International Journal of Mathematics And its Applications
Volume 5, Issue 3–C (2017), 319–326.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

# General Milloux Inequality for Algebroid Functions on Annuli

Research Article

**Renukadevi S. Dyavanal[1]* and Ashok Rathod[1]**

1 Department of Mathematics, Karnatak University, Dharwad, Karnataka, India.

**Abstract:** The purpose of this paper is to establish the general form of Milloux inequality for algebroid function on annuli when the multiple values are considered.

**MSC:** 34M10. 30D35.

**Keywords:** Nevanlinna Theory, Milloux inequality, algebroid functions, the annuli.



## 1. Introduction

The uniqueness theory of algebroid functions is an interesting problem in the value distribution theory. The uniqueness problem of algebroid functions was firstly considered by Valiron, afterwards some scholars have got several uniqueness theorems of algebroid functions in the complex plane $\mathbb{C}$ (see [3, 11]). In 2005, A. Ya. Khrystiyanyn and A. A. Kondratyuk have proposed on the Nevanlinna Theory for meromorphic functions on annuli (see [4, 5]) and after this work others have done lot of work in this area (see [8, 12, 13]). In 2009, Cao and Yi [1] investigated the uniqueness of meromorphic functions sharing some values on annuli. In 2015, Yang Tan [6], Yang Tan and Yue Wang [7] proved some interesting results on the multiple values and uniqueness of algebroid functions on annuli. Thus it is interesting to consider the uniqueness problem of algebroid functions in multiply connected domains. By Doubly connected mapping theorem [10] each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}, 0 \leq r < R \leq +\infty$. We consider only two cases : $r = 0,\ R = +\infty$ simultaneously and $0 \leq r < R \leq +\infty$. In the latter case the homothety $z \mapsto \frac{z}{rR}$ reduces the given domain to the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right) = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, in two cases every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$.

## 2. Basic Notations and Definitions

We assume that the reader is familiar with the Nevanlinna theory of meromorphic functions and algebroid functions (see [2, 9]). Let $A_v(z), A_{v-1}(z), ..., A_0(z)$ be a group of analytic functions which have no common zeros and define on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ and

$$\psi(z, W) = A_v(z)W^v + A_{v-1}(z)W^{v-1} + ... + A_1(z)W + A_0(z) = 0, \tag{1}$$

* E-mail: renukadyavanal@gmail.com

then irreducible equation (1) defines a $v$-valued algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$. Let $W(z)$ be a $v$-valued algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, we use the notations

$$m(r, W) = \frac{1}{\nu}\sum_{j=1}^{\nu} m(r, w_j) = \frac{1}{\nu}\sum_{j=1}^{\nu}\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j(re^{i\theta})|d\theta,$$

$$N_1(r, W) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{n_1(t, W)}{t}dt, \qquad N_2(r, W) = \frac{1}{\nu}\int_{1}^{r}\frac{n_2(t, W)}{t}dt,$$

$$\overline{N}_1\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{\overline{n}_1\left(t, \frac{1}{W-a}\right)}{t}dt, \qquad \overline{N}_2\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{1}^{r}\frac{\overline{n}_2\left(t, \frac{1}{W-a}\right)}{t}dt,$$

$$\overline{N}_1^{k)}\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{\overline{n}_1^{k)}\left(t, \frac{1}{W-a}\right)}{t}dt, \qquad \overline{N}_2^{k)}\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{1}^{r}\frac{\overline{n}_2^{k)}\left(t, \frac{1}{w-a}\right)}{t}dt,$$

$$\overline{N}_1^{(k}\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{\overline{n}_1^{(k}\left(t, \frac{1}{W-a}\right)}{t}dt, \qquad \overline{N}_2^{(k}\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{1}^{r}\frac{\overline{n}_2^{(k}\left(t, \frac{1}{W-a}\right)}{t}dt,$$

$$m_0(r, W) = m(r, W) + m\left(\frac{1}{r}, W\right) - 2m(1, W), \qquad N_0(r, W) = N_1(r, W) + N_2(r, W),$$

$$\overline{N}_0\left(r, \frac{1}{W-a}\right) = \overline{N}_1\left(r, \frac{1}{W-a}\right) + \overline{N}_2\left(r, \frac{1}{W-a}\right), \overline{N}_0^{k}\left(r, \frac{1}{W-a}\right) = N_1^{k}\left(r, \frac{1}{W-a}\right) + N_2^{k}\left(r, \frac{1}{W-a}\right),$$

