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Suzuki Type Fixed Point Theorems in S-metric Space

Research Article

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Abstract: Inspired by the work of Suzuki [19] and Sedghi et al. [16], we proved some Suzuki-type fixed point theorems in S-metric space which generalized the result of Sedghi et al. [16] and Edelstein [7]. Our first theorem also characterizes the completeness of S-metric space. Moreover, we gave an application for a certain class of functional equations arising in Dynamical programming and some examples to illustrate our results.

MSC: 47H10, 54H25.

Keywords: Fixed point, Suzuki contraction, S-metric space, Functional equation.

1. Introduction

The Banach contraction principle is the most celebrated and well-known result in the field of fixed point theory and has many generalization in various directions. We also know that the Banach contraction principle does not characterize the metric completeness of underlying spaces. In 2008, Suzuki [19] gave a remarkable result, which generalized the Banach contraction principle and does characterize the metric completeness. This remarkable result is given as follows:

Theorem 1.1 ([19])**.** *Let (X,d) be a complete metric space and T be a mapping on X. Define a non-increasing function ϕ from $[0,1)$ into $(\frac{1}{2},1]$ by*

$$\phi(r)=\begin{cases} 1 & \text{if } 0\le r\le \frac{\sqrt{5}-1}{2}\\ \frac{1-r}{r^2} & \text{if } \frac{\sqrt{5}-1}{2}\le r\le \frac{1}{\sqrt{2}}\\ \frac{1}{1-r} & \text{if } \frac{1}{\sqrt{2}}\le r<1.\end{cases}$$

Assume that there exists $r\in[0,1)$ such that

$$\phi(r)d(x,Tx)\le d(x,y) \text{ implies } d(Tx,Ty)\le rd(x,y)$$

for all $x,y\in X$. Then there exists a unique fixed point z of T. Moreover $\lim\limits_{n\to\infty} T^n x = z$ for all $x\in X$.

Following Theorem 1.1, Suzuki generalized the well-known result of Edelstein [7], which states “On a compact metric space, every contractive map possesses a unique fixed point ”. This result is given as follows:

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Theorem 1.2 ([20]). *Let (X, d) be a compact metric space and let T be a mapping on X. Assume that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \text{ implies } d(Tx, Ty) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point.

The work of Suzuki [19, 20] provides a new methodology of proof, which attracts several authors to do work along this line. Therefore several authors have extended and generalized the result of Suzuki in different directions (see for instance [1–4, 12, 13, 17, 18] and references therein). Recently Sedghi et al. [16] introduced S-metric space as a generalization of metric spaces. They also proved some properties of S-metric spaces and gave some fixed point theorems for self-maps. Since then, many authors proved several fixed point theorems under different contractive condition in S-metric space (see [5, 6, 8–11, 14, 15] etc.). In this paper, we proved some Suzuki-type fixed point theorems in S-metric space, which generalized the result of Sedghi et al. [16] and Edelstein [7]. Our first theorem also characterizes the completeness of S-metric space. Moreover, we gave an application for a certain class of functional equations arising in Dynamical programming and some examples to illustrate our results.

2. Preliminaries

In this section we recall, some basic definitions and results of S-metric space, which we need in this sequel. Throughout the paper, we shall assume that (X, d) be a metric space and $\mathbb{N}$ will denote the set of natural numbers.

Definition 2.1 ([16]). *Let X be a non-empty set. An S-metric on X is a function $S : X \times X \times X \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,*

(S_1) $S(x, y, z) \geq 0$,

(S_2) $S(x, y, z) = 0$ *if and only if* $x = y = z$,

(S_3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

Some immediate examples of such S-metric spaces are given in [16] as follows:

Example 2.2. *Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X.*

Example 2.3. *Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x, y, z) = \|x - z\| + \|y - z\|$ is an S-metric on X.*

Example 2.4. *Let X be a nonempty set, d is ordinary metric on X, then $S(x, y, z) = d(x, z) + d(y, z)$ is an S-metric on X.*

Lemma 2.5 ([16]). *In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.*

Definition 2.6 ([16]). *Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$, the open ball $B_S(x, r)$ and closed ball $B_S[x, r]$ with a center x and a radius r is defined as follow:*

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\},$$

$$B_S[x, r] = \{y \in X : S(y, y, x) \leq r\}.$$

Definition 2.7 ([10]). *Let (X,S) be an S-metric space and A be a nonempty subset of X. The diameter of A is defined by*

$$diam\{A\} = \sup\{S(x,y,y) : x,y \in A\}. \tag{1}$$

If A is S-bounded, then $diam\{A\} < \infty$.

