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An Extension of Common Fixed Point Theorem in D-Metric Space

Research Article

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Abstract: There have been number of generalizations of Metric spaces. D–Metric space is one such generalization initiated by Dhage [1] in 1984 and open new research area. Then many authors have obtained interesting fixed point results in D–Metric space satisfying contractive type condition. In this paper we proved some fixed point theorems in D–Metric space and also proved new fixed point theorem D–Metric space for a contractive self–maps.

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Keywords: D–Metric space, fixed point theorems.

1. Introduction

In 1984, Dhage introduce a new structure of topological space under the title "D–Metric space" and some fundamental fixed point principles proved for continuous as well as discontinuous operators. Rhoades [7] generalized Dhage's contractive condition by increasing number of factors and proved fixed point of self–mapping in D–Metric space. Then fixed point theorems in D–Metric space have been established by various authors. Our aim is to discuss about fixed point theory in D–Metric space and also established fixed point theorem in it, which is an extension of common fixed point theorem.

2. Some Preliminaries and Definitions

Definition 2.1 ([7]). *Let X be a non–empty set. Let function $d : X \times X \times X \to [0, \infty)$ is called a D–Metric if d satisfies, for all $x, y, z, a \in X$*

(D_1) $d(x, y, z) = 0$ *iff* $x = y = z$ *(coincidence)*

(D_2) $d(x, y, z) = d(p\{x, y, z\})$ *(Symmetry). Where p is a permutation of x, y, z.*

(D_3) $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$ *(Tetrahedral inequality)*

The non–empty set X together with D–Metric "d" is called D–Metric space and it is denoted by (X, d).

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Example 2.2. *Define* $d : R^n \times R^n \times R^n \to [0, \infty)$ *by,* $d(a, b, c) = \alpha \max\{\| a - b \|, \| b - c \|, \| c - a \|\}$ *for all* $a, b, c \in R^n$, $\alpha > 0$. *Where* $\| \ \|$ *is norm in Euclidean space* R^n *is a D–Metric on* R^n. *Hence* (R^n, d) *is a D–Metric space.*

Definition 2.3 ([2])**.** *Sequence* $\{a_n\}$ *in D–Metric space is called D–Cauchy if* $\lim\limits_{m,n,p} d(a_n, a_m, a_p) = 0$.

Definition 2.4 ([2])**.** *Sequence* $\{a_n\}$ *in D–Metric space (X, d) is said to be D–Convergent to* $a \in X$ *if* $\lim_{m,n} d(a_n, a_m, a) = 0$. *That is, for a point* $a \in X$, *if each* $\epsilon > 0$ *there exist positive integer* n_0 *such that* $d(a_n, a_m, a) < \epsilon$ *for all* $m, n > n_0$.

Definition 2.5 ([3])**.** *Let* (X, d) *be D–Metric space. A subset U of X is said to be bounded if these exist constant* $s > 0$ *such that,* $d(a, b, c) < s$, *for all* $a, b, c \in U$ *and s is called D–bound of U. For a bounded sequence* $\{y_n\}$ *in D–Metric space* (X, d), *let* $a_n = \delta(\{y_n, y_{n+1}, y_{n+2}, \ldots\})$ *for* $n \in N$. *Then* a_n *is finite for all* $n \in N$ *and* $\{a_n\}$ *is non increasing and* $a_n \geq 0$ *for all* $n \in N$ *so there is an* $a \geq 0$ *such that,* $\lim\limits_{n\to\infty} a_n = a$.

Definition 2.6 ([2])**.** *Every D–Cauchy sequence converges to a point in D–Metric space is called complete D–Metric space.*

Definition 2.7 ([6])**.** *Consider* (X, d) *be D–Metric space and* $f : X \to X$. *The orbit of f at the point* $a \in X$ *is the set* $O(a) = \{a, fa, f^2a, \ldots\}$.

Definition 2.8 ([6])**.** *Consider* (X, d) *be D–Metric space and* $O(a)$ *be orbit of* $f : X \to X$ *is said to be bounded if there exists a constant* $C > 0$ *such that* $d(x, y, z) \leq C$ *for all* $x, y, z \in O(a)$. *The constant C is called D–bound of* $O(a)$. *D–Metric space is said to be f–orbitally bounded if* $O(a)$ *is bounded for each* $a \in X$.