where $w_j(z)(j = 1, 2, ..., \nu)$ is one valued branch of $W(z)$, $n_1(t, W)$ is the counting functions of poles of the function $W(z)$ in $\{z : t < |z| \le 1\}$ and $n_2(t, W)$ is the counting functions of poles of the function $W(z)$ in $\{z : 1 < |z| \le t\}$ (both counting multiplicity). $\overline{n}_1\left(t, \frac{1}{W-a}\right)$ is the counting functions of poles of the function $\frac{1}{W-a}$ in $\{z : t < |z| \le 1\}$ and $\overline{n}_2\left(t, \frac{1}{W-a}\right)$ is the counting functions of poles of the function $\frac{1}{W-a}$ in $\{z : 1 < |z| \le t\}$ (both ignoring multiplicity). $n_1^k(t, a, W)$ is the number of zeros of $W - a$ in $\{z : t < |z| \le 1\}$ and $n_2^k(t, a, W)$ is the number of zeros of $W - a$ in $\{z : 1 < |z| \le t\}$, where zero of order $< k$ is counted according to it's multiplicity and a zero of order $\ge k$ is counted exactly $k$ times, respectively.

Let $W(z)$ be a $v$-valued algebroid function which determined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, when $a \in \mathbb{C}$, $n_0\left(r, \frac{1}{W-a}\right) = n_0\left(r, \frac{1}{\psi(z,a)}\right)$, $N_0\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}N_0\left(r, \frac{1}{\psi(z,a)}\right)$. In particular, when $a = 0$, $N_0\left(r, \frac{1}{W}\right) = \frac{1}{\nu}N_0\left(r, \frac{1}{A_0}\right)$. When $a = \infty$, $N_0(r, W) = \frac{1}{\nu}N_0\left(r, \frac{1}{A_v}\right)$; where $n_0\left(r, \frac{1}{W-a}\right)$ and $n_0\left(r, \frac{1}{\psi(z,a)}\right)$ are the counting function of zeros of $W(z) - a$ and $\psi(z, a)$ on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, respectively.

**Definition 2.1** ([6]). *Let $W(z)$ be an algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, the function*

$$T_0(r, W) = m_0(r, W) + N_0(r, W), \qquad 1 \le r < R_0$$

*is called Nevanlinna characteristic of $W(z)$.*

## 3. Some Lemmas

**Lemma 3.1** (The first fundamental theorem on annuli [7]). *Let $W(z)$ be $v$-valued algebroid function which is determined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, $a \in \mathbb{C}$*

$$T_0\left(r, \frac{1}{W-a}\right) = m_0\left(r, \frac{1}{W-a}\right) + N_0\left(r, \frac{1}{W-a}\right) = T_0(r, W) + O(1).$$

**Lemma 3.2** (The second fundamental theorem on annuli [13]). *Let $W(z)$ be v-valued algebroid function which is determined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, $a_k$ $(k = 1, 2, .., p)$ are $p$ distinct complex numbers (finite or infinite), then we have*

$$(p - 2v)T_0\,(r, W) \leq \sum_{k=1}^{p} N_0\left(r, \frac{1}{W - a_k}\right) - N_1(r, W) + S_0(r, W) \tag{2}$$

*where $N_1(r, W)$ is the density index of all multiple values including finite or infinite, every $\tau$ multiple value counts $\tau - 1$, and*

$$S_0(r, W) = m_0\left(r, \frac{W'}{W}\right) + \sum_{j=1}^{p} m_0\left(r, \frac{W'}{W - a_k}\right) + O(1).$$

*The remainder of the second fundamental theorem is the following formula*

$$S_0(r, W) = O\left(\log T_0(r, W)\right) + O(\log r),$$

*outside a set of finite linear measure, if $r \to R_0 = +\infty$, while*

$$S_0(r, W) = O\left(\log T_0(r, W)\right) + O\left(\log \frac{1}{R_0 - r}\right),$$

*outside a set $E$ of $r$ such that $\int_E \frac{dr}{R_0 - r} < +\infty$, when $r \to R_0 < +\infty$.*

**Lemma 3.3** ([7]). *Let $W(z)$ be $\nu$-valued algebroid function which is determined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, if the following conditions are satisfied*

$$\liminf_{r \to \infty} \frac{T_0\,(r, W)}{\log r} < \infty, \quad R_0 = +\infty,$$

$$\liminf_{r \to R_0^-} \frac{T_0\,(r, W)}{\log \frac{1}{(R_0 - r)}} < \infty, \quad R_0 < +\infty,$$

*then $W(z)$ is an algebraic function.*

# 4. Main Results

Now we prove the general form of Milloux inequality, which is our main result of this paper.

