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Domination Number and Total Domination Number of Square of Normal Product of Cycles

Research Article

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Abstract: In 1958-Claude Berge introduced the domination number of a graph which is utilized to secure the single vertices. A set $S \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) - S$ is adjacent to at least one vertex of S. The cardinality of the smallest dominating set of G is called the domination number of G. A dominating set S is called total dominating set if the induced subgraph $\langle S \rangle$ has no isolated vertex. The square G^2 of a graph G is obtained from G by adding new edges between every two vertices having distance 2 in G. In this paper, we determine the domination number and total domination number of square of normal product of cycle graphs by evaluating their minimum dominating set and minimum total dominating set and short display are also provided to understand the results.

MSC: 05C50, 05C69

Keywords: Domination Number, Total Domination Number, Cycles, Square of a Graph, Normal Product.

1. Introduction

Graph theory is a stand out applicable limb of advanced mathematics with its multi-pronged technological advancements. The beginning of extensive study of dominating sets took place in nearly 1960. De jaenisch discovered and studied in depth the difficulties related to $(n \times n)$ chessboard problem that how many number of queens are required to lead. This seems the historical roots dating back to 1862 of this subject. Berge elaborated the conception of the domination number of a graph which is known as coefficient External Stability, in 1962. For the same concept, Ore provided the name "Dominating set" and "Domination number". After this, many theories were revealed on graph theory. An interesting theory was revealed in 1977 by Cockayne and Hedetniemi, they made outstanding survey related to dominating set and total dominating set in graphs. The writing regarding this matter has been surveyed and point by point in the two excellent domination books by Hynes, Hedetniemi and Slater who made a remarkable showing with regards to of bringing together outcomes scattered through somewhere in the range of 1200 domination papers around then. They have applied the notation $\Upsilon_t(G)$ for the total domination number of a graph.

Domination applies in facility location problem like: Hospitals, Fire stations when a person needs to travel to get to the nearest facility, the number of amenities are permanent and its required to minimize the distance in one attempt. It also works on that problem where maximum space to a facility is permanent and needed one attempt to minimize the number of amenities compulsory. In case that everyone is serviced Concepts from domination is also occur in problems including

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discovering sets of representative in the administering communication or electrical network and in land reconnoitering i.e. an evaluator must remain in order to acquire the estimations of height for a whole domain meanwhile the number of places are minimized. This is a fact that mathematical study of domination is creating its own area of study and plays an important role in mathematical study. It is surely possible that undiscovered prospect of possible set of the combinations of graphs may become a key aspect of network design. A productive endeavour could be significant in the study of different methods of creating a network, different products of graphs as an example.

All graphs consider to be simple, finite and undirected throughout the paper. Graph $G = (V, E)$ [13] where $V = V(G)$ is the vertex set and $E = E(G)$ edge set. Given two vertices u and v in graph G, we say u dominates v if $u = v$ or $uv \in E$. In graph theory, many graphs has been awared but we will discuss only Cycle graph with n vertices i.e. C_n and Path graph with n vertices i.e. P_n [10]. "The open neighborhood of a vertex u is denoted by $N(u) = \{v \in V(G), uv \in E(G)\}$ and the close neighborhood of a vertex u is denoted by $N[u] = N(u) \cup \{u\}$. The open neighborhood and closed neighborhood of $S \subseteq V(G)$ are defined as $N(S) = \cup_{u\in S}N(u)$ and $N[S] = \cup_{u\in S}N[u]$". A set $S \subseteq V(G)$ is a dominating set of graph G if every vertex of $V(G) - S$ is adjacent to atleast one vertex of set S. Domination number is called the cardinality of the smallest dominating set of G which is denoted by $\Upsilon(G)$. A dominating set of cardinality $\Upsilon(G)$ is called a $\Upsilon-$ set. "A total dominating set, denoted TDS, of G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S. The total domination number of G, denoted by $\Upsilon_t(G)$ is the minimum cardinality of a TDS". Clearly $\Upsilon(G) \leq \Upsilon_t(G)$, also it has been proved that $\Upsilon_t(G) \leq 2\,\Upsilon\,(G)$. For a survey of total domination in graphs, See [8].

Given any graph G, its square graph G^2 is a graph with the same vertex set $V(G)$ and two vertices are adjacent whenever they are at distance 1 or 2 in graph G. A set $S \subseteq V(G)$ is a $2-$ distance dominating set of G if $d_G(u, s) = 1$ or 2 for each vertex of $\{V(G) - S\}$. $\Upsilon^2(G)$ is denoted as the cardinality of the smallest two distance dominating set of G and is called $2-$ distance domination number of G. In graph theory, many operations will be used like as: Normal product, Cartesian product, Tensor product, Corona product, etc. but here bring to light on the "Normal Product". The Normal product was first introduced by sabidussi [11]. The normal product of a graphs $G(V(G), E(G))$ and $H(V(H), E(H))$ denoted by $G \boxtimes H$ is a graph with vertex set $V(G \boxtimes H) = V(G) \times V(H)$, that is, the set $\{(u, v)/u \in G,\ v \in H\}$ and edge $((u, u')\ ,\ (v, v'))$ exists whenever any of the following conditions hold:

(1). $(u, u') \in E(G)$ and $v = v'$

(2). $u = u'$ and $(v, v') \in E(H)$

(3). $(u, u') \in E(G)$ and $(v, v') \in E(H)$.