Definition 2.8 ([16]). *Let (X,S) be an S-metric space.*

(1) If for every $x \in A$ there exit $r > 0$ such that $B_S(x,r) \subset A$, then the subset A is called an open subset of X.

(2) A subset A of X is said to be S-bounded if there exists $r > 0$ such that $S(x,x,y) < r$ for all $x,y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n,x_n,x) < \epsilon$ and we denote this by $\lim\limits_{n\to\infty} x_n = x$.

(4) A sequence $\{x_n\}$ in X is called Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n,x_n,x_m) < \epsilon$.

(5) The S-metric space (X,S) is said to be complete if every Cauchy sequence is convergent.

Lemma 2.9 ([16]). *Let (X,S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$, then $\lim\limits_{n\to\infty} S(x_n,x_n,y_n) = S(x,x,y)$.*

The relationship between a metric and an S-metric was shown in [9] as follow:

Lemma 2.10. *Let (X,d) be a metric space. Then the following properties are satisfied:*

(1) $S_d(x,y,z) = d(x,z) + d(y,z)$ for all $x,y,z \in X$ is an S-metric on X.

(2) $x_n \to x$ in (X,d) if only if $x_n \to x$ in (X,S_d).

(3) $\{x_n\}$ is Cauchy in (X,d) if and only if $\{x_n\}$ is Cauchy in (X,S_d).

(4) (X,d) is complete if and only if (X,S_d) is complete.

The metric S_d is called as the S-metric generated by d (see [11]).

3. Fixed Point Theorems

Now we state our main results.

Theorem 3.1. *Let (X,S) be a complete S-metric space and T be a mapping on X. Define a non-increasing function θ from $[0,1)$ onto $(\frac{1}{3},1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{1}{2} \\ \frac{1-r}{2r^2} & \text{if } \frac{1}{2} < r < \frac{1}{3} \\ \frac{1}{2+r} & \text{if } \frac{1}{\sqrt{3}} \leq r < 1. \end{cases} \tag{2}$$

Assume that there exists $r \in [0,1)$ such that

$$\theta(r)S(x,x,Tx) \leq S(x,x,y) \text{ implies } S(Tx,Tx,Ty) \leq rS(x,x,y) \tag{3}$$

for all $x,y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim\limits_{n\to\infty} T^n = z$ for all $x \in X$.

Proof. Since $\theta(r) \leq 1$, therefore $\theta(r)S(x,x,Tx) \leq S(x,x,Tx)$ holds for every $x \in X$. By hypothesis (3),

$$S(Tx,Tx,T^2x) \leq rS(x,x,Tx) \tag{4}$$

for all $x \in X$. Suppose u be an arbitrary point in X and define a sequence $\{u_n\}$ in X by $u_n = T^n u$. Then (4) yields

$$\begin{aligned} S(u_n,u_n,u_{n+1}) =& S(T^n u, T^n u, T^{n+1}u) \\ \leq& rS(T^{n-1}u, T^{n-1}u, T^n u) \\ &\vdots \\ \leq& r^n S(u,u,Tu). \end{aligned}$$

Hence

$$\begin{aligned} S(u_n,u_n,u_m) =& S(T^n u, T^n u, T^m u) \\ \leq& 2\sum_{k=n}^{m-1} S(T^k u, T^k u, T^{k+1}u) + S(T^m u, T^m u, T^{m+1}u) \\ \leq& 2\sum_{k=n}^{m} S(T^k u, T^k u, T^{k+1}u) \leq 2\sum_{k=n}^{m} r^k S(u,u,Tu) \\ \leq& \frac{2r^n}{1-r} S(u,u,Tu) \end{aligned}$$

for all $n,m \in \mathbb{N}$ with $m > n$. This implies that $\{u_n\}$ is a Cauchy sequence in X. Since X is complete, so there exists a point $z \in X$ such that $\{u_n\}$ converges to z. Now we will prove that