**Theorem 4.1** (General form of Milloux inequality). *Let $W(z)$ be a $\nu$-valued algebroid function determined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$. Let $a^{[i]}$, $b^{[j]} \in$ $(i = 1, 2, ..., p;\ j = 1, 2, ..., q)$ be distinct finite complex numbers such that $b^{[j]} \neq 0$ $(j = 1, 2, ..., q)$ and let $m_i$, $n_j$ $(i = 1, 2, ..., p;\ j = 1, 2, ..., q)$ and $l$ be any positive integers. Then*

$$\begin{aligned}&\left\{pq - \left(\sum_{i=1}^{p} \frac{kq+1}{m_i+1} + \sum_{j=1}^{q} \frac{1}{n_j+1} + \frac{1}{l+1}\left(1 + k\sum_{j=1}^{q} \frac{1}{n_j+1}\right)\right)\right\} T_0(r, W) \\ &\leq \frac{1}{l+1}\left(1 + k\sum_{j=1}^{q} \frac{1}{n_j+1}\right)\overline{N}_0^{l}(r, W) + (kq+1)\sum_{i=1}^{p} \overline{N}_0^{m_j}\left(r, \frac{1}{W - a^{[i]}}\right) + \sum_{j=1}^{q} \overline{N}_0^{n_j}\left(r, \frac{1}{W^{(k)} - b^{[j]}}\right) + S_0(r, W).\end{aligned} \tag{3}$$

*Proof.* We have

$$\begin{aligned}
T_0(r, W') &= m_0(r, W') + N_0(r, W') \\
&\leq m_0(r, W) + m_0\left(r, \frac{W'}{W}\right) + N_0(r, W') \\
&\leq m_0(r, W) + N_0(r, W) + \overline{N}_0(r, W) + N_x(r, W) + S_0(r, W) \\
&\leq m_0(r, W) + N_0(r, W) + \overline{N}_0(r, W) + 2(\nu - 1)T_0(r, W) + S_0(r, W) \\
&\leq (2\nu - 1)T_0(r, W) + \overline{N}_0(r, W) + S_0(r, W) \qquad (4) \\
&\leq 2\nu T_0(r, W) + S_0(r, W).
\end{aligned}$$

By Lemma 3.2, we get

$$S_0(r, W^{(k)}) = O(\log rT_0(r, W^{(k)})) = O(\log rT_0(r, W)) = S_0(r, W) \tag{5}$$

and hence

$$m_0\left(r, \frac{W^{(k)}}{W - a^{[i]}}\right) = S_0(r, W) \tag{6}$$

holds for any positive $a^{[i]}$. Put

$$F(z) = \sum_{i=1}^{p} \frac{1}{W(z) - a^{[i]}}.$$

Then as in [11], we have

$$\begin{aligned} m(r, F) + O(1) &\geq \sum_{i=1}^{p} m\left(r, \frac{1}{W(z) - a^{[i]}}\right). \\ m\left(\frac{1}{r}, F\right) &\geq \sum_{i=1}^{p} m\left(r, \frac{1}{W(z) - a^{[i]}}\right). \end{aligned} \tag{7}$$

In fact, (7) holds if $p = 1$. If $p \geq 2$, put

$$\delta = \min_{i \neq j} |a^{[i]} - a^{[j]}|.$$

Obviously $\delta > 0$. For fixed $z$, there exist some $k$ in $\{1, 2, ..., \nu\}$ and some $i$ in $\{1, 2, ..., q\}$, such that

$$|w_k - a^{[i]}| < \frac{\delta}{2q} \leq \frac{\delta}{4},$$

the inequality

$$|w_k(z) - a^{[j]}| \geq |a^{[i]} - a^{[j]}| - |w_k(z) - a^{[i]}| \geq \frac{3\delta}{4},$$

for $i \neq j$. Therefore the set of points on $\partial\mathbb{C}_r$ where $\mathbb{C}_r = \{z : |z| = r\}(r = r \text{ or } r = \frac{1}{r})$, which is determined by $|w_k(z) - a^{[i]}| < \frac{\delta}{2q}$ is either empty or any two such sets for different $i$ have empty intersection. In any case

