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An New Extended Beta Function

Research Article

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Abstract: In this paper, we will present a new and modified extended classical Gamma, Beta functions. Some properties basic are studied, such as the integral representations, functional relations and we will evaluate the action of the Mellin transform of the new modified extended Beta function.

Keywords: Extended Beta, Gamma functions, Mittag-Leffler function, Mellin Transform.

1. Introduction

Recently the generalized function $B_p^{(\alpha,\beta,\kappa,\mu)}(x,y)$ is introduced by Srivastava, with is the most gereralized extension of the classical beta function and is defined as (see [4, 5]):

$$B_p^{(\alpha,\beta,\kappa,\mu)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha,\beta,-\frac{p}{t^{\kappa}(1-t)^{\mu}}\right)dt \tag{1}$$

where

$$\kappa \geq 0, \mu \geq 0, min\left\{R_e(\alpha), R_e(\beta)\right\} > 0, R_e(x) > R_e(\kappa), R_e(y) > R_e(\mu)$$

and ${}_1F_1(,)$ is the confluent hypergeometric function, which a is special case of the well know generalized hypergeometric series ${}_pF_q(,)$. The generalized hypergeometric series ${}_pF_q(,)$, $p,q \in \mathbb{N}$ is defined as:

$${}_pF_q\left[\alpha_1,\alpha_2\dots\alpha_p\right] = \begin{bmatrix} \alpha_1 & \alpha_2 & \dots\alpha_p \\ \beta_1 & \beta_2 & \dots\beta_q, \end{bmatrix}(z) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\dots(\alpha_p)_n}{(\beta_1)_n\dots(\beta_p)_n}\frac{z^n}{n!} \tag{2}$$

where $(\lambda)_n$ is the Pochhammer symbol defined by:

$$(\lambda)_n = \begin{cases} 1 & \text{if } n=0 \\ \lambda(\lambda+1\dots(\lambda+n-1)) & \text{if } n\in\mathbb{N} \end{cases}$$

If $\kappa=\mu$ (1) reduced to the generalized extended beta function defined by (see [3, 4, 6])

$$B_p^{(\alpha,\beta,\mu)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha,\beta,-\frac{p}{t^{\mu}(1-t)^{\mu}}\right)dt \tag{3}$$

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where

$$R_e(p) \geq 0, min\left\{R_e(x), R_e(y), R_e(\alpha), R_e(\beta), R_e(\mu)\right\} > 0$$

If $\mu = 1$ (3) reduced to generalized beta type function as follows (see [2])

$$B_p^{(\alpha,\beta,1)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} {}_1F_1\left(\alpha, \beta, -\frac{p}{t(1-t)}\right) dt \tag{4}$$

where

$$R_e(p) \geq 0, min\left\{R_e(x), R_e(y), R_e(\alpha), R_e(\beta)\right\} > 0$$

If $\alpha = \beta$ (4) reduces to the generalized beta type function due to Chaudhry et al (see [6]) and defined as:

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}} dt \tag{5}$$

where

$$R_e(p) \geq 0, min\left\{R_e(x), R_e(y)\right\} > 0$$

Note that if $p = 0$ all the above extensions reduced to the classical beta function $B(x,y)$ which is defined by:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt \tag{6}$$

where

$$R_e(x) > 0, R_e(y) > 0$$

In particular, Euler's beta function $B(x,y)$ has a close relationship to the gamma function.

$$B(x,y) = \frac{\Gamma(x) \cdot \Gamma(y)}{\Gamma(x+y)} \tag{7}$$

2. Main Result

In this section, we introduce a new modified extension of the classical the gamma and beta functions. Consider some of its properties such as functional relation,summations relations and integral representations and we will evaluate the action of the Mellin transform of the new modified extended beta function.

Definition 2.1 ([1]). *Let $\alpha \in \mathbb{R}^+$, $x \in \mathbb{C}$ be such that $R_e(x) > 0$. Then, the new classical gamma function is defined as:*

$$\Gamma^{\alpha}(x) = \int_0^{\infty} t^{x-1} E_{\alpha}(-t) dt \tag{8}$$

where

$$E_{\alpha}(-t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{\Gamma(\alpha n + 1)} \quad \textit{Mittag-Leffler function}$$

Note that if $\alpha = 1$ we have to $\Gamma^{\alpha}(x) = \Gamma(x)$

Lemma 2.2. *Let $x \in \mathbb{C}$, $R_e(x) > 0$ and $\alpha \in \mathbb{R}^+$ Then:*

$$\Gamma^{\alpha}(x) = \frac{\Gamma(x+1) \cdot \Gamma(1-(x+1))}{\Gamma(1-\alpha(x+1))} \tag{9}$$

