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Elzaki Transform for Exponential Growth and Decay

Research Article

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Abstract: In this paper, we show the solutions to the problems occurred in applications of first order ordinary differential equations based on the principle of Law of natural growth and decay by using the new integral transform called Elzaki Transform.

Keywords: Elzaki Transform, Law of natural growth and decay, Differential equations.

1. Introduction

Differential equations play an important role in science and engineering. In order to solve the differential equations, the integral transforms were extensively used. The importance of an Integral Transforms is that they provide powerful operational methods for solving initial value problems. There are several techniques like Laplace Transform, Fourier Transform, etc. are available to solve differential equations. Elzaki Transform was introduced by Tarig Elzaki [1] in 2011. We apply this new transform technique for solving the problems on law of natural growth and decay. Elzaki Transform defined for function of exponential order, we consider function on the set A defined by

$$A = \left\{ f(t) : \exists\, M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}},\ if\ t \in (-1)^j \times [0, \infty) \right\}$$

For a given function in the set A the constant M must be a finite number, k_1 and k_2 may be finite or infinite. The Elzaki transform denoted by the operator E, defined by an integral equation.

$$E[f(t)] = v \int_0^{\infty} f(t) e^{\frac{-t}{v}} dt = T(V), \quad t \geq 0, k_1 \leq v \leq k_2.$$

The sufficient conditions for the existence of the Elzaki transform are that $f(t)$ for $t \geq 0$ be piecewise continuous and of the exponential order otherwise Elzaki transform may (or) may not exist.

1.1. Elzaki Transform of Some Functions

1. If $f(t) = 1$. Now $E\{f(t)\} = v \int\limits_0^{\infty} e^{\frac{-t}{v}} f(t) dt = v \int\limits_0^{\infty} e^{\frac{-t}{v}} dt = v \left(\frac{e^{\frac{-t}{v}}}{\frac{-1}{v}} \right)_0^{\infty} = v^2$. Therefore $E\{1\} = v^2$.
2. If $f(t) = t$, then $E\{f(t)\} = v \int\limits_0^{\infty} e^{\frac{-t}{v}} f(t) dt \Rightarrow E\{t\} = v \int_0^{\infty} t\, e^{\frac{-t}{v}} dt = v \left\{ t \left(\frac{e^{\frac{-t}{\nu}}}{\frac{-1}{\nu}} \right) - (1) \frac{e^{\frac{-t}{\nu}}}{\frac{1}{\nu^2}} \right\}_0^{\infty} = v(v^2) = v^3$.

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3. Similarly, we get $E(t^n) = n!\, v^{n+2}$.

4. $E(e^{at}) = v \int\limits_0^\infty e^{at} e^{-\frac{t}{v}} dt = \frac{v^2}{1-av}$.

Theorem 1.1. *Let f(t) be the given function and* $E\{f(t)\} = T(v)$ *then* $E\{f^1(t)\} = \frac{T(v)}{v} - vf(0)$.

Proof. Given $E\{f(t)\} = T(v)$. By Definition of Elzaki Transform $E\{f(t)\} = v \int\limits_0^\infty f(t)e^{\frac{-t}{v}} dt$. Now

$$\begin{aligned} E\{f^1(t)\} &= v \int_0^\infty f^1(t) e^{\frac{-t}{v}} dt \\ &= v \left\{ \left(e^{\frac{-t}{v}} f(t) \right)_0^\infty - \int_0^\infty e^{\frac{-t}{v}} \left(\frac{-1}{v} \right) f(t) dt \right\} \\ &= v \left\{ -f(0) + \frac{1}{v} \int_0^\infty f(t) e^{\frac{-t}{v}} dt \right\} \\ &= -vf(0) + \int_0^\infty f(t) e^{\frac{-t}{v}} dt = -vf(0) + \frac{T(v)}{v} \\ &= \frac{T(v)}{v} - vf(0) \end{aligned}$$

□

1.2. Law of Natural Growth and Decay

Let $x(t)$ be the amount of substance at time 't' and it is getting converted chemically. A law of chemical conversion states that the rate of change of amount $x(t)$ is proportional to the amount of the substance available at that time 't'. i.e. $\frac{dx}{dt}\, \alpha\, x \Rightarrow \frac{dx}{dt} = kx$ where '$k > 0$' is the rate constant. This is a first order first degree differential equation which represents natural growth. Similarly, the differential equation for natural decay is $\frac{dx}{dt} = -kx$ where '$k > 0$' is the rate constant.