In this paper, we provide an approach to finding Dominating set and Total Dominating Set of Square of Normal Product of Cycle Graph. It is based on the cardinality of the smallest dominating set and smallest total dominating set of G. This paper is organized as follows: In Section 2, related work is given. Section 3 and Section 4 contains the main result of the paper and its standard graphs shown by example. In the Section 5, discussion and conclusion is given. Section 6 contains references.

2. Related Work

Haynes et al. [6] examined the dominating number of distinct graphs and indicated various problems in this concept. Further, Jacobson and Kinch [9] proposed the domination number of products of graphs. It is based on the different products of graphs and took different approaches to the problem. Chaluvaraju and Appajigowda [3] established the dominating set of normal product of paths and cycles. They identified the various results of normal product of paths and its dominating sets.

Alishahi et al.[1] deduced the square of graphs mainly and contradicted an important resolution regarding the Cartesian product of cycles and paths. Atapour M. et al.[2] gave important results over total dominating sets of different graphs which was used in their paper. Some results at high level and algorithms for total dominating set were analysed by Henning [8]

3. Domination Number of $(C_m \boxtimes C_n)^2$

In this section, We investigate the domination number of square of normal products including two cycles C_m and C_n for $m \geq 3$ and $n \geq 3$, respectively. It is easy to check the following:

Theorem 3.1. *For $m = 5k_1 - 2$ or $5k_1 - 1$ or $5k_1$, $k_1 \geq 1$ and $m = 5k_1 - 3$, $k_1 \geq 2$, $\gamma[(C_m \boxtimes C_n)]^2 = k_1k_2$, if $n = 5k_2 - 4$ or $5k_2 - 3$ or $5k_2 - 2$ or $5k_2 - 1$ or $5k_2$, $k_2 \geq 1$.*

Proof. Let C_m and C_n be the Cycle graphs with vertex sets $\{u_1, u_2, ..., u_m\}$ and $\{v_1, v_2, ..., v_n\}$ respectively. Then the following cases arise.

Case 1: For $m = 5k_1 - 2$ or $5k_1 - 1$ or $5k_1$, $k_1 \geq 1$. In this case, we consider the two subcases as follows:

Subcase 1: For $n = 5k_2 - 2$ or $5k_2 - 1$ or $5k_2$, $k_2 \geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $A = \{(u_{5t_1-2}, v_{5t_2-2})\ ;\ 1 \leq t_1 \leq k_1\ ;\ 1 \leq t_2 \leq k_2\}$ is the $\gamma-$ set. Clearly, no dominating set with cardinality less than $|A|$ exists in the graph. Thus A is minimum dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma[(C_m \boxtimes C_n)]^2 = |A| = k_1k_2$.

Subcase 2: For $n = 5k_2 - 4$ or $5k_2 - 3$, $k_2 \geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $B = \{(u_{5t_1-3}, v_{5t_2-4})\ ;\ 1 \leq t_1 \leq k_1\ ;\ 1 \leq t_2 \leq k_2\}$ is the $\gamma-$ set. Clearly, no dominating set with cardinality less than $|B|$ exists in the graph. Thus B is minimum dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma[(C_m \boxtimes C_n)]^2 = |B| = k_1k_2$.

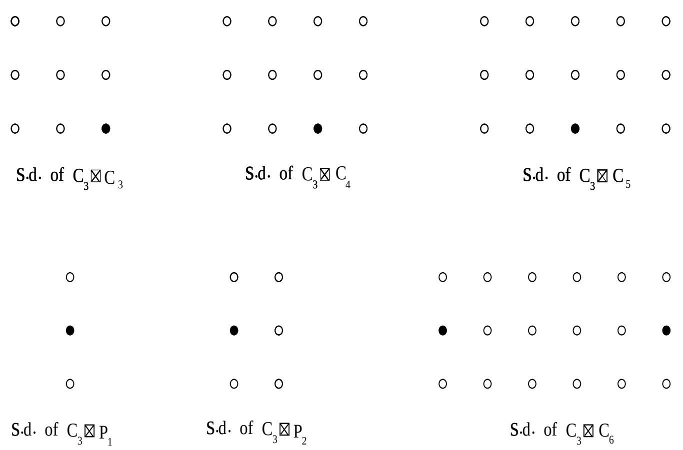


Figure 1: A dominating set for $(C_m \boxtimes C_n)^2$ (Case 1)

Case 2: For $m = 5k_1 - 3$, $k_1 \geq 2$. In this case, we consider the two subcases as follows:

Subcase 1: For $n = 5k_2 - 2$ or $5k_2 - 1$ or $5k_2$, $k_2 \geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $C = \{(u_{5t_1-3}, v_{5t_2-2})\ ;\ 1 \leq t_1 \leq k_1\ ;\ 1 \leq t_2 \leq k_2\}$ is the $\gamma-$ set. Clearly, no dominating set with cardinality less than $|C|$ exists in the graph. Thus C is minimum dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma[(C_m \boxtimes C_n)]^2 = |C| = k_1k_2$.

