w||^{2q} + ||z||^{2q}\right\}\right\}, \end{cases} \tag{78}$$

for all $w, z \in \mathcal{W}$, then there exists a unique cubic function $\mathcal{C}_\beta : \mathcal{W} \to \mathcal{Z}$ such that

$$\|f(z) - \mathcal{C}_\beta(z)\| \leq \begin{cases} \dfrac{p}{2\beta|\beta^3-1|}, & \\ \dfrac{p||z||^q}{2\beta|\beta^3-\beta^q|}, & q \neq 3, \\ \dfrac{p||z||^{2q}}{2\beta|\beta^3-\beta^{2q}|}, & 2q \neq 3, \end{cases} \tag{79}$$

for all $w \in \mathcal{W}$.

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