2. Problems on Exponential Growth and Decay

Example 2.1. *The number N of bacteria on a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What was the value of N after 90 minutes?*

Solution. Given, at $t = 0$, $N = 100$. At $t = 60$min, $N = 332$. Then at $t = 90$min, $N =$?. From law of Natural growth we have $\frac{dN}{dt} \propto N \Rightarrow \frac{dN}{dt} = kN \Rightarrow N^1(t) = kN \Rightarrow N^1(t) - kN = 0$. Now taking Elzaki Transform on both sides,

$$E\{N^1(t) - kN\} = E\{0\} \Rightarrow E\{N^1(t)\} - E\{kN\} = E\{0\} \Rightarrow \left[\frac{\bar{N}(v)}{v} - vN(0) \right] - k\bar{N}(v) = 0$$

Applying the initial condition $N(0) = 100$, then we have $\frac{\bar{N}(v)}{v} - 100v - k\bar{N}(v) = 0$.

$$\Rightarrow \left(\frac{1}{v} - k \right) \bar{N}(v) = 100v \Rightarrow (1-kv)\bar{N}(v) = 100v^2 \Rightarrow \bar{N}(v) = \frac{100v^2}{(1-kv)}$$

Taking inverse Elzaki Transform on either side,

$$\Rightarrow N(t) = E^{-1}\left\{ \frac{100v^2}{(1-kv)} \right\} \Rightarrow N(t) = 100e^{kt} \tag{1}$$

At $t = 60$min, $N(t) = 332$. From (1),

$$332 = 100e^{60k} \Rightarrow e^k = \left(\frac{332}{100} \right)^{\frac{1}{60}} \tag{2}$$

Now to find $N(t)$ at $t = 90$min, From (1) we have

$$N(t) = 100e^{90k} = 100(e^k)^{90} \tag{3}$$

Substituting (2) in (3), we get

$$N(t) = 100\left(\frac{332}{100}\right)^{\frac{90}{60}} = 604.93 \Rightarrow N(t) \cong 605.$$

□

Example 2.2. *The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number is doubles in 2 hours, in how many hours will it be triple?.*

Solution. Let N represent the number of bacteria at any time t. At the initial time $t = 0$, $N = x$. From Law of natural growth we have $\frac{dN}{dt} \propto N \Rightarrow \frac{dN}{dt} = kN \Rightarrow N^1(t) = kN \Rightarrow N^1(t) - kN = 0$. Now taking Elzaki Transform both sides, $E\left\{N^1(t) - kN\right\} = E\left\{0\right\} \Rightarrow E\left\{N^1(t)\right\} - E\left\{kN\right\} = E\left\{0\right\} \Rightarrow \left[\frac{\bar{N}(v)}{v} - vN(0)\right] - k\bar{N}(v) = 0 \Rightarrow \left[\frac{1}{v} - k\right]\bar{N}(v) = xv \Rightarrow (1 - kv)\,\bar{N}(v) = xv^2 \Rightarrow \bar{N}(v) = \frac{xv^2}{1-kv}$. Taking inverse Elzaki transform, we have

$$N\left(t\right) = xe^{kt} \tag{4}$$

Given that at $t = 2$hours, $N = 2x$, From (4),

$$2x = xe^{2k} \Rightarrow 2 = e^{2k} \Rightarrow e^k = 2^{\frac{1}{2}} \tag{5}$$

If $N = 3x$, from (4), we get

$$3x = xe^{kt} \;\Rightarrow 3 = e^{kt} \;\Rightarrow 3 = \left(e^k\right)^t \;\Rightarrow 3 = (2)^{\frac{t}{2}}$$

Taking both sides logarithm, then we have $\log 3 = \frac{t}{2}\log 2 \Rightarrow t = \frac{2\log 3}{\log 2}hours.$ □

Example 2.3. *If 30% radioactive disappear in 10days then how long it will take to disappears 90%?.*