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |F(re^{i\theta})| d\theta &\geq \frac{1}{2\pi} \sum_{i=1}^{q} \int_{|w_k - a^{[i]}| < \frac{\delta}{2q}, |z| = r} \log^+ |F(re^{i\theta})| d\theta \\ &\geq \frac{1}{2\pi} \sum_{i=1}^{q} \int_{|w_k - a^{[i]}| < \frac{\delta}{2q}, |z| = r} \log^+ \frac{1}{|w_k(re^{i\theta}) - a^{[i]}|} d\theta. \end{aligned}$$

Because of

$$\begin{aligned} \frac{1}{2\pi} \sum_{i=1}^{q} \int\limits_{|w_k - a^{[i]}| < \frac{\delta}{2q}, |z| = r} \log^+ \frac{1}{|w_k(re^{i\theta}) - a^{[i]}|} d\theta &= m\left(r, \frac{1}{W(z) - a^{[i]}}\right) \frac{1}{2\pi} \sum_{i=1}^{q} \int\limits_{|w_k - a^{[i]}| \geq \frac{\delta}{2q}, |z| = r} \log^+ \frac{1}{|w_k(re^{i\theta}) - a^{[i]}|} d\theta \\ &\geq m\left(r, \frac{1}{W(z) - a^{[i]}}\right) - q \log^+ \frac{2q}{\delta}. \end{aligned}$$

we deduce

$$\begin{aligned} m(r, F) &= \frac{1}{\nu} \sum_{k=1}^{\nu} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |F(re^{i\theta})| d\theta \geq \frac{1}{\nu} \sum_{k=1}^{\nu} \sum_{i=1}^{q} m\left(r, \frac{1}{W(z) - a^{[i]}}\right) - \frac{1}{\nu} \log^+ \frac{2q}{\delta} \\ &= \sum_{i=1}^{q} m\left(r, \frac{1}{W(z) - a^{[i]}}\right) - \frac{1}{\nu} \log^+ \frac{2q}{\delta}. \end{aligned}$$

Hence (7) follows from above inequality under the case of $r = r$ and $r = \frac{1}{r}$. Since

$$\begin{aligned} m(r,F) &= m(r,W^{(k)}F) + m\left(r,\frac{1}{W^{(k)}}\right) \\ &\leq \sum_{i=1}^{p} m\left(r,\frac{W^{(k)}}{W-a^{[i]}}\right) + m\left(r,\frac{1}{W^{(k)}}\right) \end{aligned}$$

and

$$\begin{aligned} m\left(\frac{1}{r},F\right) &= m\left(\frac{1}{r},W^{(k)}F\right) + m\left(\frac{1}{r},\frac{1}{W^{(k)}}\right) \\ &\leq \sum_{i=1}^{p} m\left(\frac{1}{r},\frac{W^{(k)}}{W-a^{[i]}}\right) + m\left(\frac{1}{r},\frac{1}{W^{(k)}}\right). \end{aligned}$$

Therefore

$$m_0\left(r,F\right) \leq \sum_{i=1}^{p} m_0\left(r,\frac{W^{(k)}}{W-a^{[i]}}\right) + m_0\left(r,\frac{1}{W^{(k)}}\right). \tag{8}$$

It follows from (5), (8) and Lemma 3.1 that

$$\begin{aligned} \sum_{i=1}^{p} m_0\left(r,\frac{1}{W(z)-a^{[i]}}\right) &\leq m_0\left(r,\frac{1}{W^{(k)}}\right) + S_0(r,W) \\ &\leq T_0(r,W^{(k)}) - N_0\left(r,\frac{1}{W^{(k)}}\right) + S_0(r,W). \end{aligned} \tag{9}$$

Thus

$$pT_0(r,W) \leq \sum_{i=1}^{p} N_0\left(r,\frac{1}{W(z)-a^{[i]}}\right) + T_0(r,W^{(k)}) - N_0\left(r,\frac{1}{W^{(k)}}\right) + S_0(r,W). \tag{10}$$

