Some Theorems on Anti $T$–Fuzzy Ideal of $\ell$–Ring

Research Article

J.Prakashmaninaran$^{1,*}$, B.Chellappa$^{2}$ and M.Jeyakumar$^{3}$

1 Research Scholar (Part Time-Mathematics), Manonmaniam Sundharanar University, Tirunelveli, Tamilnadu, India.
2 Principal, Nachiappa Swamical Arts and Science College, Koviloor, Tamilnadu, India.
3 Assistant Professor, Department of Mathematics, Alagappa University Evening College, Rameswaram, Tamilnadu, India.

Abstract: In this paper, we made an attempt to study the properties of anti $T$–fuzzy ideal of $\ell$–ring and we introduce some definitions and theorems of product of anti $T$–fuzzy ideal of $\ell$–ring.


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1. Introduction

The concept of fuzzy sets was initiated by L.A.Zadeh [9] in 1965. After the introduction of fuzzy sets several researchers explored on the generalization of the concept of fuzzy sets. In this paper we define, characterize and study the anti $T$–fuzzy right and left ideals. Z. D. Wang introduced the basic concepts of TL-ideals. We introduced anti $T$–fuzzy right ideals of $\ell$–ring. We compare fuzzy ideal introduced by Liu to anti $T$–fuzzy ideals. We have shown that ring is regular if and only if union of any anti $T$–fuzzy right ideal with anti $T$–fuzzy left ideal is equal to its product. We discuss some theorems. We have shown that the product of anti $T$–fuzzy ideal of $\ell$–ring.

2. Main Results

Definition 2.1. A non-empty set $R$ is called lattice ordered ring or $\ell$–ring if it has four binary operations $+,-,\lor,\land$ defined on it and satisfy the following

(1). $(R,+,\cdot)$ is a ring

(2). $(R,\lor,\land)$ is a lattice

(3). $x + (y \lor z) = (x + y) \lor (x + z)$; $x + (y \land z) = (x + y) \land (x + z)$

$(y \lor z) + x = (y + x) \lor (z + x)$; $(y \land z) + x = (y + x) \land (z + x)$

(4). $x \cdot (y \lor z) = (xy) \lor (xz)$; $x \cdot (y \land z) = (xy) \land (xz)$

$(y \lor z) \cdot x = (yx) \lor (zx)$; $(y \land z) \cdot x = (yx) \land (zx)$.

* E-mail: prakashmani1982@gmail.com
for all $x$, $y$, $z$ in $R$ and $x \geq 0$.

**Example 2.2.** $(\mathbb{Z}, +, \cdot, \lor, \land)$ is a $\ell$–ring, where $\mathbb{Z}$ is the set of all integers.

**Example 2.3.** $(n\mathbb{Z}, +, \cdot, \lor, \land)$ is a $\ell$–ring, where $\mathbb{Z}$ is the set of all integers and $n \in \mathbb{Z}$.

**Definition 2.4.** A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm [t-norm] if and only if it satisfies the following conditions:

1. $T(x, 1) = T(1, x) = x$, for all $x \in [0, 1]$.
2. $T(x, y) = T(y, x)$, for all $x, y \in [0, 1]$.
3. $T(x, T(y, z)) = T(T(x, y), z)$ for all $x, y, z \in [0, 1]$.
4. $T(x, y) \leq T(x, z)$, whenever $y \leq z$, for all $x, y, z \in [0, 1]$.

**Definition 2.5.** A mapping from a nonempty set $X$ to $[0, 1]; \mu : X \to [0, 1]$ is called a fuzzy subset of $X$.

**Definition 2.6.** A fuzzy subset $\mu$ of a ring $R$ is called anti $T$–fuzzy right ideal if

1. $\mu(x - y) \leq T(\mu(x), \mu(y))$
2. $\mu(xy) \leq \mu(x)$, for all $x, y$ in $R$

**Definition 2.7.** A fuzzy subset $\mu$ of a ring $R$ is called anti $T$–fuzzy left ideal if

1. $\mu(x - y) \leq T(\mu(x), \mu(y))$
2. $\mu(xy) \leq \mu(y)$, for all $x, y$ in $R$

**Theorem 2.8.** Every fuzzy right ideal of a ring $R$ is an anti $T$–fuzzy right ideal.

**Proof.** Let $\mu$ be a fuzzy right ideal of $R$. Then $\mu(x - y) \leq T(\mu(x), \mu(y))$ and $\mu(xy) \leq \mu(x)$, for all $x, y \in R$. Hence $\mu$ is an anti $T$–fuzzy ideal. □

