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# On a Simple Inequality Problem

Research Article

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**Abstract:** This note introduces five different proofs of a simple inequality problem applying five well-known inequalities.

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## 1. Introduction

Mathematical Inequality is a useful and powerful tool in many mathematical areas. In mathematical contests we can also find many problems requiring proving inequalities. However, this topic is almost untouched in intro Math courses, and, in most cases, only slightly mentioned in advanced Math courses. As a result, students sometimes use lots complicated calculation to prove or calculate a fairly simple problem, due to their lack of these knowledge. In Cvetkovski's book [2], he provided the following example.

**Example 1.1.** *Prove that for every positive real number $a, b, c$ we have*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq a+b+c.$$

This seemingly easy problem can be proved in many ways. However, if one plans to prove it without using any already known inequalities, it becomes extremely difficult. Using this as an example, I will in this note first introduce five well-known inequalities. I will then use them one by one and provide five different proofs.

## 2. Proofs of the Problem

All the inequalities introduced in this section, together with their proofs, can be found in many inequality focused textbooks, like [4, 5], and the classic book [3] by Hardy, Littlewood, and Pólya. Here I refer to [2] as my main source of notations.

**Theorem 2.1** (AM-GM Inequality)**.** *Let $a_1, a_2, \cdots, a_n$ be positive real numbers. The numbers $AM = \frac{a_1+a_2+\cdots+a_n}{n}$ and $GM = \sqrt[n]{a_1a_2\cdots a_n}$ are called the arithmetic mean and geometric mean for the numbers $a_1, a_2, \cdots, a_n$, respectively, and we have $AM \geq GM$. Equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.*

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Using this inequality, we may find the first proof of Example 1.1. This proof was provided in [2] at p.16.

*Proof.* Applying AM-GM inequality, we have

$$\frac{a^2}{b} + b \geq 2a, \qquad \frac{b^2}{c} + c \geq 2b, \qquad \frac{c^2}{a} + a \geq 2c.$$

Summing them together we have $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + (a+b+c) \geq 2(a+b+c)$, which can be easily simplified to the claimed inequality in Example 1.1. □

**Theorem 2.2** (Rearrangement Inequality). *Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be real numbers. For any permutation $(x_1, x_2, \cdots, x_n)$ of $(a_1, a_2, \cdots, a_n)$ we have the following inequalities:*

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq x_1b_1 + x_2b_2 + \cdots + x_nb_n \geq a_nb_1 + a_{n-1}b_2 + \cdots + a_1b_n.$$

The next proof was summarized from part of a proof in [2] at p.66.

*Proof.* Without loss of generality, we may assume that $a \leq b \leq c$. Therefore, $a^2 \leq b^2 \leq c^2$, and $\frac{1}{a} \geq \frac{1}{b} \geq \frac{1}{c}$. Applying rearrangement inequality,

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{a^2}{a} + \frac{b^2}{b} + \frac{c^2}{c} = a + b + c.$$

□

Since it sometimes was mentioned in Vector Analysis or Linear Algebra courses, the next inequality may seem familiar for some students.

**Theorem 2.3** (Cauchy-Schwarz Inequality). *Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be real numbers. Then we have*

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_ib_i\right)^2.$$

*Proof.* Applying Cauchy-Schwarz Inequality, we have

$$(b+c+a)\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right) \geq (a+b+c)^2,$$

which can be easily simplified to the claimed inequality by dividing $(a+b+c)$ at both sides. □

The next inequality is very useful when considering sums of fractions.

**Theorem 2.4** (Radon's Inequality). *Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be positive real numbers. If $p$ is also a positive real number, then*

$$\frac{a_1^{p+1}}{b_1^p} + \frac{a_2^{p+1}}{b_2^p} + \cdots + \frac{a_n^{p+1}}{b_n^p} \geq \frac{(a_1 + a_2 + \cdots + a_n)^{p+1}}{(b_1 + b_2 + \cdots + b_n)^p}.$$

In our next proof, we need to use a special case of Radon's Inequality when $p = 1$. This special case is sometimes referred to as Bergström's Inequality (see [1]).

*Proof.* Applying Radon's Inequality,

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{(a+b+c)^2}{b+c+a} = a + b + c.$$

□

The next proof requires an important but not so commonly used inequality.

**Definition 2.5.** *We say that the sequence* $(b_i)_{i=1}^n$ *is majorized by* $(a_i)_{i=1}^n$*, denoted* $(b_i) \prec (a_i)$*, if we can rearrange the terms of the sequences* $(a_i)$ *and* $(b_i)$ *in such a way as to satisfy the following conditions:*

*(1).* $b_1 + b_2 + \cdots + b_n = a_1 + a_2 + \cdots + a_n$.

*(2).* $b_1 \geq b_2 \geq \cdots \geq b_n$ *and* $a_1 \geq a_2 \geq \cdots \geq a_n$

*(3).* $b_1 + b_2 + \cdots + b_s \leq a_1 + a_2 + \cdots + a_s$ *for any* $1 \leq s \leq n$.

**Theorem 2.6** (Karamata's Inequality)**.** *Let* $f : I \to R$ *be a convex function on the interval* $I \subseteq R$ *and let* $(a_i)_{i=1}^n, (b_i)_{i=1}^n$*, where* $a_i, b_i \in I, i = 1, 2, \cdots, n$*, are two sequences, such that* $(a_i) \succ (b_i)$*. Then*

$$f(a_i) + f(a_2) + \cdots + f(a_n) \geq f(b_1) + f(b_2) + \cdots + f(b_n).$$

*Proof.* Let $x_1 = \ln a$, $x_2 = \ln b$, and $x_3 = \ln c$. Then the claimed inequality becomes

$$e^{2x_1 - x_2} + e^{2x_2 - x_3} + e^{2x_3 - x_1} \geq e^{x_1} + e^{x_2} + e^{x_3}.$$

Since $f(x) = e^x$ is a convex function on $R$, if we consider the sequences $(a_i) = \{2x_1 - x_2, 2x_2 - x_3, 2x_3 - x_1\}$ and $(b_i) = \{x_1, x_2, x_3\}$, we only need to prove that $(a_i) \succ (b_i)$ (ordered in some way) according to Karamata's Inequality. Let us assume that $2x_{m_1} - x_{m_1+1} \geq 2x_{m_2} - x_{m_2+1} \geq 2x_{m_3} - x_{m_3+1}$, and $x_{k_1} \geq x_{k_2} \geq x_{k_3}$ for some indexes $m_i, k_i \in \{1, 2, 3\}$. Therefore,

$$2x_{m_1} - x_{m_1} \geq 2x_{k_1} - x_{k_1+1} \geq x_{k_1},$$
$$(2x_{m_1} - x_{m_1+1}) + (2x_{m_2} - x_{m_2+1}) \geq (2x_{k_1} - x_{k_1+1}) + (2x_{k_2} - x_{k_2+1}) \geq x_{k_1} + x_{k_2},$$

and

$$(2x_{m_1} - x_{m_1+1}) + (2x_{m_2} - x_{m_2+1}) + (2x_{m_3} - x_{m_3+1}) = x_1 + x_2 + x_3 = x_{k_1} + x_{k_2} + x_{k_3}.$$

So, $(a_i) \succ (b_i)$, and it finishes the proof. □

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