$$S(z,z,Tx) \leq rS(z,z,x) \text{ for all } x \in X - \{z\}. \tag{5}$$

Since sequence $\{u_n\}$ converges to z, therefore for fix $x \in X - \{z\}$, there exists $n_0 \in \mathbb{N}$ such that

$$S(u_n,u_n,z) \leq \frac{1}{7}S(x,x,z), \text{ for all } n \geq n_0.$$

Then in view of Lemma 2.5, we have

$$\begin{aligned} \theta(r)S(u_n,u_n,Tu_n) \leq& S(u_n,u_n,Tu_n) = S(u_n,u_n,u_{n+1}) \\ \leq& 2S(u_n,u_n,z) + S(u_{n+1},u_{n+1},z) \\ \leq& \frac{3}{7}S(x,x,z) = \frac{1}{2}\left[S(x,x,z) - \frac{1}{7}S(x,x,z)\right] \\ \leq& \frac{1}{2}\left[S(x,x,z) - S(u_n,u_n,z)\right] \\ \leq& S(u_n,u_n,x) \text{ for all } n \geq n_0. \end{aligned}$$

By hypothesis (3),

$$S(u_{n+1},u_{n+1},Tx) \leq rS(u_n,u_n,x) \text{ for all } n \geq n_0.$$

Making $n \to \infty$, we get

$$S(z,z,Tx) \leq rS(z,z,x) \text{ for all } x \in X - \{z\}.$$

Thus we obtained (5). Now we will prove that z is a fixed point of T. Arguing by contradiction, we assume that $T^j z \neq z$ for all $j \in \mathbb{N}$. Then (5) yields

$$S(z,z,T^{j+1}z) \leq r^j S(z,z,Tz) \text{ for all } j \in \mathbb{N}. \tag{6}$$

Now we consider the following three cases:

(i) $0 \leq r \leq \frac{1}{2}$

(ii) $\frac{1}{2} < r < \frac{1}{\sqrt{3}}$

(iii) $\frac{1}{\sqrt{3}} \leq r < 1$

In case (i), where $0 \leq r \leq \frac{1}{2}$, we note that $2r^2 + r - 1 \leq 0$ and $3r^2 < 1$. If we assume that $S(z,z,T^2z) < S(T^3z,T^3z,T^2z)$. Then in view of Lemma 2.5, we have

$$\begin{aligned}
S(z,z,Tz) &\leq 2S(z,z,T^2z) + S(Tz,Tz,T^2z) \\
&< 2S(T^3z,T^3z,T^2z) + S(Tz,Tz,T^2z) \\
&\leq 2r^2 S(Tz,Tz,z) + rS(z,z,Tz) \\
&= 2r^2 S(z,z,Tz) + rS(z,z,Tz) \\
&\leq S(z,z,Tz),
\end{aligned}$$

which is a contradiction. Hence

$$S(z,z,T^2z) \geq S(T^3z,T^3z,T^2z) = \theta(r)S(T^3z,T^3z,T^2z).$$

By the hypothesis (3) and (6), we have

$$\begin{aligned}
S(z,z,Tz) &\leq 2S(z,z,T^3z) + S(Tz,Tz,T^3Z) \\
&\leq 2r^2 S(z,z,Tz) + rS(z,z,T^2z) \\
&\leq 2r^2 S(z,z,Tz) + r^2 S(z,z,Tz) \\
&< S(z,z,Tz).
\end{aligned}$$

This is a contradiction. In the case (ii), where $\frac{1}{2} < r < \frac{1}{\sqrt{3}}$, we note that $3r^2 < 1$. If we assume $S(z,z,T^2z) < \theta(r)S(T^3z,T^3z,T^2z)$. Then in view of (4) and Lemma 2.5, we have

$$\begin{aligned}
S(z,z,Tz) &\leq 2S(z,z,T^2z) + S(Tz,Tz,T^2z) \\
&< 2\theta(r)S(T^3z,T^3z,T^2z) + S(Tz,Tz,T^2z) \\
&\leq 2\theta(r)r^2 S(Tz,Tz,z) + rS(z,z,Tz) \\
&= 2\theta(r)r^2 S(z,z,Tz) + rS(z,z,Tz) \\
&= S(z,z,Tz),
\end{aligned}$$