**Definition 2.9.** A fuzzy subset $\mu$ of a lattice ordered ring (or $\ell$–ring) $R$ is called an anti fuzzy sub $\ell$–ring of $R$, if the following conditions are satisfied

1. $\mu(x - y) \leq \max\{\mu(x), \mu(y)\}$
2. $\mu(xy) \leq \max\{\mu(x), \mu(y)\}$
3. $\mu(x \lor y) \leq \max\{\mu(x), \mu(y)\}$
4. $\mu(x \land y) \leq \max\{\mu(x), \mu(y)\}$, for all $x, y$ in $R$

**Example 2.10.** Consider an anti-fuzzy subset $\mu$ of the $\ell$–ring $(\mathbb{Z}, +, \cdot, \lor, \land)$

$$
\mu_1(x) = \begin{cases} 
0.4, & \text{if } x \in \langle 2 \rangle; \\
0.7, & \text{if } Z - \langle 2 \rangle.
\end{cases}
$$

Then $\mu_1$ is an anti-fuzzy sub $\ell$–ring.
Definition 2.11. A fuzzy subset $\mu$ of an $\ell$–ring $R$ is called an anti fuzzy $\ell$–ideal (or) fuzzy $\ell$–ideal of $R$, if for all $x$, $y$ in $R$ the following conditions are satisfied

(1). $\mu(x - y) \leq \max\{\mu(x), \mu(y)\}$

(2). $\mu(xy) \leq \min\{\mu(x), \mu(y)\}$

(3). $\mu(x \lor y) \leq \max\{\mu(x), \mu(y)\}$

(4). $\mu(x \land y) \leq \min\{\mu(x), \mu(y)\}$, for all $x$, $y$ in $R$

Definition 2.12. A fuzzy subset $\mu$ of a ring $R$ is called an anti $T$–fuzzy ideal, if the following conditions are satisfied,

(1). $\mu(x - y) \leq T(\mu(x), \mu(y))$

(2). $\mu(xy) \leq \mu(x); \mu(xy) \leq \mu(y)$, for all $x, y \in R$.

Definition 2.13. A fuzzy subset $\mu$ of an $\ell$–ring $R$ is called an anti $T$–fuzzy ideal, if the following conditions are satisfied,

(1). $\mu(x - y) \leq T(\mu(x), \mu(y))$

(2). $\mu(xy) \leq \mu(x); \mu(xy) \leq \mu(y)$

(3). $\mu(x \lor y) \leq T(\mu(x), \mu(y))$

(4). $\mu(x \land y) \leq T(\mu(x), \mu(y))$, for all $x$, $y$ in $R$.

Definition 2.14. Now $(R = \{a, b, c\}, +, \cdot, \lor, \land)$ is a $\ell$–ring. The operations $+, \cdot, \lor$ and $\land$ defined by the following. Consider an anti-fuzzy subset $\mu_{A}$ of the $\ell$–ring $R$.

\[
\mu(x) = \begin{cases} 
0.2, & \text{if } x = a; \\
0.5, & \text{if } x = b; \\
0.8, & \text{if } x = c. 
\end{cases}
\]

Then $\mu$ is an anti $T$–fuzzy ideal of $\ell$–ring $R$.

Theorem 2.15. If $\mu$ and $\lambda$ are any two anti $T$–fuzzy ideal of $\ell$–rings $R_{1}$ and $R_{2}$ then the product of $\mu \times \lambda$ is also anti $T$–fuzzy ideal of $\ell$–ring $R_{1} \times R_{2}$.

Proof. Given $\mu$ and $\lambda$ are any two anti $T$–fuzzy ideal of $\ell$–rings $R_{1}$ and $R_{2}$ respectively. Let $x, y \in R$.

(1). $(\mu \times \lambda)(x - y) = T(\mu(x - y), \lambda(x - y))$

\[
\leq T(T(\mu(x), \mu(y)), T(\lambda(x), \lambda(y)))
\]

\[
= T(T(T(\mu(x), \mu(y)), \lambda(x)), \lambda(y))
\]

\[
= T(T(\mu \times \lambda(x)), (\lambda \times \lambda)(y))
\]

Therefore, $(\mu \times \lambda)(x - y) \leq T((\mu \times \lambda)(x), (\mu \times \lambda)(y))$ for all $x, y \in R$.

(2). Since $\mu(xy) \leq \mu(x)$ and $\lambda(xy) \leq (x)$. Now $(\mu \times \lambda)(xy) \leq T(\mu(xy), \lambda(xy)) \leq T(\mu(x), \lambda(x)) \leq (\mu \times \lambda)(x)$. Therefore $(\mu \times \lambda)(xy) \leq (\mu \times \lambda)(x)$, for all $x, y \in R$. 