Subcase 2: For $n = 5k_2 - 4$ or $5k_2 - 3$, $k_2 \geq 1$. In $[(C_m \boxtimes C_{n)}]^2$, the subset $D = \{(u_{5t_1-3}, v_{5t_2-4})\ ;\ 1 \leq t_1 \leq k_1\ ;\ 1 \leq t_2 \leq k_2\}$ is the $\gamma-$ set. Clearly, no dominating set with cardinality less than $|D|$ exists in the graph. Thus D is minimum dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma[(C_m \boxtimes C_n)]^2 = |D| = k_1k_2$.

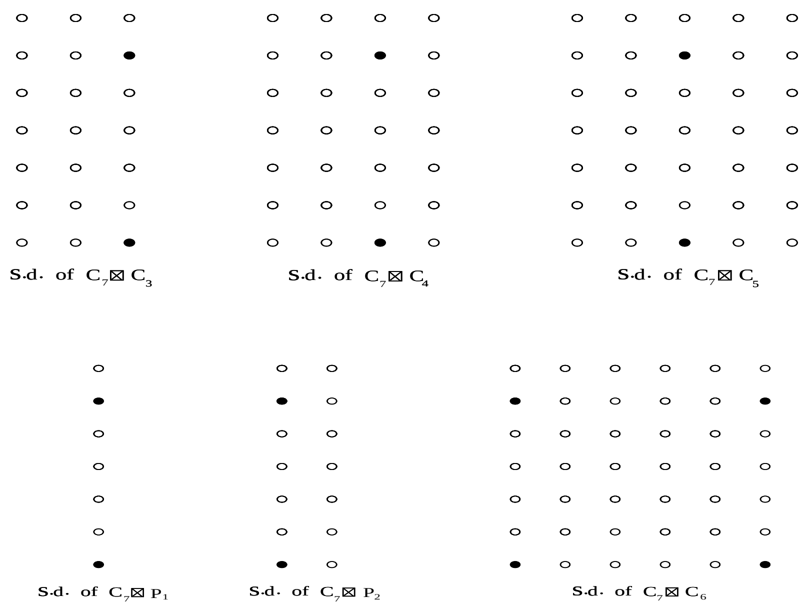


Figure 2: A dominating set for $(C_m \boxtimes C_n)^2$ (Case 2)

$\square$

Theorem 3.2. *For* $m = 5k_1 - 4, \quad k_1 \geq 2,$

$$\gamma[(C_m \boxtimes C_n)]^2 = \begin{cases} k_1k_2 & \text{if } n = 5k_2 - 3 \text{ or } 5k_2 - 2 \text{ or } 5k_2 - 1 \text{ or } 5k_2, \; k_2 \geq 1 \\ k_1(k_2+1) - 1 & \text{if } n = 5k_2 + 1, \; k_2 \geq 1. \end{cases}$$

Proof. Let C_m and C_n be the Cycle graphs with vertex sets $\{u_1, u_2, ..., u_m\}$ and $\{v_1, v_2, ..., v_n\}$ respectively. Then the following cases arise.

Case 1: If $m = 5k_1 - 4, \; k_1 \geq 2$. In this case, we consider the two subcases as follows:

Subcase 1: For $n = 5k_2 - 3$ or $5k_2 - 2, \; k_2 \geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $A = \{(u_{3t_1}, v_{5t_2-4}) \; ; \; 1 \leq t_1 \leq k_1 \; ; \; 1 \leq t_2 \leq k_2\}$ is the $\gamma-$ set. Clearly, no dominating set with cardinality less than $|A|$ exists in the graph. Thus A is minimum dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma[(C_m \boxtimes C_n)]^2 = |A| = k_1k_2$.

Subcase 2: For $n = 5k_2 - 1$ or $5k_2, \; k_2 \geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $B = \{(u_{3t_1}, v_{5t_2-2}) \; ; \; 1 \leq t_1 \leq k_1 \; ; \; 1 \leq t_2 \leq k_2\}$ is the $\gamma-$ set. Clearly, no dominating set with cardinality less than $|B|$ exists in the graph. Thus B is minimum dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma[(C_m \boxtimes C_n)]^2 = |B| = k_1k_2$.