which is a contradiction. Hence $S(z,z,T^2z) \geq \theta(r)S(T^3z,T^3z,T^2z)$. As in the previous case, we can prove that

$$S(z,z,Tz) \leq 3r^2 S(z,z,Tz) < S(z,z,Tz).$$

This is a contradiction. In case (iii), where $\frac{1}{\sqrt{3}} \leq r < 1$, we note that for all $x, y \in X$, either

$$\theta(r)S(x,x,Tx) \leq S(x,x,y) \text{ or } \theta(r)S(Tx,Tx,T^2x) \leq S(Tx,Tx,y)$$

holds. Indeed if

$$\theta(r)S(x,x,Tx) > S(x,x,y) \text{ and } \theta(r)S(Tx,Tx,T^2x) > S(Tx,Tx,y).$$

Then we have

$$\begin{aligned}
S(x,x,Tx) &\leq 2S(x,x,y) + S(Tx,Tx,y) \\
&< \theta(r)\left[2S(x,x,Tx) + S(Tx,Tx,T^2x)\right] \\
&\leq \theta(r)\left[2S(x,x,Tx) + rS(x,x,Tx)\right] \\
&= S(x,x,Tx)
\end{aligned}$$

which is a contradiction. Hence either

$$\theta(r)S(u_{2n},u_{2n},u_{2n+1}) \leq S(u_{2n},u_{2n},z) \text{ or } \theta(r)S(u_{2n+1},u_{2n+1},u_{2n+2}) \leq S(u_{2n+1},u_{2n+1},z)$$

holds for every $n \in \mathbb{N}$. By (3), it follows that either

$$S(u_{2n+1},u_{2n+1},Tz) \leq rS(u_{2n},u_{2n},z) \text{ or } S(u_{2n+2},u_{2n+2},Tz) \leq rS(u_{2n+1},u_{2n+1},z)$$

holds for every $n \in \mathbb{N}$. Since sequence $\{u_n\}$ converges to z, the above inequalities imply there exits a sub-sequence of $\{u_n\}$ which converges to Tz. This implies $Tz = z$, which is a contradiction. Therefore in all the cases there exists $j \in \mathbb{N}$ such that $T^j z = z$. That is z is a fixed point of T. Since $\{T^n z\}$ is a Cauchy sequence, we obtain $Tz = z$.

Now we shall show that T has a unique fixed point. Arguing by contradiction, we assume that w is another fixed point of T, that is, $Tw = w$ and $w \neq z$. Since

$$\theta(r)S(z,z,Tz) = 0 \leq S(z,z,w).$$

Hence by hypothesis (3),

$$S(z,z,w) = S(Tz,Tz,Tw) \leq rS(z,z,w) < S(z,z,w).$$

which is a contradiction. Thus T has unique fixed point, that is $z = w$. □

Example 3.2. *Let $X = [0,2]$ be endowed with S-metric defined by*

$$S(x,y,z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x,y,z\} & \text{if } x \neq y \neq z. \end{cases}$$

Then it is easy to show that (X,S) be a complete S-metric space. We define a function $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{4} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } 1 < x \leq 2. \end{cases}$$

Now, for $0 \le x \le 1$, *we have*

$$\theta(r)S(x,x,Tx) = \theta(r)S(x,x,x/4) = \theta(r)x \le \max\{x,y\} = S(x,x,y),$$

and

$$S(Tx,Tx,Ty) = S(x/4,x/4,Ty) \le \max\{x/4,y/4\} \le 1/4\max\{x,y\} = rS(x,x,y).$$

Similarly, for $1 < x \le 2$, *we have*

$$\theta(r)S(x,x,Tx) = \theta(r)S(x,x,0) = \theta(r)x \le \max\{x,y\} = S(x,x,y)$$

and

$$S(Tx,Tx,Ty) = S\left(0,0,Ty\right) \le y/4 \le 1/4\max\{x,y\} = rS(x,x,y)$$

Hence T *satisfied all the conditions of Theorem 3.1 and has a fixed point at* $x = 0$.