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(3). \((\mu \times \lambda)(x \vee y) = T(\mu(x \vee y), \lambda(x \vee y))\)
\[
\leq T(T(\mu(x), \mu(y)), T(\lambda(x), \lambda(y)))
\]
\[
= T(T(T(\mu(x), \mu(y)), \lambda(x)), \lambda(y))
\]
\[
= T(T(T(\mu \times \lambda)(x)), \mu(y)), \lambda(y))
\]
\[
= T(T(\mu \times \lambda)(x), T(\mu \times \lambda)(y)))
\]
\[
= T((\mu \times \lambda)(x), (\mu \times \lambda)(y))
\]
Therefore, \((\mu \times \lambda)(x \vee y) \leq T((\mu \times \lambda)(x), (\mu \times \lambda)(y))\) for all \(x, y \in R\).

(4). \((\mu \times \lambda)(x \wedge y) = T(\mu(x \wedge y), \lambda(x \wedge y))\)
\[
\leq T(T(\mu(x), \mu(y)), T(\lambda(x), \lambda(y)))
\]
\[
= T(T(T(\mu(x), \mu(y)), \lambda(x)), \lambda(y))
\]
\[
= T(T(T(\mu \times \lambda)(x)), \mu(y)), \lambda(y))
\]
\[
= T(T(\mu \times \lambda)(x), T(\mu \times \lambda)(y)))
\]
\[
= T((\mu \times \lambda)(x), (\mu \times \lambda)(y))
\]
Therefore, \((\mu \times \lambda)(x \wedge y) \leq T((\mu \times \lambda)(x), (\mu \times \lambda)(y))\) for all \(x, y \in R\). Thus, \(\mu \times \lambda\) is an anti \(T\)-fuzzy right ideal of \(\ell\)-ring \(R_1 \times R_2\).

\[\square\]

**Theorem 2.16.** If \(\mu_i\) are anti \(T\)-fuzzy ideal of \(\ell\)-rings \(R_i\), then \(\prod \mu_i\) is an anti \(T\)-fuzzy ideal of \(\ell\)-ring \(\prod R_i\).

**Proof.** If \(\mu_i\) are anti \(T\)-fuzzy ideal of \(\ell\)-rings \(R_i\). Let \(x, y \in R\) and let \(\mu_i = \mu_1 \times \mu_2 \times \ldots \times \mu_n\)

(1). \((\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x - y) = T(\mu_1(x - y), \mu_2(x - y), \ldots, \mu_n(x - y))\)
\[
\leq T(T(\mu_1(x), \mu_1(y)), T(\mu_2(x), \mu_2(y)), \ldots, T(\mu_n(x), \mu_n(y)))
\]
\[
= T(T((\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x)), T((\mu_1 \times \mu_2 \times \ldots \times \mu_n)(y)))
\]
\[
= T((\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x), (\mu_1 \times \mu_2 \times \ldots \times \mu_n)(y))
\]
Therefore, \((\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x - y) \leq T((\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x), (\mu_1 \times \mu_2 \times \ldots \times \mu_n)(y))\), for all \(x, y \in R\).

(2). Since \(\mu_i(xy) \leq \mu_i(x)\) and \(\lambda_i(xy) \leq \lambda_i(x)\). Now,

\[
(\mu_1 \times \mu_2 \times \ldots \times \mu_n)(xy) = T(\mu_1(xy), \mu_2(xy), \ldots, \mu_n(xy))
\]
\[
\leq T(\mu_1(x), \mu_2(x), \ldots, \mu_n(x))
\]
\[
\leq (\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x)
\]
Therefore, \((\mu_1 \times \mu_2 \times \ldots \times \mu_n)(xy) \leq (\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x)\), for all \(x, y \in R\).

(3). \((\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x \vee y) = T(\mu_1(x \vee y), \mu_2(x \vee y), \ldots, \mu_n(x \vee y))\)
\[
\leq T(T(\mu_1(x), \mu_1(y)), T(\mu_2(x), \mu_2(y)), \ldots, T(\mu_n(x), \mu_n(y)))
\]
\[
= T(T((\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x)), T((\mu_1 \times \mu_2 \times \ldots \times \mu_n)(y)))
\]
\[
= T((\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x), (\mu_1 \times \mu_2 \times \ldots \times \mu_n)(y))
\]
Therefore, \((\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x \vee y) \leq T((\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x), (\mu_1 \times \mu_2 \times \ldots \times \mu_n)(y))\) for all \(x, y \in R\).
Theorem 2.17. Let \( R_1 \) and \( R_2 \) be \( \ell \)-rings. If \( \mu_1 \) and \( \mu_2 \) are any two anti \( T \)-fuzzy ideal of \( \ell \)-ring \( R_1 \) and \( R_2 \) respectively, then \( \mu = \mu_1 \times \mu_2 \) is an anti \( T \)-fuzzy ideal of the direct product of \( R_1 \times R_2 \).