Proof. Let $v = x + 1$

$$\Gamma^{\alpha}(x+1) = \Gamma^{\alpha}(v) = \int_0^{\infty} t^{v-1} E_{\alpha}(-t)dt = M\left\{E_{\alpha}(-t)\right\}(v) = \frac{\Gamma(v) \cdot \Gamma(1-v)}{\Gamma(1-\alpha v)}$$

where $M\left\{f(t)\right\}(s)$ is Mellin transform. Reeplacing $v = x + 1$ this completes the proof. □

Definition 2.3. *Let $b \geq 0$, $\alpha \in \mathbb{R}^+$, x, $y \in \mathbb{C}$ be such that $R_e(x) > 0$, $R_e(y) > 0$. Then, the new and modified extension of classical beta function is defined as:*

$$B_b^{\alpha}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} E_{\alpha}(-bt(1-t))dt \tag{10}$$

Note that if $\alpha = 1$ and $b = 0$ $B_b^{\alpha}(x,y) = B(x,y)$ the classical beta function.

Lemma 2.4 (Functional Relations)**.** *Let $b \geq 0$, $\alpha \in \mathbb{R}^+$, x, $y \in \mathbb{C}$ be such that $R_e(x+1) > 0$ and $R_e(y+1) > 0$. Then:*

$$B_b^{\alpha}(x, y+1) + B_b^{\alpha}(x+1, y) = B_b^{\alpha}(x,y) \tag{11}$$

Proof.

$$\begin{aligned} B_b^{\alpha}(x, y+1) + B_b^{\alpha}(x+1, y) &= \int_0^1 t^{x-1}(1-t)^{y} E_{\alpha}(-bt(1-t))dt + \int_0^1 t^{x}(1-t)^{y-1} E_{\alpha}(-bt(1-t))dt \\ &= \int_0^1 \left(t^{x-1}(1-t)^{y} + t^{x}(1-t)^{y-1}\right) E_{\alpha}(-bt(1-t))dt \\ &= \int_0^1 t^{x-1}(1-t)^{y-1} E_{\alpha}(-bt(1-t))dt \\ &= B_b^{\alpha}(x,y) \end{aligned}$$

□

Lemma 2.5 (Relations Summations)**.** *Let $b \geq 0$, $\alpha \in \mathbb{R}^+$, x, $y \in \mathbb{C}$ be such that $R_e(x) > 0$, $R_e(1-y) > 0$. Then, the new modified extension classical Beta function have the following summation relation:*

$$B_b^{\alpha}(x, 1-y) = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B_b^{\alpha}(x+n, 1) \tag{12}$$

Proof.

$$B_b^{\alpha}(x, 1-y) = \int_0^1 t^{x-1}(1-t)^{-y} E_{\alpha}(-bt(1-t))dt$$

Using the binomial series expansion:

$$(1-t)^{-y} = \sum_{n=0}^{\infty} \frac{(y)_n \cdot t^n}{n!}, \quad if|t| < 1$$

we obtain

$$B_b^{\alpha}(x, 1-y) = \int_0^1 t^{x-1} \sum_{n=0}^{\infty} \frac{(y)_n \cdot t^n}{n!} E_{\alpha}(-bt(1-t))dt$$

By the uniform convergence of the series, we can interchange the orden of integration and summation, we obtain

$$\begin{aligned} B_b^{\alpha}(x, 1-y) &= \sum_{n=0}^{\infty} \frac{(y)_n}{n!} \int_0^1 t^{x+n-1} E_{\alpha}(-bt(1-t))dt \\ &= \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B(x+n, 1) \end{aligned}$$

□

Lemma 2.6. *Let $b \geq 0$, $\alpha \in \mathbb{R}^+$, x, $y \in \mathbb{C}$ be such that $R_e(x) > 0$, $R_e(y) > 0$. Then the new modified extension classical Beta function have the following relation:*

$$B_b^\alpha(x,y) = \sum_{n=0}^{\infty} B_b^\alpha(x+n, y+1) \tag{13}$$

Proof.

$$\begin{aligned} B_b^\alpha(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}E_\alpha(-bt(1-t))dt \\ &= \int_0^1 t^{x-1}(1-t)^{y}(1-t)^{-1}E_\alpha(-bt(1-t))dt \end{aligned}$$

Using the binomial serieS expansion $(1-t)^{-1} = \sum\limits_{n=0}^{\infty} t^n, \quad if\ |t| < 1$

$$B_b^\alpha(x,y) = \int_0^1 t^{x-1}(1-t)^{y} \sum_{n=0}^{\infty} t^n E_\alpha(-bt(1-t))dt$$