Theorem 3.3. *Let* (X,S) *be a compact* S*-metric space and let* T *be a mapping on* X. *Assume that*

$$\frac{1}{3}S(x,x,Tx) < S(x,x,y) \text{ implies } S(Tx,Tx,Ty) < S(x,x,y) \tag{7}$$

for all $x, y \in X$. *Then* T *has a unique fixed point.*

Proof. We put $\beta = \inf\{S(x,x,Tx) : x \in X\}$ and choose a sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} S(x_n,x_n,Tx_n) = \beta$. Since X is a compact S-metric space, therefore we may assume that sequences $\{x_n\}$ and $\{Tx_n\}$ converges to some elements (say) u, v in X respectively. Now we shall show that $\beta = 0$. Arguing by contradiction, we assume that $\beta > 0$. Then

$$\lim_{n\to\infty} S(x_n,x_n,v) = S(u,u,v) = \lim_{n\to\infty} S(x_n,x_n,Tx_n) = \beta.$$

We can choose $n_0 \in \mathbb{N}$ such that

$$\frac{4}{9}\beta < S(x_n,x_n,v) \text{ and } S(x_n,x_n,Tx_n) < \frac{4}{3}\beta$$

for all $n \in \mathbb{N}$ with $n \ge n_0$. Thus

$$\frac{1}{3}S(x_n,x_n,Tx_n) < \frac{4}{9}\beta < S(x_n,x_n,v)$$

for all $n \ge n_0$. By assumption (7),

$$S(Tx_n,Tx_n,Tv) < S(x_n,x_n,v)$$

holds for all $n \ge n_0$. This implies

$$S(v,v,Tv) = \lim_{n\to\infty} S(Tx_n,Tx_n,Tv) \le \lim_{n\to\infty} S(x_n,x_n,v) = \beta.$$

From the definition of β , we obtain $S(v,v,Tv) = \beta$. Since

$$\frac{1}{3}S(v,v,Tv) < S(v,v,Tv),$$

we have by assumption (7)

$$S(Tv,Tv,T^2v) < S(v,v,Tv) = \beta$$

which contradict the definition of β, therefore we obtain $\beta = 0$. Further we shall show that T has a fixed point. Assume that T does not have a fixed point. Then

$$\frac{1}{3}S(x_n, x_n, Tx_n) < S(x_n, x_n, Tx_n)$$

implies

$$S(Tx_n, Tx_n, T^2x_n) < S(x_n, x_n, Tx_n), \text{ for all } n \in \mathbb{N}.$$

Now, we have

$$\lim_{n\to\infty} S(u, u, Tx_n) = S(u, u, v) = \lim_{n\to\infty} S(x_n, x_n, Tx_n) = \beta = 0$$

which implies that $\{Tx_n\}$ converges to u. We also have

$$\begin{aligned} S(u, u, T^2x_n) &\leq 2S(u, u, Tx_n) + S(Tx_n, Tx_n, T^2x_n) \\ &\leq 2S(u, u, Tx_n) + S(x_n, x_n, Tx_n). \end{aligned}$$

Making $n \to \infty$, we get

$$\lim_{n\to\infty} S(u, u, T^2x_n) = 0.$$

Thus the sequence $\{T^2x_n\}$ also converges to u. If

$$\frac{1}{3}S(x_n, x_n, Tx_n) \geq 2S(x_n, x_n, u) \text{ and } \frac{1}{3}S(Tx_n, Tx_n, T^2x_n) \geq S(Tx_n, Tx_n, u).$$

Then we have,

$$\begin{aligned} S(x_n, x_n, Tx_n) &\leq 2S(x_n, x_n, u) + S(Tx_n, Tx_n, u) \\ &\leq \frac{2}{3}S(x_n, x_n, Tx_n) + \frac{1}{3}S(Tx_n, Tx_n, T^2x_n) \\ &< \frac{2}{3}S(x_n, x_n, Tx_n) + \frac{1}{3}S(x_n, x_n, Tx_n) \\ &= S(x_n, x_n, Tx_n), \end{aligned}$$

which is a contradiction. Hence for every $n \in \mathbb{N}$, either

$$\frac{1}{3}S(x_n, x_n, Tx_n) < S(x_n, x_n, Tx_n) \text{ or } \frac{1}{3}S(Tx_n, Tx_n, T^2x_n) < S(Tx_n, Tx_n, u)$$

holds. Hence one of the following holds.