Case 2: If $m = 5k_1 - 4, \; k_1 \geq 2$. In this case, we consider the two subcases as follows:

Subcase 1: For $n = 6, \; k_2 \geq 2$. In $[(C_m \boxtimes C_n)]^2$, the subset $C = \{(u_1, v_1)\} \bigcup \{(u_{5t_1+1}, v_3) \; ; \; 1 \leq t_1 \leq k_1 - 1\} \bigcup \{(u_{5t_1-2}, v_{5t_2+1}) \; ; \; 1 \leq t_1 \leq k_1 - 1 \; ; \; 1 \leq t_2 \leq k_2\}$ is the $\gamma-$ set. Clearly, no dominating set with cardinality less than $|C|$ exists in the graph. Thus C is minimum dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma[(C_m \boxtimes C_n)]^2 = |C| = \{k_1(k_2+1) - 1\}$.

Subcase 2: For $n = 5k_2 + 1, \; k_2 \geq 2$. In $[(C_m \boxtimes C_n)]^2$, the subset $D = \{(u_1, v_1)\} \bigcup \{(u_{5t_1+1}, v_3) \; ; \; 1 \leq t_1 \leq k_1 - 1\} \bigcup \{(u_{5t_1-2}, v_{5t_2+1}) \; ; \; 1 \leq t_1 \leq k_1 - 1 \; ; \; 1 \leq t_2 \leq k_2\} \bigcup \{(u_{5t_1+1}, v_{5t_2+3}) \; ; \; t_1 = k_1 - 1 \; ; \; 1 \leq t_2 \leq k_2\}$ is the $\gamma-$ set. Clearly, no dominating set with cardinality less than $|D|$ exists in the graph. Thus D is minimum dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma[(C_m \boxtimes C_n)]^2 = |D| = \{k_1(k_2+1) - 1\}$.

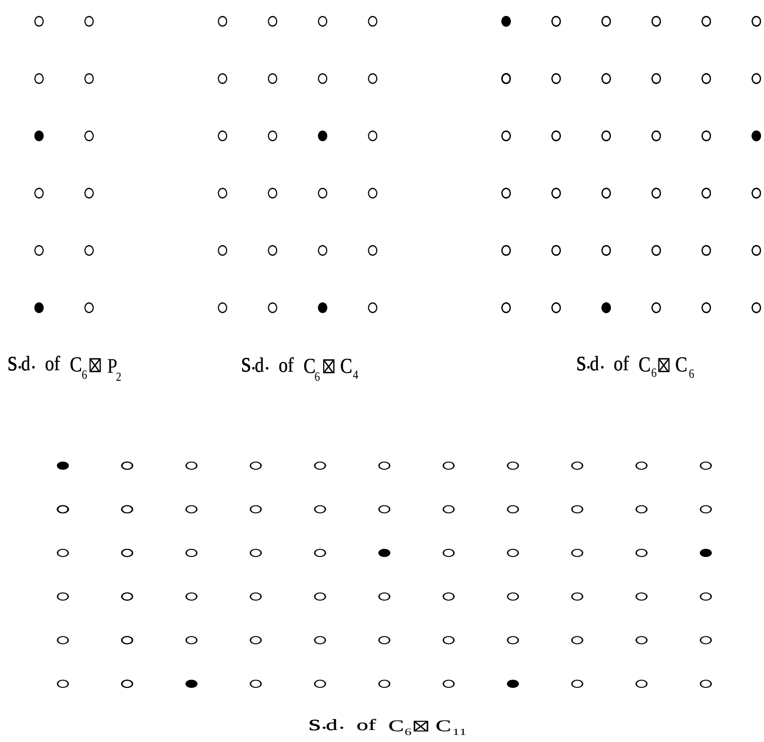


Figure 3: A dominating set for $(C_m \boxtimes C_n)^2$

$\square$

4. Total Domination Number of $(C_m \boxtimes C_n)^2$

In this section, We give the values of $\Upsilon_t(C_m \boxtimes C_n)^2$ when $m, n \geq 3$.

Theorem 4.1. *For $m = 5k_1 - 2$ or $5k_1 - 1$ or $5k_1,\ k_1 = 1$,*

$$\Upsilon_t[(C_m \boxtimes C_n)]^2 = \begin{cases} 2k_1k_2 & \text{if } n = 7k_2 - 4 \text{ or } 7k_2 - 3 \text{ or } 7k_2 - 2 \text{ or } 7k_2 - 1 \text{ or } 7k_2,\ k_2 \geq 1 \\ 2k_1k_2 + 1 & \text{if } n = 7k_2 + 1 \text{ or } 7k_2 + 2,\ k_2 \geq 1. \end{cases}$$

Proof. Let C_m and C_n be the Cycle graphs with vertex sets $\{u_1, u_2, ..., u_m\}$ and $\{v_1, v_2, ..., v_n\}$ respectively. Then the following cases arise.