By the uniform convergence of the series, we can interchange the orden of integration and summation, we obtain

$$\begin{aligned} B_b^\alpha(x,y) &= \sum_{n=0}^{\infty} \int_0^1 t^{x+n-1}(1-t)^{y}E_\alpha(-bt(1-t))dt \\ &= \sum_{n=0}^{\infty} B_\alpha^\alpha(x+n, y+1) \end{aligned}$$

□

Lemma 2.7 (Mellin Transform)**.** *Let $b \geq 0$, $\alpha \in \mathbb{R}^+$ $s \in \mathbb{C}$ such that $R_e(s) > 0$, $R_e(x-s) > 0$ and $R_e(y-s) > 0$. Then Mellin transform representation the new modified extension classical beta function is given by*

$$M\left\{B_b^\alpha(x,y)\right\}(s) = B(x-s, y-s)\Gamma^\alpha(s) \tag{14}$$

Proof.

$$M\left\{B_b^\alpha(x,y)\right\}(s) = \int_0^\infty b^{s-1}\left(\int_0^1 t^{x-1}(1-t)^{y-1}E_\alpha(-bt(1-t))dt\right)db \tag{15}$$

From the uniform convergence of the integral, the orden of integration can be interchanged. Thus, we have

$$M\left\{B_b^\alpha(x,y)\right\}(s) = \int_0^1 t^{x-1}(1-t)^{y-1}b^{s-1}\int_0^\infty b^{s-1}E_\alpha(-bt(1-t))dtdb \tag{16}$$

Making the following change of variable $v = bt(1-t)$ and $\lambda = t$ and replacing in (16) it result

$$\begin{aligned} M\left\{B_b^\alpha(x,y)\right\}(s) &= \int_0^1 \lambda^{x-s-1}(1-\lambda)^{y-s-1}d\lambda \times \int_0^\infty v^{s-1}E_\alpha(-v)dv \\ &= \int_0^1 \lambda^{x-s-1}(1-\lambda)^{y-s-1}d\lambda \cdot \Gamma^\alpha(s) \\ &= B(x-s, y-s)\cdot\Gamma^\alpha(s) \end{aligned} \tag{17}$$

□

Remark 2.8. *Putting $s = 1$, in (17) we get*

$$M\left\{B_b^\alpha(x,y)\right\}(s) = \int_0^\infty B_b^\alpha(x,y)db = B(x-1,y-1)\Gamma^\alpha(1)$$

Thus, if $\alpha = 1$

$$\int_0^\infty B_b^\alpha(x,y)db = B(x-1,y-1) \tag{18}$$

where $R_e(x-1) > 1, R_e(y-1) > 1$

Lemma 2.9 (Integral Representations)**.** *Let $b \geq 0$, $\alpha \in \mathbb{R}^+$, $x,\ y \in \mathbb{C}$ be such that $R_e(x) > 0,\ R_e(y) > 0$. Then, the new modified extension classical Beta function have the following integral representations:*

$$B_b^\alpha(x,y) = 2\int_0^{\frac{\pi}{2}} cos^{2x-1}\theta sin^{2y-1}\theta E_\alpha(-bcos^2\theta sin^2\theta)d\theta \tag{19}$$

$$B_b^\alpha(x,y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}} \cdot E_\alpha(-b\frac{u}{(1+u)^2})du \tag{20}$$

$$B_b^\alpha(x,y) = (c-a)^{1-x-y}\int_a^c (u-a)^{x-1}(c-u)^{y-1} \times E_\alpha\left(-b\frac{(u-a)(c-u)}{(c-a)^2}\right)du \tag{21}$$

$$B_b^\alpha(x,y) = 2^{1-x-y}\int_{-1}^1 (1+u)^{x-1}(1-u)^{y-1}E_\alpha\left(-b\frac{(1-u)^2}{4}\right)du \tag{22}$$

Proof. In (10) letting $t = cos^2\theta$, then $dt = -2cos\theta \cdot sin\theta d\theta$ when $t = 0 : \theta = \frac{\pi}{2}$ and $t = 1 : \theta = 0$. Therefore

$$\begin{aligned} B_b^\alpha(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}E_\alpha(-bt(1-t))dt \\ &= 2\int_0^{\frac{\pi}{2}} cos^{2x-2}\theta sin^{2y-2}\theta E_\alpha(-bcos^2\theta sin^2\theta)cos\theta sin\theta d\theta \\ &= 2\int_0^{\frac{\pi}{2}} cos^{2x-1}\theta sin^{2y-1}\theta E_\alpha(-bcos^2\theta sin^2\theta)d\theta \end{aligned}$$