(i) There exists an infinite subset I of $\mathbb{N}$ such that $S(Tx_n, Tx_n, Tu) < S(x_n, x_n, u)$ for all $n \in I$.

(ii) There exists an infinite subset J of $\mathbb{N}$ such that $S(T^2x_n, T^2x_n, Tu) < S(Tx_n, Tx_n, u)$ for all $n \in J$.

In the first case, we obtain

$$S(u, u, Tu) = \lim_{n\in I, n\to\infty} S(Tx_n, Tx_n, Tu) \leq \lim_{n\in I, n\to\infty} S(x_n, x_n u) = 0$$

which implies $Tu = u$. Also in the second case, we obtain

$$S(u, u, Tu) = \lim_{n\in J, n\to\infty} S(T^2x_n, T^2x_n Tu) \leq \lim_{n\in J, n\to\infty} S(Tx_n, Tx_n, u) = 0.$$

Hence, we have shown that u is a fixed point of T in both cases, which is a contradiction. Therefore there exists a $z \in X$ such that $Tz = z$. The uniqueness of fixed of T follows easily from previous theorem. $\square$

Example 3.4. *Let $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be endowed with S-metric defined by*

$$S(x, y, z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x, y, z\} & \text{if } x \neq y \neq z. \end{cases}$$

Then it is not difficult to prove that (X, S) be a compact S-metric space. We define a function $T : X \to X$ by

$$Tx = \begin{cases} 0 & \text{if } x \in \{0, 1, 2, 3, 4, 5\} \\ \frac{1}{x} & \text{if } x \in \{6, 7, 8, 9, 10\}. \end{cases}$$

Now, for $x \in \{1, 2, 3, 4, 5\}$, we obtain

$$\frac{1}{3}S(x, x, Tx) = \frac{1}{3}S(x, x, 0) = \frac{x}{3} < \max\{x, y\} = S(x, x, y),$$

and

$$S(Tx, Tx, Ty) = S(0, 0, Ty) \leq \frac{1}{y} < \max\{x, y\} = S(x, x, y).$$

Similarly for $x \in \{6, 7, 8, 9, 10\}$, we obtain

$$\frac{1}{3}S(x, x, Tx) = \frac{1}{3}S(x, x, \frac{1}{x}) = \frac{x}{3} < \max\{x, y\} = S(x, x, y)$$

and

$$S(Tx, Tx, Ty) = S\left(\frac{1}{x}, \frac{1}{x}, Ty\right) \leq \max\left\{\frac{1}{x}, \frac{1}{y}\right\} < \max\{x, y\} = S(x, x, y).$$

4. S-metric Completeness

In this section, we discuss the completeness of S-metric space using Theorem 3.1.

Theorem 4.1. *Let (X, S) be an S-metric space and define a function θ as in Theorem 3.1. For $r \in [0, 1)$ and $\eta \in (0, \theta(r)]$. Let $A_{r,\eta}$ be the family of mappings T on X satisfying the following:*

(a) For $x, y \in X$,

$$\eta S(x, x, Tx) \leq S(x, x, y) \text{ implies } S(Tx, Tx, Ty) \leq rS(x, x, y).$$

Let $B_{r,\eta}$ be the family of mappings T on X satisfying

(b) $T(X)$ is countably infinite.

(c) Every subset of $T(X)$ is closed.

Then the following are equivalent:

(i) X is complete.

(ii) Every mapping $T \in A_{r,\theta(r)}$ has a fixed point for all $r \in [0, 1)$.

(iii) There exist $r \in (0, 1)$ and $\eta \in (0, \theta(r)]$ such that every mapping $T \in B_{r,\eta}$ has a fixed point.

Proof. By Theorem 3.1, (i) implies (ii). Since $B_{r,\eta} \subset A_{r,\theta(r)}$ for $r \in [0,1)$ and $\eta \in (0, \theta(r)]$ hence (ii) implies (iii). Now we prove that (iii) implies (i). We assume that (iii) holds. Arguing by contradiction, we also assume that X is not complete, that is, there exists a Cauchy sequence $\{u_n\}$ which does not converges in X. We define a function $f : X \to [0, \infty)$ by

$$f(x) = \lim_{n\to\infty} S(x, x, u_n) \text{ for all } x \in X.$$

Then the function $f(x)$ is well defined because $\{S(x, x, u_n)\}$ is also a Cauchy sequence for every $x \in X$ and the following are obvious:

- $f(x) > 0$ for all $x \in X$.
- $\lim_{n\to\infty} f(u_n) = 0$.
- $\frac{f(y)-f(x)}{2} \leq S(x, x, y) \leq 2f(x) + f(y)$ for all $x, y \in X$.