Now it follows from Lemma 3.1 and Lemma 3.2 and (5) that

$$\begin{aligned} qT_0(r,W^{(k)}) &\leq \sum_{i=1}^{q} N_0\left(r,\frac{1}{W^{(k)}(z)-b^{[j]}}\right) + N_0\left(r,\frac{1}{W^{(k)}}\right) + N_0(r,W^{(k)}) \\ &\quad -(N_0\left(r,\frac{1}{W^{(k+1)}}\right) + 2N_0(r,W^{(k)}) - N_0(r,W^{(k+1)}) + S_0(r,W^{(k)}) \\ &= \sum_{i=1}^{q} N_0\left(r,\frac{1}{W^{(k)}(z)-b^{[j]}}\right) + N_0\left(r,\frac{1}{W^{(k)}}\right) - N_0(r,W^{(k)}) \\ &\quad +N_0(r,W^{(k+1)}) - N_0\left(r,\frac{1}{W^{(k+1)}}\right) + S_0(r,W) \\ &\leq \sum_{i=1}^{q} N_0\left(r,\frac{1}{W^{(k)}(z)-b^{[j]}}\right) + N_0\left(r,\frac{1}{W^{(k)}}\right) + \overline{N}_0(r,W) - N_0\left(r,\frac{1}{W^{(k+1)}}\right) + S_0(r,W) \end{aligned} \tag{11}$$

Next, multiplying $q$ on both sides of equation (10) then we get

$$pqT_0(r,W) \leq q\sum_{i=1}^{p} N_0\left(r,\frac{1}{W(z)-a^{[i]}}\right) + qT_0(r,W^{(k)}) - qN_0\left(r,\frac{1}{W^{(k)}}\right) + S_0(r,W). \tag{12}$$

It follows from (12) and (11) that

$$\begin{aligned} pqT_0(r,W) &\leq q\sum_{i=1}^{p} N_0\left(r,\frac{1}{W(z)-a^{[i]}}\right) + \sum_{i=1}^{q} N_0\left(r,\frac{1}{W^{(k)}(z)-b^{[j]}}\right) + N_0\left(r,\frac{1}{W^{(k)}}\right) + \overline{N}_0(r,W) \\ &\quad -N_0\left(r,\frac{1}{W^{(k+1)}}\right) - qN_0\left(r,\frac{1}{W^{(k)}}\right) + S_0(r,W) \\ pqT_0(r,W) &\leq \overline{N}_0(r,W) + (q-1)\left\{\sum_{i=1}^{p} N_0\left(r,\frac{1}{W(z)-a^{[i]}}\right) - N_0\left(r,\frac{1}{W^{(k)}}\right)\right\} \\ &\quad +\left\{\sum_{i=1}^{p} N_0\left(r,\frac{1}{W(z)-a^{[i]}}\right) + \sum_{i=1}^{q} N_0\left(r,\frac{1}{W^{(k)}(z)-b^{[j]}}\right) - N_0\left(r,\frac{1}{W^{(k+1)}}\right)\right\} + S_0(r,W). \end{aligned} \tag{13}$$

A zero of $W(z)-a^{[i]}$ of order $s>k$ is a zero of $W^{(k+1)}$ of order $s-(k+1)$ and a zero of $W^{(k)}-b^{[j]}$ of order $s$ is a zero of $W^{(k+1)}$ of order $s-1$. Further, zeros of $W-a^{[i]}$ of order $>k$ are zeros of $W^{(k)}$ and so are not zeros of $W^{(k)}-b^{[j]}$. Hence

$$\begin{aligned}\sum_{i=1}^{p}N_0\left(r,\frac{1}{W(z)-a^{[i]}}\right)+\sum_{i=1}^{q}N_0\left(r,\frac{1}{W^{(k)}(z)-b^{[j]}}\right)-N_0\left(r,\frac{1}{W^{(k+1)}}\right)\\ \leq\sum_{i=1}^{p}N_0^{k+1}\left(r,\frac{1}{W(z)-a^{[i]}}\right)+\sum_{i=1}^{q}N_0\left(r,\frac{1}{W^{(k)}(z)-b^{[j]}}\right)\end{aligned}\tag{14}$$

and

$$\sum_{i=1}^{q}N_0\left(r,\frac{1}{W(z)-a^{[i]}}\right)-N_0\left(r,\frac{1}{W^{(k)}}\right)\leq\sum_{i=1}^{p}N_0^{k}\left(r,\frac{1}{W-a^{[i]}}\right).\tag{15}$$

Substituting (14) and (15) to (13), we obtain

$$pqT_0(r,W)\leq\overline{N}_0(r,W)+(q-1)\sum_{i=1}^{p}N_0^{k}\left(r,\frac{1}{W-a^{[i]}}\right)\sum_{i=1}^{p}N_0^{k+1}\left(r,\frac{1}{W(z)-a^{[i]}}\right)+\sum_{i=1}^{q}N_0\left(r,\frac{1}{W^{(k)}(z)-b^{[j]}}\right).\tag{16}$$