Case 1: If $m = 5k_1 - 2$ or $5k_1 - 1$ or $5k_1,\ k_1 = 1$. In this case, we consider the two subcases as follows:

Subcase 1: For $n = 7k_2 - 4$ or $7k_2 - 3$ or $7k_2 - 2,\ k_2 \geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $A = \{(u_3, v_{7t-6})\ ;\ 1 \leq t \leq k_2\} \bigcup \{(u_3, v_{7t-4})\ ;\ 1 \leq t \leq k_2\}$ is the γ_t- set. Clearly, no total dominating set with cardinality less than $|A|$ exists in the graph. Thus A is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |A| = 2k_1k_2$.

Subcase 2: For $n = 7k_2 - 1$ or $7k_2,\ k_2 \geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $B = \{(u_3, v_{9t-6})\ ;\ 1 \leq t \leq k_2\} \bigcup \{(u_3, v_{8t-5})\ ;\ 1 \leq t \leq k_2\}$ is the γ_t- set. Clearly, no total dominating set with cardinality less than $|B|$ exists in the graph. Thus B is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |B| = 2k_1k_2$.

Case 2: If $m = 5k_1 - 2$ or $5k_1 - 1$ or $5k_1,\ k_1 = 1$. In this case, For $n = 7k_2 + 1$ or $7k_2 + 2,\ k_2 \geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $C = \{u_3, v_3\} \bigcup \{(u_3, v_{7t-2})\ ;\ 1 \leq t \leq k_2\} \bigcup \{(u_3, v_{7t})\ ;\ 1 \leq t \leq k_2\}$ is the γ_t- set. Clearly, no total dominating set with cardinality less than $|C|$ exists in the graph. Thus C is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |C| = 2k_1k_2 + 1$.

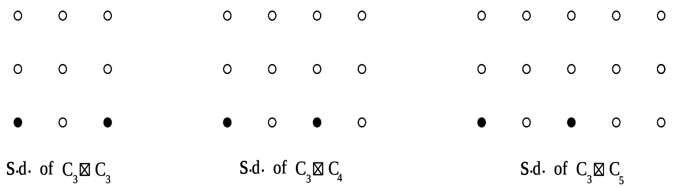


Figure 4: A total dominating set for $(C_m \boxtimes C_n)^2$

□

Theorem 4.2. *For* $m = 5k_1 + 1$ *or* $5k_1 + 2$, $k_1 = 1$,

$$\gamma_t[(C_m \boxtimes C_n)]^2 = \begin{cases} 2 & \text{if } n = 3 \text{ or } 4 \text{ or } 5 \\ 2k_2 & \text{if } n = 5k_2+1 \text{ or } 5k_2+2 \text{ or } 5k_2+3 \text{ or } 5k_2+4 \text{ or } 5k_2+5,\ k_2 \geq 1. \end{cases}$$

Proof. Let C_m and C_n be the Cycle graphs with vertex sets $\{u_1, u_2, ..., u_m\}$ and $\{v_1, v_2, ..., v_n\}$ respectively. Then the following cases arise.

Case 1: For $n = 3$ or 4 or 5. In $[(C_m \boxtimes C_n)]^2$, the subset $A = \{(u_3, v_1), (u_3, v_3)\}$ is the γ_t- set. Clearly, no total dominating set with cardinality less than $|A|$ exists in the graph. Thus A is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |A| = 2$.

Case 2: If $m = 5k_1 + 1$ or $5k_1 + 2$, $k_1 = 1$. In this case, we consider the two subcases as follows:

Subcase 1: For $n = 5k_2 + 1$ or $5k_2 + 2$, $k_2 \geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $B = \{(u_3, v_{5t-4})\ ;\ 1 \leq t \leq k_2 + 1\} \bigcup \{(u_5, v_{5t-4})\ ;\ 1 \leq t \leq k_2 + 1\}$ is the γ_t- set. Clearly, no total dominating set with cardinality less than $|B|$ exists in the graph. Thus B is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |B| = 2k_2$.

Subcase 2: For $n = 5k_2 + 3$ or $5k_2 + 4$ or $5k_2 + 5$, $k_2 \geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $C = \{(u_3, v_{5t-2})\ ;\ 1 \leq t \leq k_2 + 1\} \bigcup \{(u_5, v_{5t-2})\ ;\ 1 \leq t \leq k_2 + 1\}$ is the γ_t- set. Clearly, no total dominating set with cardinality less than $|C|$ exists in the graph. Thus C is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |C| = 2k_2$.