  It follows from Lemma 2.5 and

  (i) $S(x, x, y) \leq 2S(x, x, u_n) + S(y, y, u_n)$.

  (ii) $S(y, y, u_n) - S(u_n, u_n, x) \leq 2S(y, y, x)$, implies
  $S(y, y, u_n) - S(x, x, u_n) \leq 2S(x, x, y)$.

Since $f(x) > 0$, for all $x \in X$ and $f(u_n) \to 0$ as $n \to \infty$. Therefore, for each $x \in X$, there exist $v_x \in \mathbb{N}$ such that

$$f(u_{v_x}) \leq \frac{\eta r}{8 + \eta r} f(x), x \in X.$$

We define a mapping $T : X \to X$ by $Tx = u_{v_x}$ for all $x \in X$. Then it is obvious that

$$f(Tx) \leq \frac{\eta r}{8 + \eta r} f(x) \text{ and } Tx \in \{u_n : n \in \mathbb{N}\} \text{ for all } x \in X.$$

Then $Tx \neq x$ for all $x \in X$, because $f(Tx) < f(x)$, that is, T does not have a fixed point. Since $T(X) \subset \{u_n : n \in \mathbb{N}\}$ =countably infinite set, therefore (b) holds. Also it is easy to prove that every subset of $T(X)$ is closed. Now we shall prove (a). For $x, y \in X$ with $\eta S(x, x, Tx) \leq S(x, x, y)$. In case when $f(y) > 2f(x)$, we have

$$\begin{aligned}
S(Tx, Tx, Ty) &\leq 2f(Tx) + f(Ty) \\
&\leq \frac{\eta r}{8 + \eta r}(2f(x) + f(y)) \leq \frac{r}{8}(2f(x) + f(y)) \\
&\leq \frac{r}{8}(2f(x) + f(y)) + \frac{3r}{8}(f(y) - 2f(x)) \\
&= r\left[\frac{f(y) - f(x)}{2}\right] \leq rS(x, x, y).
\end{aligned}$$

In the other case, when $f(y) \leq 2f(x)$, we have

$$\begin{aligned}
S(x, x, y) &\geq \eta S(x, x, Tx) \geq \eta\,[f(x) - f(y)] \\
&\geq \frac{\eta}{2}\left[f(x) - \frac{\eta r}{8 + \eta r} f(x)\right] = \frac{\eta}{2}\left[1 - \frac{\eta r}{8 + \eta r}\right] f(x) \\
&= \left(\frac{4\eta}{8 + \eta r}\right) f(x).
\end{aligned}$$

Hence

$$\begin{aligned} S(Tx, Tx, Ty) &\leq 2f(Tx) + f(Ty) \leq \frac{\eta r}{8 + \eta r}(2f(x) + f(y)) \\ &\leq \frac{\eta r}{8 + \eta r}(2f(x) + 2f(x)) = \frac{4\eta r}{8 + \eta r} f(x) \\ &\leq rS(x, x, y). \end{aligned}$$

Therefore we have shown (a), that is, $T \in B_{r,\eta}$. By (iii), T has a fixed point which yields a contradiction. Hence we obtain that X is complete. This completes the proof. □

If we take $\eta = \frac{1}{1000}$ in the above theorem then we obtain the following corollary as a direct consequence of Theorem 4.1

Corollary 4.2. *For an S-metric space (X, S), the following are equivalent:*

(i) X is complete.

(ii) There exists $r \in (0, 1)$ such that every mapping T on X satisfying the following

$$\frac{1}{1000} S(x, x, Tx) \leq S(x, x, y) \text{ implies } S(Tx, Tx, Ty) \leq rS(x, x, y)$$

then T has a fixed point.