Since

$$\begin{aligned}N_0^{k+1}\left(r,\frac{1}{W-a^{[i]}}\right)&\leq(k+1)\overline{N}_0\left(r,\frac{1}{W-a^{[i]}}\right)\\ &\leq\frac{k+1}{m_i+1}\left\{m_i\overline{N}_0^{m_i}\left(r,\frac{1}{W-a^{[i]}}\right)+N_0\left(r,\frac{1}{W-a^{[i]}}\right)\right\}\\ &\leq\frac{k+1}{m_i+1}\left\{m_i\overline{N}_0^{m_i}\left(r,\frac{1}{W-a^{[i]}}\right)+T_0(r,W)\right\}+O(1),\end{aligned}\tag{17}$$

and

$$\begin{aligned}N_0^{k}\left(r,\frac{1}{W-a^{[i]}}\right)&\leq k\overline{N}_0\left(r,\frac{1}{W-a^{[i]}}\right)\\ &\leq\frac{k}{m_i+1}\left\{m_i\overline{N}_0^{m_i}\left(r,\frac{1}{W-a^{[i]}}\right)+N_0\left(r,\frac{1}{W-a^{[i]}}\right)\right\}\\ &\leq\frac{k}{m_i+1}\left\{m_i\overline{N}_0^{m_i}\left(r,\frac{1}{W-a^{[i]}}\right)+T_0(r,W)\right\}+O(1),\end{aligned}\tag{18}$$

Similarly, we can get

$$\overline{N}_0\left(r,\frac{1}{W^{(k)}-b^{[j]}}\right)\leq\frac{1}{n_j+1}\left\{n_j\overline{N}_0^{n_j}\left(r,\frac{1}{W^{(k)}-b^{[j]}}\right)+T_0(r,W^{(k)})\right\}+O(1)\tag{19}$$

and

$$\overline{N}_0(r,W)\leq\frac{1}{l+1}\left\{l\overline{N}_0^{l}(r,W)+T_0(r,W)\right\}.\tag{20}$$

By (6), we have

$$\begin{aligned}T_0(r,W^{(k)})&=m_0(r,W^{(k)})+N_0(r,W^{(k)})\\ &\leq m_0(r,W)+m_0\left(r,\frac{W^{(k)}}{W}\right)+N_0(r,W^{(k)})\\ &\leq m_0(r,W)+N_0(r,W)+k\overline{N}_0(r,W)+N_x(r,W)+S_0(r,W)\\ &\leq m_0(r,W)+N_0(r,W)+k\overline{N}_0(r,W)+2(\nu-1)(2k-1)T_0(r,W)+S_0(r,W)\\ &\leq[2\nu(2k-1)-3(k-1)]T_0(r,W)+S_0(r,W).\end{aligned}\tag{21}$$

From (17), (21) and (16), we obtain

$$\begin{aligned}
pqT_0(r,W) \ &\leq \ \overline{N}_0(r,W) + (q-1)\sum_{i=1}^{p}\frac{k}{m_i+1}\left\{m_i\overline{N}_0^{m_i}\left(r,\frac{1}{W-a^{[i]}}\right) + T_0(r,W)\right\} \\
&\quad + \sum_{i=1}^{p}\frac{k+1}{m_i+1}\left\{m_i\overline{N}_0^{m_i}\left(r,\frac{1}{W-a^{[i]}}\right) + T_0(r,W)\right\} \\
&\quad + \sum_{j=1}^{q}\frac{1}{n_j+1}\left\{n_j\overline{N}_0^{n_j}\left(r,\frac{1}{W^{(k)}-b^{[j]}}\right) + T_0(r,W^{(k)})\right\} + S_0(r,W) \\
&\leq \left(1+\sum_{j=1}^{q}\frac{k}{n_j+1}\right)\overline{N}_0(r,W) + (kq+1)\sum_{i=1}^{p}\frac{m_i}{m_i+1}\overline{N}_0^{m_i}\left(r,\frac{1}{W-a^{[i]}}\right) \\
&\quad + \sum_{i=1}^{p}\frac{kq+1}{m_i+1}T_0(r,W) + \sum_{j=1}^{q}\frac{n_j}{n_j+1}\overline{N}_o^{n_j}\left(r,\frac{1}{W^{(k)}-b^{[j]}}\right) \\
&\quad + \sum_{j=1}^{q}\frac{1}{n_j+1}T_0(r,W) + S_0(r,W) \qquad (22) \\
&\leq \left(1+\sum_{j=1}^{q}\frac{k}{n_j+1}\right)\frac{l}{l+1}\overline{N}_0^{l}(r,W) + (kq+1)\sum_{i=1}^{p}\overline{N}_0^{m_i}\left(r,\frac{1}{W-a^{[i]}}\right) \\
&\quad + \left\{\sum_{i=1}^{p}\frac{kq+1}{m_i+1} + \sum_{j=1}^{q}\frac{1}{n_j+1} + \frac{1}{l+1}\left(1+k\sum_{j=1}^{q}\frac{1}{n_j+1}\right)\right\}T_0(r,W) \\
&\quad + \sum_{j=1}^{q}\frac{1}{n_j+1}T_0(r,W) + \sum_{j=1}^{q}\overline{N}_0^{n_j}\left(r,\frac{1}{W^{(k)}-b^{[j]}}\right) + S_0(r,W).
\end{aligned}$$