Definition 2.9 ([5])**.** *An orbit* $O(a)$ *is said to be f–orbitally complete if every D–Cauchy sequence in* $O(a)$ *converges to a point in X.*

Definition 2.10 ([5])**.** *Let* $f : X \to X$ *is called* α*–condensing if, for any bounded set* $B \subseteq X$, $f(B)$ *is bounded and* $\alpha(f(B)) < \alpha(B)$ *if* $\alpha(B) > 0$. *Many authors refer* α*– condensing maps as densifying mapping.*

3. Some Fixed Point Theorems on D-Metric Space

Theorem 3.1 ([6])**.** *Let f be a self map on X and f–orbitally bounded, X be complete D–Metric space also* α*–condensing. Then* $\overline{0(a)}$ *is compact for each* $a \in X$.

Proof. Consider $a \in X$ and define $B \subset X$ by $B = \{a_n\}$ where $a_n = f^n a$ then,

$$B = \{a, fa, f^2a, \ldots\} = \{a\} \cup \{fa, f^2a, \ldots\} = \{a\} \cup f(B). \tag{1}$$

Therefore if B is not precompact, then $\alpha(B) = \alpha(f(B)) < \alpha(B)$ which is contradiction. Therefore $\overline{B} = \overline{O\,(a)}$ is compact. □

Theorem 3.2. *Let* $f : X \to X$, *where f is continuous compact of a bounded D–Metric space such that,*

$$d(f^x a, f^y b, f^z c) < \delta(a, b, c) \quad for\ a, b, c \in X \tag{2}$$

with two of $\{a, b, c\}$ *are distinct and* $\delta(a, b, c)$ *defined as* $\delta(a, b, c) = \delta(O(a) \bigcup O(b) \bigcup O(c))$ *also* x, y, z *are fixed positive integer. Then f has a unique fixed point.*

Proof. Since f is compact, there is a compact subset A of X containing $f(X)$. Then $f(A) \subset A$ and $B = \bigcap_{n=1}^{\infty} f^n(A) (\neq \emptyset)$ be compact f-invariant subset of X. Since B is compact there is $a, b, c \in B$ such that, $\delta(B) = d(a, b, c)$ and let $\delta(B) > 0$. Since $f(B) = B$, there is $a_1, b_1, c_1 \in B$ such that $a = f_{a_1}^x, b = f_{b_1}^y$ and $c = f_{c_1}^z$, form (2)

$$\delta(B) = d(a, b, c) = d(f_{a_1}^x, f_{b_1}^y, f_{c_1}^z) < \delta(a, b, c) = \delta(B) \tag{3}$$

Which is a contradiction, therefore B contain single point, which is fixed point of f. Now, to show uniqueness, suppose s and t are fixed point of f such that $s \neq t$ Then from (2)

$$0 < d(s, s, t) = d(f_s^x, f_s^y, f_t^z) < d(s, s, t) \tag{4}$$

Which is a contradiction. Therefore there is a unique fixed point. □

Corollary 3.3 ([7]). *Let f be a continuous self map on X, where X is compact D–Metric space satisfying,*

$$d(f_a, f_b, f_c) < \max\{d(a, b, c), d(a, fa, c), d(b, f_b, c), d(a, f_b, c), d(b, fa, c), d(q, q, p)\} \tag{5}$$

For all $a, b, c \in X$ with $a \neq fa$, $b \neq f_b$, or $c \neq f_c$. Then f has a unique fixed point q in X.

Proof. From (5) we get $d(f_a, f_b, f_c) < \delta(a, b, c)$ and from Theorem 3.2 we get the existence and uniqueness of a fixed point q. □

Theorem 3.4 ([4]). *Let $\{b_n\}$ be a bounded sequence in D–Metric space X with ℓ as D–bound such that,*

$$d(b_n, b_{n+1}, b_m) \leq \emptyset^n(\ell) \tag{6}$$

For all $m > n \in N$, where $\emptyset : R^+ \to R^+$ satisfies $\sum_{n=1}^{\infty} \emptyset^n(s) < \infty$ for each $s \in R^+$. Then $\{b_n\}$ is D–Cauchy.

Let ψ denote the class of all real functions $\emptyset : R^+ \to R^+$ satisfying

1. $\emptyset$ is continuous
2. $\emptyset$ is increasing
3. $\emptyset(s) < s$ if $s > 0$ and
4. $\sum_{n=1}^{\infty} \emptyset^n(s) < \infty$ for $s \in R^+$.