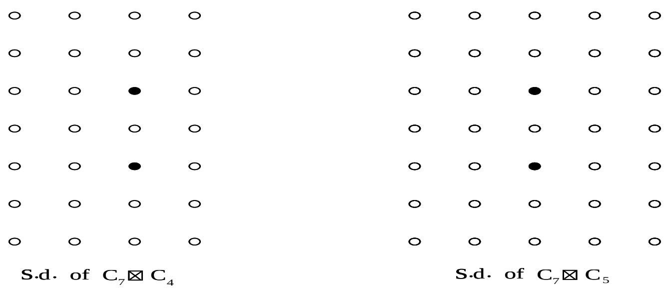


Figure 5: A total dominating set for $(C_m \boxtimes C_n)^2$

□

Theorem 4.3. *For* $m = 5k_1 + 3$ *or* $5k_1 + 4$, $k_1 = 1$,

$$\gamma_t[(C_m \boxtimes C_n)]^2 = \begin{cases} 3k_1k_2 & \text{if } n = 5k_2-2 \text{ or } 5k_2-1 \text{ or } 5k_2,\ k_2 = 1 \text{ or } 2 \\ 4 & \text{if } n = 6 \text{ or } 7 \\ k_2(k_1+2)+3 & \text{if } n = 5k_2+1 \text{ or } 5k_2+2 \text{ or } 5k_2+3 \text{ or } 5k_2+4 \text{ or } 5k_2+5,\ k_2 \geq 2. \end{cases}$$

Proof. Let C_m and C_n be the Cycle graphs with vertex sets $\{u_1, u_2, ..., u_m\}$ and $\{v_1, v_2, ..., v_n\}$ respectively. Then the following cases arise.

Case 1: For $n = 5k_2 - 2$ or $5k_2 - 1$ or $5k_2$, $k_2 = 1$ or 2. In $[(C_m \boxtimes C_n)]^2$, the subset $A = \{(u_3, v_{5t-2})\ ;\ 1 \leq t \leq k_2\} \bigcup \{(u_5, v_{5t-2})\ ;\ 1 \leq t \leq k_2\} \bigcup \{(u_7, v_{5t-2})\ ;\ 1 \leq t \leq k_2\}$ is the γ_t- set. Clearly, no total dominating set with cardinality less than $|A|$ exists in the graph. Thus A is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |A| = 3k_1k_2$.

Case 2: For $n = 6$ or 7. In $[(C_m \boxtimes C_n)]^2$, the subset $B = \{(u_3, v_{2t+1})\ ;\ 1 \leq t \leq k_2 + 1\} \bigcup \{(u_8, v_{2t+1})\ ;\ 1 \leq t \leq k_2 + 1\}$ is the γ_t- set. Clearly, no total dominating set with cardinality less than $|B|$ exists in the graph. Thus B is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |B| = 4$.

Case 3: If $m = 5k_1 + 3$ or $5k_1 + 4$, $k_1 = 1$. In this case, we consider the two subcases as follows:

Subcase 1: For $n = 5k_2 + 1$ or $5k_2 + 2$ or $5k_2 + 3$, $k_2 \geq 2$. In $[(C_m \boxtimes C_n)]^2$, the subset $C = \{(u_3, v_{5t-4})\ ;\ 1 \leq t \leq k_2 + 1\} \bigcup \{(u_5, v_{5t-4})\ ;\ 1 \leq t \leq k_2 + 1\} \bigcup \{(u_7, v_{5t-4})\ ;\ 1 \leq t \leq k_2 + 1\}$ is the γ_t- set. Clearly, no total dominating set with cardinality less than $|C|$ exists in the graph. Thus C is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |C| = k_2(k_1 + 2) + 3$.

Subcase 2: For $n = 5k_2 + 4$ or $5k_2 + 5$, $k_2 \geq 2$. In $[(C_m \boxtimes C_n)]^2$, the subset $D = \{(u_3, v_{5t-2})\ ;\ 1 \leq t \leq k_2 + 1\} \bigcup \{(u_5, v_{5t-2})\ ;\ 1 \leq t \leq k_2 + 1\} \bigcup \{(u_7, v_{5t-2})\ ;\ 1 \leq t \leq k_2 + 1\}$ is the γ_t- set. Clearly, no total dominating set with cardinality less than $|D|$ exists in the graph. Thus D is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |D| = k_2(k_1 + 2) + 3$.

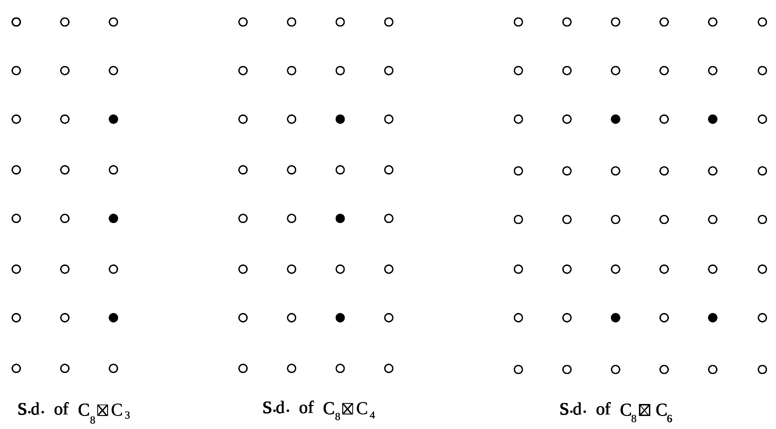


Figure 6: A total dominating set for $(C_m \boxtimes C_n)^2$

$\square$

Theorem 4.4. *For $m = 5k_1 + 1$ or $5k_1 + 2$ or $5k_1 + 3$ or $5k_1 + 4$ or $5k_1 + 5$, $k_1 \geq 2$,*