Hence (3) follows from (22). □

Put $p=q=1$ and $l, m_i, n_j$ tend to infinity in (3), we get Milloux inequality as follows

**Theorem 4.2** (Milloux inequality). *Let $W(z)$ be a $\nu$-valued algebroid function determined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\leq+\infty)$, respectively. Let $a$, $b$ be two distinct complex number and $b\neq 0$. Then for any $0<r<R_0$, we have*

$$T_0(r,W) \leq \overline{N}(r,W) + (k+1)\overline{N}_0\left(r,\frac{1}{W-a}\right) + \overline{N}_0\left(r,\frac{1}{W^{(k)}-b}\right) + S_0(r,W).$$

# Acknowledgement

The first author is supported by the UGC SAP-DRS-III, Ref. No. F.510/3/DRS-III/2016(SAP-I) Dated : $29^{th}$ Feb, 2016 and second author is supported by the UGC- Rajiv Gandhi National Fellowship (no. F1-17.1/2013-14-SC-KAR-40380) of India.

## References

[1] T.B.Cao and Z.S.Deng, *On the uniqueness of meromorphic functions that share three or two finite sets on annuli*, Proceeding Mathematical Sciences, 122(2012), 203-220.

[2] S.Daochun and G.Zongsheng, *Value disribution theory of algebroid functions*, beijing: Science Press; (2014).

[3] W.K.Hayman, *Meromorphic functions*, Oxford: Oxford University Press, (1964).

[4] A.Ya.Khrystiyanyn and A.A.Kondratyuk, *On the Nevanlinna Theory for meromorphic functions on annuli-I*, Mathematychin Studii, 23(2005), 19-30.

[5] A.Ya.Khrystiyanyn and A.A.Kondratyuk, *On the Nevanlinna Theory for meromorphic functions on annuli-II*, Mathematychin Studii, 24(2005), 57-68.

[6] Yang Tan, *Several uniqueness theorems of algebroid functions on annuli*, Acta Mathematica Scientia, 36B(1)(2016), 295-316.

[7] T.Yang and Z.Qingcai, *The fundamental theorems of algebroid functions on annuli*, Turk. J. Math., 39(2015), 293-312.

[8] Yang Tan and Yue Wang, *On the multiple values and uniqueness of algebroid functions on annuli*, Complex Variable and Elliptic Equations, 60(9)(2015), 1254-1269.

[9] C.C.Yang and H.X.Yi, *Uniqueness theory of meromorphic functions*, Science Press, (1995); Kluwer, (2003).

[10] S.Axler, *Harmonic functions from a complex analysis view point*, Amer Math Monthly, 93(4)(1986), 246-258.

[11] Meili Liang, *On the value distribution of algebroid functions*, Sop transactions on applied mathematics, 1(1)(2014).

[12] Renukadevi S.Dyavanal, Ashwini M.Hattikal and Madhura M.Mathai, *Uniqueness of meromorphic functions sharing a set on annuli*, CUBO A Mathematical Journal, 18(01)(2016), 0114.

[13] Renukadevi S.Dyavanal and Ashok Rathod, *Uniqueness theorems for meromorphic functions on annuli*, IJMMS, 12(1)(2016), 1-10.