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Hyers Type Stability of a Radical Reciprocal Quadratic Functional Equation Originating From 3 Dimensional Pythagorean Means

Research Article

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Abstract: In this paper, authors introduce a 3 dimensional Pythagorean mean functional equation

$$f\left(\sqrt{x^2+y^2}\right)+f\left(\sqrt{y^2+z^2}\right)+f\left(\sqrt{z^2+x^2}\right)=\frac{f(x)f(y)}{f(x)+f(y)}+\frac{f(y)f(z)}{f(y)+f(z)}+\frac{f(z)f(x)}{f(z)+f(x)}$$

which relates the three classical Pythagorean mean and investigate its generalized Hyers-Ulam stability.

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Keywords: Pythagorean Means, Arithmetic mean, Geometric mean and Harmonic mean, Generalized Hyers-Ulam stability.

1. Introduction

The stability problem of functional equations originates from the basic question of S.M. Ulam [26] in 1940 The initial clarification to Ulam's query was given by D.H. Hyers [13]. He considered the case of approximately additive mappings. Further between 1951 to 2007, T. Aoki [2], Th.M. Rassias [23], J.M.Rassias [21], P.Gavruta [10] and J.M.Rassias et.al., [25] provided a generalized version of the theorem of Hyers which permitted the Cauchy difference to become unbounded. For the past seven decades, the generalized Ulam stability for various functional equations have been extensively investigated by numerous authors; one can refer to [3–5, 7, 9, 12, 14–19, 22, 24]. There is a legend that one day when Pythagoras (c.500 BCE) was passing a blacksmiths shop, he heard harmonious music ringing from the hammers. When he enquired, he was told that the weights of the hammers were 6, 8, 9, and 12 pounds. These ratios produce a fundamental and its fourth, fifth and octave. This was evidence that the elegance of mathematics is manifested in the harmony of nature. Returning to music, these ratios are indeed a foundation of music as noted by Archytus of Tarentum(c.350 BCE): There are three means in music: one is the arithmetic, the second is the geometric and the third is the sub contrary, which they call harmonic. The arithmetic mean is when there are three terms showing successively the same excess: the second exceeds the third by the same amount as the first exceeds the second. In this proportion, the ratio of the larger number is less, that of the smaller

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mumbers greater. The geometric mean is when the second is to the third as the first is to the second; in this, the greater numbers have the same ratio as the smaller numbers. The sub contrary, which we call harmonic, is as follows: by whatever part of itself the first term exceeds the second, the middle term exceeds the third by the same part of the third. In this proportion, the ratio of the larger numbers is larger and of the lower numbers less [8].

Definition 1.1 (Pythagorean Means [6])**.** *In Mathematics, the three classical Pythagorean means are the Arithmetic Mean(A.M.), the Geometric Mean(G.M.), and the Harmonic Mean(H.M.). They are defined for N values by*

(1). ***Arithmetic Mean (A.M.):*** *Arithmetic mean is the total of all the items divided by their total number of items.*

$$A.M. = \frac{X_1 + X_2 + \cdots + X_N}{N}.$$

(2). ***Geometric Mean (G.M.):*** *Geometric mean of N values is the Nth root of the product of N values.*

$$G.M. = \sqrt[N]{X_1 \cdot X_2 \cdots X_N}.$$

(3). ***Harmonic Mean (H.M.):*** *Harmonic mean is the reciprocal of the means of the reciprocals of the values.*

$$H.M. = \frac{N}{\frac{1}{X_1} + \frac{1}{X_2} + \cdots + \frac{1}{X_N}}.$$

Lemma 1.2. *For any two items a and b, we have $G.M. = \sqrt{A.M. \times H.M.}$*

Proof. Let a and b be two items. Then, we have

$$A.M. = \frac{a+b}{2}, \tag{1}$$

$$G.M. = \sqrt{ab}, \tag{2}$$

$$H.M. = \frac{2}{\frac{1}{a} + \frac{1}{b}}. \tag{3}$$

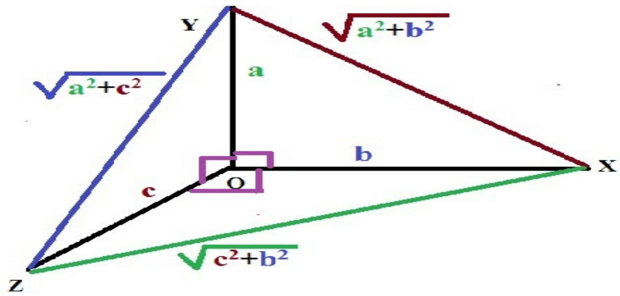
From (1) and (3), we arrive

$$\begin{aligned} A.M. \times H.M. &= \frac{a+b}{2} \times \frac{2}{\frac{1}{a} + \frac{1}{b}} \\ &= \frac{a+b}{2} \times \frac{2}{\frac{b+a}{ab}} \\ &= \frac{a+b}{2} \times \frac{2ab}{a+b} \\ &= ab \\ &= (G.M.)^2. \end{aligned}$$

Hence we derived our result. □

2. Geometrical Interpretation of Functional Equation

Let O be the center and X, Y, Z be any point on the three perpendicular axises. Assume that $OX = \mathrm{b}, OY = \mathrm{a}, OZ = \mathrm{c}$.