$$\gamma_t[(C_m \boxtimes C_n)]^2 = \left\{ (k_1 + 1)(k_2 + 1) \quad \text{if } n = 2k_2 + 4 \text{ or } 2k_2 + 5,\ k_2 \geq 1. \right.$$

Proof. Let C_m and C_n be the Cycle graphs with vertex sets $\{u_1, u_2, ..., u_m\}$ and $\{v_1, v_2, ..., v_n\}$ respectively. Then the following cases arise.

Case 1: If $m = 5k_1 + 1$ or $5k_1 + 2$ or $5k_1 + 3$ or $5k_1 + 4$ or $5k_1 + 5$, $k_1 \geq 2$. In this case, we consider the four subcases as follows:

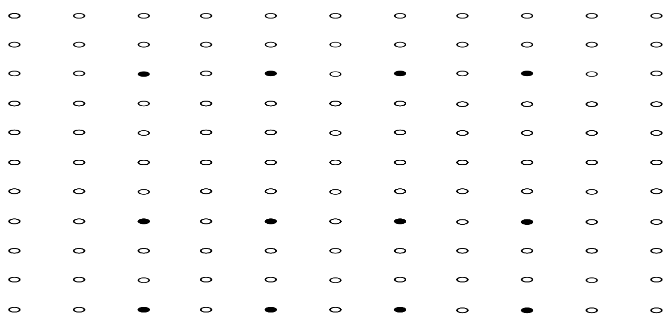
Subcase 1: For $m = 5k_1 + 1$ or $5k_1 + 2$, $k_1 \geq 2$ and $n = 2k_2 + 4$, ≥ 1. In $[(C_m \boxtimes C_n)]^2$, the subset $A = \{(u_3, v_{2t})\ ;\ 1 \leq t \leq k_2 + 1\} \bigcup \{(u_8, v_{2t})\ ;\ 1 \leq t \leq k_2 + 1\} \bigcup \{(u_{11}, v_{2t})\ ;\ 1 \leq t \leq k_2 + 1\}$ is the γ_t- set. Clearly, no total dominating set

set with cardinality less than $|A|$ exists in the graph. Thus A is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |A| = (k_1+1)(k_2+1)$.

Subcase 2: For $m = 5k_1+3$ or $5k_1+4$ or $5k_1+5,\ k_1 \geq 2$ and $n = 2k_2+4,\ k_2 \geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $B = \{(u_3, v_{2t})\ ;\ 1 \leq t \leq k_2+1\} \bigcup \{(u_8, v_{2t})\ ;\ 1 \leq t \leq k_2+1\} \bigcup \{(u_{11}, v_{2t})\ ;\ 1 \leq t \leq k_2+1\}$ is the γ_t- set. Clearly, no total dominating set with cardinality less than $|B|$ exists in the graph. Thus B is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |B| = (k_1+1)(k_2+1)$.

Subcase 3: For $m = 5k_1+1$ or $5k_1+2,\ k_1 \geq 2$ and $n = 2k_2+5,\ k_2 \geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $C = \{(u_3, v_{2t+1})\ ;\ 1 \leq t \leq k_2+1\} \bigcup \{(u_8, v_{2t+1})\ ;\ 1 \leq t \leq k_2+1\} \bigcup \{(u_{11}, v_{2t+1})\ ;\ 1 \leq t \leq k_2+1\}$ is the γ_t- set. Clearly, no total dominating set with cardinality less than $|C|$ exists in the graph. Thus C is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |C| = (k_1+1)(k_2+1)$.

Subcase 4: For $m = 5k_1+3$ or $5k_1+4$ or $5k_1+5, k_1 \geq 2$ and $n = 2k_2+5,\ k_2 \geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $D = \{(u_3, v_{2t+1})\ ;\ 1 \leq t \leq k_2+1\} \bigcup \{(u_8, v_{2t+1})\ ;\ 1 \leq t \leq k_2+1\} \bigcup \{(u_{11}, v_{2t+1})\ ;\ 1 \leq t \leq k_2+1\}$ is the γ_t- set. Clearly, no total dominating set with cardinality less than $|D|$ exists in the graph. Thus D is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |D| = (k_1+1)(k_2+1)$.