Proof. Let \( \mu_1 \) and \( \mu_2 \), are any two anti \( T \)-fuzzy ideal of \( \ell \)-rings \( R_1 \) and \( R_2 \) respectively. Let \((x_1, x_2), (y_1, y_2), (z_1, z_2) \in R_1 \times R_2,\)

1. \( \mu((x_1, x_2) - (y_1, y_2)) = \mu(x_1 - y_1, x_2 - y_2) \)
   \[ = (\mu_1 \times \mu_2)(x_1 - y_1, x_2 - y_2) \]
   \[ = T(\mu_1(x_1 - y_1), \mu_1(x_2 - y_2)) \]
   \[ \leq T(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_1(x_2), \mu_1(y_2))) \]
   \[ \geq T(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_1(x_2), \mu_1(y_2))) \]
   \[ = T((\mu_1 \times \mu_2)(x_1, x_2), (\mu_1 \times \mu_2)(y_1, y_2)) \]
   \[ = T(\mu(x_1, x_2), \mu(y_1, y_2)) \]

Therefore, \( \mu((x_1, x_2) - (y_1, y_2)) \leq T(\mu(x_1, x_2), \mu(y_1, y_2)), \) for all \((x_1, x_2), (y_1, y_2) \in R_1 \times R_2.\)

2. Since \( \mu_i(xy) \leq \mu_i(x) \) and \( \lambda_i(xy) \leq \lambda_i(x) \)

\[ \mu((x_1, x_2)(y_1, y_2)) = \mu(x_1 y_1, x_2 y_2) \]
\[ = (\mu_1 \times \mu_2)(x_1 y_1, x_2 y_2) \]
\[ \leq T(\mu_1(x_1, y_1), \mu_2(x_2, y_2)) \]
\[ = (\mu_1 \times \mu_2)(x_1, x_2) \]

Therefore, \( \mu((x_1, x_2)(y_1, y_2)) \leq (\mu_1 \times \mu_2)(x_1, x_2), \) for all \( x, y \in R.\)

3. \( \mu((x_1, x_2) \lor (y_1, y_2)) = \mu(x_1 \lor y_1, x_2 \lor y_2) \)
\[ = (\mu_1 \times \mu_2)(x_1 \lor y_1, x_2 \lor y_2) \]
\[ = T(\mu_1(x_1 \lor y_1), \mu_1(x_2 \lor y_2)) \]
\[ \leq T(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_1(x_2), \mu_1(y_2))) \]
\[ \geq T(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_1(x_2), \mu_1(y_2))) \]
\[ = T((\mu_1 \times \mu_2)(x_1, x_2), (\mu_1 \times \mu_2)(y_1, y_2)) \]
\[ = T(\mu(x_1, x_2), \mu(y_1, y_2)) \]

Therefore \( \mu((x_1, x_2) \lor (y_1, y_2)) \leq T(\mu(x_1, x_2), \mu(y_1, y_2)), \) for all \((x_1, x_2), (y_1, y_2) \in R_1 \times R_2.\)
(4). $\mu((x_1, x_2) \land (y_1, y_2)) = \mu(x_1 \land y_1, x_2 \land y_2)$

$= (\mu_1 \times \mu_2)(x_1 \land y_1, x_2 \land y_2)$

$= T(\mu_1(x_1 \land y_1), \mu_1(x_2 \land y_2))$

$\leq T(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_1(x_2), \mu_1(y_2)))$

$\geq T(T(\mu_1(x_1), \mu_1(x_2)), T(\mu_1(y_1), \mu_1(y_2)))$

$= T((\mu_1 \times \mu_2)(x_1, x_2), (\mu_1 \times \mu_2)(y_1, y_2))$

$= T(\mu(x_1, x_2), \mu(y_1, y_2))$

Therefore $\mu_A((x_1, x_2) \land (y_1, y_2)) \leq T(\mu(x_1, x_2), \mu(y_1, y_2))$, for all $(x_1, x_2), (y_1, y_2) \in R_1 \times R_2$.

Thus $\mu = \mu_1 \times \mu_2$ is an anti $T$–fuzzy ideal of the direct product of $R_1 \times R_2$.

References