The element of the class ψ is called control function and commonly used control function as $\emptyset(s) = ks$, $0 \leq k < 1$. Now using Definition 2.5, 2.8 and 2.9 we generalized Theorem 3.2 and Corollary 3.3 as follows.

4. Main Result

Theorem 4.1. *Consider f, g be self mapping on X such that,*

$$d(f_a, f_b, f_c) \leq \alpha\emptyset(\max\{d(ga, gb, gc) + d(gb, f_a, gc), d(gb, f_b, gc) + d(ga, f_b, gc), d(ga, f_a, gc) + d(ga, gb, gc)\}) \tag{7}$$

for $a, b, c \in X$ with $0 \leq \alpha < \frac{1}{2}$ and $\emptyset \in \psi$. Also,

1. $f(X) \subseteq g(X)$
2. $g(X)$ *is complete and* $f(X)$ *is bounded.*
3. g *and* f *are commuting.*

Then g *and* f *have unique common fixed point* $t \in X$.

Proof. Consider $a_0 = a \in X$ and define $\{b_n\}$ in X by

$$b_0 = ga_0, \quad b_{n+1} = fa_n = ga_{n+1}, n = 0, 1, 2, \dots \tag{8}$$

because $g(X) \supseteq f(X)$. If $b_r = b_{r+1}$ for $r \in N$ then, $b_{r=}f_{a_{r-1}} = f_{a_r} = ga_r = ga_{r+1} = b_{r+1} = t$, for some $t \in X$. Now, we prove that, t is a common fixed point of g and f. Because $f_{a_r} = ga_r$ and g, f are coincidentally commuting. Therefore $ft = fga_r = gfa_r = gt$. Now,

$$\begin{aligned}
d(ft, gt, t) &= d(ft, gfa_r, fa_r) \\
&= d(ft, ft, fa_r) \\
&\leq \alpha\emptyset(\max\{d(gt, gt, ga_r) + d(gt, ft, ga_r), d(gt, ft, ga_r) + d(gt, ft, ga_r), d(gt, ft, ga_r) + d(gt, gt, ga_r)\}) \\
&\leq \alpha\emptyset(\max\{d(gt, ft, t) + d(gt, ft, t), d(gt, ft, t) + d(gt, ft, t), d(gt, ft, t) + d(gt, ft, t)\}) \\
&\leq 2\alpha\emptyset(d(gt, ft, t)) \\
&\leq \emptyset(d(ft, gt, t))
\end{aligned}$$

Gives $ft = gt = t$. Therefore t is a common fixed point of g and f. Therefore assume that, $b_n \neq b_{n+1}$ for all n in N. Now, we prove that, $\{b_n\}$ is D–Cauchy. For $m > 1$,

$$\begin{aligned}
d(b_1, b_2, b_m) &= d(f_a, f_{a_1}, f_{a_{m-1}}) \\
&\leq \alpha\emptyset(\max\{d(ga_0, ga_1, ga_{m-1}) + d(ga_1, fa_0, ga_{m-1}), d(ga_1, fa_1, ga_{m-1}) \\
&\quad + d(ga_0, fa_1, ga_{m-1}), d(ga_0, fa_0, ga_{m-1}) + d(ga_0, ga_1, ga_{m-1})\}) \\
&\leq \alpha\emptyset(\max\{d(b_0, b_1, b_{m-1}) + d(b_1, b_1, b_{m-1}), d(b_1, b_2, b_{m-1}) + d(b_0, b_2, b_{m-1}), d(b_0, b_1, b_{m-1}) + d(b_0, b_2, b_{m-1})\}) \\
&\leq \alpha\emptyset\left(\max_{x,y,z}\left(d\left(b_x, b_y, b_z\right) + d\left(b_x, b_y, b_z\right)\right)\right), \text{ where } 0 \leq x \leq 1,\ 1 \leq y \leq 2,\ 1 \leq z \leq m \\
&\leq 2\alpha\emptyset\left(\max_{x,y,z} d\left(b_x, b_y, b_z\right)\right) \leq 2\alpha\emptyset(k) \leq \emptyset(k)
\end{aligned}$$