S.d. of $C_{11} \boxtimes C_{11}$

Figure 7: A total dominating set for $(C_m \boxtimes C_n)^2$

□

Theorem 4.5. *For $n = 3$ or 4 or 5,*

$$\gamma_t[(C_m \boxtimes C_n)]^2 = \begin{cases} 2(k_1+1) & \text{if } m = 7k_1+3 \text{ or } 7k_1+4 \text{ or } 7k_1+5 \text{ or } 7k_1+6 \text{ or } 7k_1+7,\ k_1 \geq 1 \\ 2k_1+3 & \text{if } m = 7k_1+8 \text{ or } 7k_1+9,\ k_1 \geq 1. \end{cases}$$

Proof. Let C_m and C_n be the Cycle graphs with vertex sets $\{u_1, u_2, ..., u_m\}$ and $\{v_1, v_2, ..., v_n\}$ respectively. Then the following cases arise.

Case 1: If $n = 3$ or 4 or 5. In this case, we consider the two subcases as follows:

Subcase 1: For $m = 7k_1+3$ or $7k_1+4,\ k_1 \geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $A = \{(u_3, v_3) \bigcup (u_5, v_3) \bigcup (u_{7t+1}, v_3)\ ;\ 1 \leq t \leq k_1\} \bigcup \{(u_{7t+3}, v_3)\ ;\ 1 \leq t \leq k_1\}$ is the γ_t- set. Clearly, no total dominating set with cardinality less than $|A|$ exists in the graph. Thus A is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |A| = 2(k_1+1)$.

Subcase 2: For $m = 7k_1+5$ or $7k_1+6$ or $7k_1+7, k_1 \geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $B = \{(u_3, v_3) \bigcup (u_5, v_3) \bigcup (u_{7t+3}, v_3)\ ;\ 1 \leq t \leq k_1\} \bigcup \{(u_{7t+5}, v_3)\ ;\ 1 \leq t \leq k_1\}$ is the γ_t- set. Clearly, no total dominating set with cardinality less than $|B|$ exists in the graph. Thus B is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |B| = 2k_1+3$.

Case 2: If $n = 3$ or 4 or 5, For $m = 7k_1+8$ or $7k_1+9\ k_1 \geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $C =$

$\{(u_3, v_3) \bigcup (u_5, v_3) \bigcup (u_7, v_3) \bigcup (u_{7t+5}, v_3)\ ;\ 1 \leq t \leq k_1\} \bigcup \{(u_{7t+7}, v_3)\ ;\ 1 \leq t \leq k_1\}$ is the γ_t- set. Clearly, no total dominating set with cardinality less than $|C|$ exists in the graph Thus C is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |C| = 2k_1 + 3$.

Figure 8: A total dominating set for $(C_m \boxtimes C_n)^2$

□

5. Conclusions

In this paper, an approach is presented to find the domination number and the total domination number of square of normal product of cycles. Our main emphasis is an obtaining the domination number and the total domination number which are related to square of normal product of cycle graphs, that give the main results of $(C_m \boxtimes C_n)^2$.

References

[1] M.Alishahi and S.H.Shalmaee, *Domination number of square of Cartesian products of cycles*, Open Journal of Discrete Mathematics, 5(04)(2015), 88.

[2] M.Atapour and N.Soltankhah, *On total dominating sets in graphs*, Int. J Contemp. Math, 4(6)(2009), 253-257.

[3] B.Chaluvaraju and C.Appajigowda, *Split domination in normal product of paths and cycles*, Gen. Math. Notes, 28(1)(2015), 72-80.

[4] R.Hammack, W.Imrich and S.Klavzar, *Hand book of product graph*, CRC Press, Taylor and Francis Group, LLC, (2011).

[5] F.Harary, *Graph Theory*, Addison-Wesley, Reading Mass, (1969).

[6] T.Haynes, S.Hedetniemi and P.J.Slater, *Fundamentals of Dominations in Graphs*, M. dekker. Inc., Newyork, (1997).

[7] M.A.Henning, C.Lowenstein, D.Rautenbach and J.Southey, *Disjoint dominating and total dominating sets in graphs*, Ars Combin, 158(15)(2010), 1615-1623.

[8] M.A.Henning, *A survey of selected recent results on total domination in graphs*, Ars Combin, 309(1)(2009), 32-63.

[9] M.S.Jacobson and L.F.Kinch, *On the gomination number of products of a graph-1*, Ars Comb, 18(1983), 33-44.

[10] O.Ore, *Theory of graphs*, American Mathematical Society Colloquium Publications, 38(American Mathematical Society, Providence, RI), (1962).

[11] G.Sabidussi, *Graph Multiplication*, Mathematische Zeitschrift, 72(1)(1960), 446-457.

[12] H.B.Walikar, B.D.Acharya and E.Sampathkumar, *Recent developments in the theory of domination in graphs*, Mehta Research Institute, Allahabad, MRI Lecture Notes in Math, (1979).

[13] D.B.West, *Introduction to graph theory*, 2nd Edition, Prentice-Hall, Upper Saddle River, (2001).