In (10) letting $t = \frac{u}{1+u}$, then $dt = \frac{1}{(1+u)^2}du$ when $t = 0 : u = 0$ and $t = 1 : u \to \infty$. Therefore

$$\begin{aligned} B_b^\alpha(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}E_\alpha(-bt(1-t))dt \\ &= \int_0^\infty \frac{u^{x-1}}{(1+u)^{x-1}} \cdot \frac{1}{(1+u)^{y-1}} E_\alpha\left(-b\frac{u}{(1+u)^2}\right)\frac{1}{(1+u)^2}du \\ &= \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}} \cdot E_\alpha\left(-b\frac{u}{(1+u)^2}\right)du \end{aligned}$$

In (10) letting $t = \frac{(u-a)}{(c-a)}$ then $dt = \frac{1}{(c-a)}du$ when $t = 0 : u = a$ and $t = 1 : u = c$. Therefore

$$\begin{aligned} B_b^\alpha(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}E_\alpha(-bt(1-t))dt \\ &= \int_a^c \frac{(u-a)^{x-1}}{(c-a)^{x-1}} \cdot \frac{(c-u)^{y-1}}{(c-a)^{y-1}} E_\alpha\left(-b\frac{(u-a)(c-u)}{(c-a)^2}\right)\frac{1}{(c-a)}du \\ &= (c-a)^{1-x-y}\int_a^c (u-a)^{x-1}(c-u)^{y-1}E_\alpha\left(-b\frac{(u-a)(c-u)}{(c-a)^2}\right)du \end{aligned}$$

If (10) taking $a = -1$ and $c = 1$, we obtain

$$\begin{aligned} B_b^\alpha(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}E_\alpha(-bt(1-t))dt \\ &= 2^{1-x-y}\int_{-1}^1 (u+1)^{x-1}(1-u)^{y-1}E_\alpha\left(-b\frac{(1-u^2)}{4}\right)du \end{aligned}$$

□

Theorem 2.10. *For the product of two new modified extension gamma function, we have the following integral representation:*

$$\Gamma^\alpha(x)\Gamma^\alpha(y) \;=\; 4\int_0^{\frac{\pi}{2}}\int_0^\infty r^{2(x+y-1)} rcos^{2x-1}\theta sin^{2y-1}\theta \times E_\alpha\left(-bcos^2\theta\right) E_\alpha\left(-bsin^2\theta\right) drd\theta$$

Proof. Substituting $t=\eta^2$ and $t=\xi^2$ in (10), we obtain

$$\Gamma^\alpha(x) = 2\int_0^\infty \eta^{2x-1}E_\alpha(-\eta^2)d\eta \text{ and} \tag{23}$$

$$\Gamma^\alpha(x) = 2\int_0^\infty \xi^{2y-1}E_\alpha(-\xi^2)d\xi \tag{24}$$

From (23) and (24), we obtain

$$\Gamma^\alpha(x)\Gamma^\alpha(y) = 4\int_0^\infty\int_0^\infty \xi^{2x-1}\xi^{2y-1}E_\alpha(-\eta^2)E_\alpha(-\xi^2)d\eta d\xi$$

Letting $\eta = rcos\theta$ and $\xi = rsin\theta$ in the above equality, we get

$$\Gamma^\alpha(x)\Gamma^\alpha(y) \;=\; 4\int_0^{\frac{\pi}{2}}\int_0^\infty r^{2(x+y-1)} rcos^{2x-1}\theta sin^{2y-1}\theta \times E_\alpha\left(-bcos^2\theta\right) E_\alpha\left(-bsin^2\theta\right) drd\theta$$

□

Remark 2.11. *Note that if* $\alpha = 1$ *(23) is reduced*

$$\Gamma(x)\Gamma(y) = 4\int_0^{\frac{\pi}{2}}\int_0^\infty r^{2(x+y-1)} rcos^{2x-1}\theta sin^{2y-1}\theta\cdot e^{-r^2}drd\theta$$

Letting $t=r^2$, *we get*

$$\begin{aligned}\Gamma(x)\Gamma(y) &= 4\int_0^{\frac{\pi}{2}}\int_0^\infty r^{2(x+y-1)} rcos^{2x-1}\theta sin^{2y-1}\theta\cdot e^{-r^2}drd\theta \\ &= 2\int_0^{\frac{\pi}{2}}\int_0^\infty t^{(x+y-1)} cos^{2x-1}\theta sin^{2y-1}\theta\cdot e^{-t}drd\theta \\ &= 2\int_0^{\frac{\pi}{2}} cos^{2x-1}\theta sin^{2y-1}\theta d\theta\cdot\int_0^\infty t^{(x+y-1)}e^{-t}dt \\ &= B(x,y)\cdot\Gamma(x+y)\end{aligned}$$

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