For $m > 2$,

$$\begin{aligned}
d(b_2, b_3, b_m) &= d(fa_1, fa_2, fa_{m-1}) \\
&\leq \alpha\emptyset(\max\{d(ga_1, ga_2, ga_{m-1}) + d(ga_2, fa_1, ga_{m-1}), d(ga_1, fa_2, ga_{m-1}) + d(ga_1, fa_2, ga_{m-1}), d(ga_1, fa_1, ga_{m-1}) \\
&\quad + d(ga_1, ga_2, ga_{m-1})\}) \\
&\leq \alpha\emptyset(\max\{d(b_1, b_2, b_{m-1}) + d(b_2, b_2, b_{m-1}), d(b_1, b_3, b_{m-1}) + d(b_1, b_3, b_{m-1}), d(b_1, b_2, b_{m-1}) + d(b_1, b_2, b_{m-1})\} \\
&\leq \alpha\emptyset^2\left(\max_{x,y,z}\left(d\left(b_x, b_y, b_z\right) + d\left(b_x, b_y, b_z\right)\right)\right) \\
&\leq 2\alpha\emptyset^2\left(\max_{x,y,z} d\left(b_x, b_y, b_z\right)\right), \quad \text{where } 0 \leq x \leq 2,\ 1 \leq y \leq 3,\ 2 \leq z \ \leq m \\
&\leq 2\alpha\emptyset^2(k) \leq \emptyset^2(k)
\end{aligned}$$

Ingeneral for $m > n$,

$$d(b_n, b_{n+1}, b_m) \leq 2\alpha\emptyset^{n-1}\left(\emptyset\left(\max_{i,j} d(b_i, b_j, b_{m-n})\right)\right), \quad \text{where } 0 \leq i \leq n \text{ and } 1 \leq j \leq n+1.$$
$$\leq 2\alpha\emptyset^n(k) \leq \emptyset^n(k)$$

By Theorem 3.4 gives $\{b_n\}$ is D–Cauchy. Since $g(X)$ is complete there exist point $t \in g(X)$ such that, $\lim b_n = t$ i.e. $\lim f_{a_n} = \lim ga_n = t$. Now to prove that t is a common fixed point of g and f. Because $t \in g(X)$ there is a point $q \in X$ such that $gq = t$. Now, we show that, $fq = gq = t$.

$$\begin{aligned} d(f_q, gq, gq) &= \lim_n d(f_q, f_{a_n}, f_{a_n}) \\ &\leq \lim_n \alpha\emptyset(\max\{d(gq, ga_n, ga_n) + d(ga_n, f_q, ga_n), d(ga_n, f_{a_n}, ga_n) \\ &\quad + d(gq, f_{a_n}, ga_n), d(gq, f_q, ga_n) + d(gq, ga_n, ga_n)\}) \\ &\leq \alpha\emptyset(\max\{0 + d(t, f_q, t), 0 + 0, d(t, f_q, t) + 0\}) \\ &\leq \alpha\emptyset(d(t, f_q, t)) \leq \emptyset(d(t, f_q, t)) \end{aligned}$$

Which gives $f_q = t$ since $\emptyset \in \psi$ then $t = f_q = gq$ is a common fixed point of g and f. To prove uniqueness, let $s(\neq t)$ be common fixed point of f and g then,

$$\begin{aligned} d(t, t, s) &= d(ft, ft, fs) \\ &\leq \alpha\emptyset(\max\{d(gt, gt, gs) + d(gt, ft, gs), d(gt, ft, gs) + d(gt, ft, gs), d(gt, ft, gs) + d(gt, gt, gs)\}) \\ &\leq \alpha\emptyset(\max\{d(t, t, s) + d(t, t, s), d(t, s, s) + d(t, s, s), d(t, t, s) + d(t, t, s)\}) \\ &\leq 2\alpha\emptyset(\max\{d(t, t, s), d(t, s, s)\}) \\ &\leq 2\alpha\emptyset(d(t, t, s)) < \emptyset(d(t, t, s)) \end{aligned}$$

Because $d(t, t, s) < \emptyset(d(t, t, s))$ is not possible. Therefore

$$d(t, t, s) \leq \emptyset(d(t, s, s)) \tag{9}$$

Now, interchanging the role of t and s we get,

$$d(t, s, s) \leq \emptyset(d(t, t, s)) \tag{10}$$

From (9) and (10) we get,

$$d(t, t, s) \leq \emptyset^2(d(t, t, s))$$

which is contradiction. Therefore $t = s$. □

5. Conclusion

In this paper we generalized fixed point theorem in D-Metric space and find out fixed point by using some contractive conditions.

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