5. Application to Dynamic Programming

Throughout the section we assume that U and V are Banach spaces, $W \subset U, D \subset V$ and $\mathbb{R}$ denote the field of real numbers. Let $\tau : W \times D \to W, f : W \times D \to \mathbb{R}, F : W \times D \times \mathbb{R} \to \mathbb{R}$ and $B(W)$ denotes the set of all bounded real valued function on W.

In this section, we will study the existence and uniqueness of a solution of the following functional equation:

$$p(x) = \sup_{y \in D}\{f(x, y) + F(x, y, p(\tau(x, y)))\}, x \in W, \tag{8}$$

where f and F are bounded functions, x and y represents the state and decision vectors, respectively, τ represents the transformation of the process and $p(x)$ represents the optimal return function with initial state x.

Now we define a mapping $T : B(W) \to B(W)$ by

$$T(h(x)) = \sup_{y \in D}\{f(x, y) + F(x, y, h(\tau(x, y)))\} \tag{9}$$

and a S-metric $S : B(W) \times B(W) \times B(W) \to [0, \infty)$ by

$$S(h, g, k) = d(h, g) + d(g, k), \tag{10}$$

where $h, g, k \in B(W)$ and $d : B(W) \times B(W) \to [0, \infty)$ is defined as

$$d(h, g) = \sup_{t \in W} |h(t) - g(t)|.$$

Now we will prove the existence and uniqueness of the solution of functional equation (8) using Theorem 3.1.

Theorem 5.1. *Suppose that there exists* $r \in [0,1)$ *such that for every* $(x,y) \in W \times D, h, k \in B(W)$ *and* $t \in W$,

$$\theta(r)|h_1(t) - T(h_1(t))| \leq |h_1(t) - h_2(t)| \tag{11}$$

implies

$$|F(x,y,h_1(t)) - F(x,y,h_2(t))| \leq r|h_1(t) - h_2(t)|. \tag{12}$$

Then the functional equation (8) has a unique bounded solution in $B(W)$.

Proof. Let T be a self map defined in (9) and $(B(W), S)$ is a complete S-metric space, where S is defined by (10). Let λ be an arbitrary positive number and $h_1, h_2 \in B(W)$. Let $x \in W$ be arbitrary and choose $y_1, y_2 \in D$ such that

$$T(h_1(x)) < f(x,y_1) + F(x,y_1,h_1(\tau_1) + \lambda \tag{13}$$

$$T(h_2(x)) < f(x,y_2) + F(x,y_2,h_2(\tau_2)) + \lambda \tag{14}$$

where $\tau_1 = \tau(x,y_1)$ and $\tau_2 = \tau(x,y_2)$. Further, by definition of T, we know

$$T(h_1(x)) \geq f(x,y_2) + F(x,y_2,h_1(\tau_2)) \tag{15}$$

$$T(h_2(x)) \geq f(x,y_1) + F(x,y_1,h_2(\tau_1)). \tag{16}$$

If the inequality (11) holds, then from (13) and (16), we obtain

$$\begin{aligned} T(h_1(x)) - T(h_2(x)) &\leq F(x,y_1,h_1(\tau_1)) - F(x,y_1,h_2(\tau_1)) + \lambda \\ &\leq |F(x,y_1,h_1(\tau_1)) - F(x,y_1,h_2(\tau_1))| + \lambda \\ &\leq r|h_1(x) - h_2(x)| + \lambda. \end{aligned} \tag{17}$$

Similarly from (14) and (15), we obtain

$$T(h_2(x)) - T(h_1(x)) \leq r|h_1(x) - h_2(x)| + \lambda. \tag{18}$$

Hence from (17) and (18), we have

$$|T(h_1(x)) - T(h_2(x))| \leq r|h_1(x) - h_2(x)| + \lambda.$$

Since $x \in W$ and $\lambda > 0$ is arbitrary hence we find from inequality (11) that

$$\theta(r)S(h_1(t), h_1(t), T(h_1(t))) \leq S(h_1(t), h_1(t), h_2(t))$$

implies

$$S(T(h_1(t)), T(h_1(t)), T(h_2(t))) \leq rS(h_1(t), h_1(t), h_2(t)).$$

Thus all the conditions of Theorem 3.1 are satisfied for the mapping T. Hence T has a unique fixed point $h^*(t)$, that is, $h^*(t)$ is a bounded solution of the functional equation (8). □

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