Solution. Let $x(t)$ be the amount of substance at any time t. Given that at time $t = 0$, $x = 100\% = 100$. Also given that at time $t = 10$days, $x = 70\% = 70$. Now we have to find the time required to become $x = 10\% = 10$. From Law of natural decay, we have $\frac{dx}{dt} = -kx \Rightarrow x^1(t) + kx = 0$, where 'k' is the rate constant. Now taking Elzaki Transform both sides, then we have

$$E\left\{x^1(t) + kx\right\} = E\{0\} \Rightarrow E\left\{x^1(t)\right\} + kE\left\{x\right\} = E\{0\} \Rightarrow \left[\frac{\bar{x}(v)}{v} - vx(0)\right] + k\bar{x}(v) = 0$$

Applying the initial condition at $t = 0$, $x(t) = 100$.

$$\left(\frac{1}{v} + k\right)\bar{x}(v) = 100v \Rightarrow (1 + kv)\bar{x}(v) = 100v^2 \Rightarrow \bar{x}(v) = \frac{100v^2}{(1 + kv)}.$$

Taking inverse Elzaki Transform on either side,

$$\Rightarrow x(t) = E^{-1}\left\{\frac{100v^2}{(1 + kv)}\right\} \Rightarrow x(t) = 100e^{-kt} \tag{6}$$

Also given that at $t = 10$days, $x(t) = 70$. From (6), we get

$$70 = 100e^{-10k} \Rightarrow 0.7 = e^{-10k} \Rightarrow e^{-k} = (0.7)^{\frac{1}{10}} \tag{7}$$

If $x(t) = 10$, from (6), $10 = 100e^{-kt} \Rightarrow\ 0.1 = (e^{-k})^t = (0.7)^{\frac{t}{10}}$. Taking logarithm on both sides, then we have

$$\log(0.1) = \frac{t}{10}\log(0.7) \Rightarrow t = \frac{10\log(0.1)}{\log(0.7)} \simeq 64.557 days$$

□

Example 2.4. *The rate at which a certain substance decomposes in a certain solution at any instant is proportional to the amount of it present in the solution at that instant. Initially, there are 27 grams and three hours later, it is found that 8 grams are left. How much substance will be left one more hour?.*

Solution. Let 'm' grams is the amount of the substance left in the solution at any time't'. From Law of natural decay, we have $\frac{dx}{dt} = -kx \Rightarrow x^1(t) + kx = 0$, where 'k' is the rate constant. Now taking Elzaki Transform both sides, then we have

$$E\left\{x^1(t) + kx\right\} = E\{0\} \Rightarrow E\left\{x^1(t)\right\} + kE\left\{x\right\} = E\{0\} \Rightarrow \left[\frac{\bar{x}(v)}{v} - vx(0)\right] + k\bar{x}(v) = 0$$

Applying the initial condition at $t = 0$, $x(t) = 27$grams.

$$\left(\frac{1}{v} + k\right)\bar{x}(v) = 27v \Rightarrow (1 + kv)\bar{x}(v) = 27v^2 \Rightarrow \bar{x}(v) = 27\frac{v^2}{(1 + kv)}$$

Taking inverse Elzaki Transform on either side,

$$x(t) = 27E^{-1}\left\{\frac{v^2}{(1 + kv)}\right\} \Rightarrow x(t) = 27e^{-kt} \tag{8}$$

Using the another given condition, at $t = 3$hours, $x(t) = 8$grams, from (8) we have

$$8 = 27e^{-3k} \Rightarrow e^{-3k} = \frac{8}{27} \Rightarrow e^{-k} = \left(\frac{8}{27}\right)^{\frac{1}{3}} \tag{9}$$

Now to find $x(t)$ after the time $t = 4$hours, from (8), we get

$$x\,(t) = 27e^{-4k} \Rightarrow x\,(t) = 27\left(e^{-k}\right)^4 \Rightarrow x\,(t) = 27\left(\frac{8}{27}\right)^{\frac{4}{3}} \Rightarrow x\,(t) = \frac{16}{3} grams.$$

This shows that after the time t=4 hours the quantity of the substance is 5.333 grams. □

3. Conclusion

Elzaki Transforms is applicable to solve the problems related to exponential growth and decay.

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