From the Triangle XOY, YOZ and XOZ, we have by Pythagoras Theorem

$$YX^2 = OY^2 + OX^2 = \mathrm{a}^2 + \mathrm{b}^2 \Rightarrow YX = \sqrt{\mathrm{a}^2 + \mathrm{b}^2}, \tag{4}$$

$$ZY^2 = OY^2 + OZ^2 = \mathrm{a}^2 + \mathrm{c}^2 \Rightarrow ZY = \sqrt{\mathrm{a}^2 + \mathrm{c}^2}, \text{ and} \tag{5}$$

$$XZ^2 = OZ^2 + OX^2 = \mathrm{c}^2 + \mathrm{b}^2 \Rightarrow XZ = \sqrt{\mathrm{c}^2 + \mathrm{b}^2}. \tag{6}$$

Adding (4), (5) and (6), we obtain

$$YX + ZY + XZ = \sqrt{\mathrm{a}^2 + \mathrm{b}^2} + \sqrt{\mathrm{a}^2 + \mathrm{c}^2} + \sqrt{\mathrm{c}^2 + \mathrm{b}^2}. \tag{7}$$

The above equation can be transformed into a radical reciprocal quadratic functional equation of the following form

$$f\left(\sqrt{x^2 + y^2}\right) + f\left(\sqrt{y^2 + z^2}\right) + f\left(\sqrt{z^2 + x^2}\right) = \frac{f(x)f(y)}{f(x) + f(y)} + \frac{f(y)f(z)}{f(y) + f(z)} + \frac{f(z)f(x)}{f(z) + f(x)} \tag{8}$$

having solution

$$f(x) = \frac{k}{x^2} \tag{9}$$

for any constant k.

3. General Solution of the Functional Equation (8)

In this section, motivated by the work of Roman Ger [11], we present the general solution of the Pyhagorean mean functional equation in the simplest case and also we give the differentiable solution of (8). The following Theorem gives the solution of (8) in the simplest case.

Theorem 3.1. *The only nonzero solution of a function $f : (0, \infty) \to \mathbb{R}$, admitting a finite limit of the quotient $\frac{f(x)}{\frac{1}{x^2}}$ at zero, of the equation (8) is of the form $\frac{k}{x^2}$ for all $x, y \in (0, \infty)$.*

Proof. Proof. Put $x = y = z = x$ (8), we obtain

$$3f\left(\sqrt{2}x\right) = 3\frac{1}{2f(x)} \Rightarrow f\left(\sqrt{2}x\right) = \frac{1}{2f(x)} \tag{10}$$

for all $x \in (0, \infty)$. The rest of the proof is similar to that of [20]. □

The following Theorem gives the differentiable solution of the Pythagorean mean functional equation (8).

Theorem 3.2. *Let $f : (0, \infty) \to \mathbb{R}$ be continuously differentiable functions with nowhere vanishing derivatives f'. Then f yields a solution to the functional equation (8) if and only if there exists nonzero real constants k such that $\frac{k}{x^2}$ for all $x, y \in (0, \infty)$.*

Proof. Differentiate equation (8) with respect to x on both sides, we obtain

$$f'\left(\sqrt{x^2+y^2}\right)\frac{x}{\sqrt{x^2+y^2}}+f'\left(\sqrt{z^2+x^2}\right)\frac{x}{\sqrt{z^2+x^2}}=\frac{f'(x)f^2(y)}{(f(x)+f(y))^2}+\frac{f^2(z)f'(x)}{(f(z)+f(x))^2} \tag{11}$$

for all $x,y,z\in(0,\infty)$. Setting $x=y=z=x$ in (8) and (11), we get

$$f\left(\sqrt{2}x\right)=\frac{1}{2f(x)} \tag{12}$$

$$f'\left(\sqrt{2}x\right)=\frac{1}{2\sqrt{2}}f'(x) \tag{13}$$

for all $x\in(0,\infty)$. Replacing $y=z=\sqrt{2}x$ in (11) and using (12), (14), we arrive

$$f'\left(\sqrt{3}x\right)=\frac{1}{3\sqrt{3}}f'(x) \tag{14}$$

for all $x\in(0,\infty)$. The rest of the proof is similar to that of [20]. □

4. Generalized Hyers-Ulam Stability of Equation (8)

Through out section, let E be a linear space and F be a Banach space. Define a difference operation $Df(x,y,z)$ by

$$Df(x,y,z)=f\left(\sqrt{x^2+y^2}\right)+f\left(\sqrt{y^2+z^2}\right)+f\left(\sqrt{z^2+x^2}\right)-\frac{f(x)f(y)}{f(x)+f(y)}-\frac{f(y)f(z)}{f(y)+f(z)}-\frac{f(z)f(x)}{f(z)+f(x)}$$

for all $x,y,z\in E$. Now, we investigate the generalized Hyers - Ulam stability of the functional equation (8) in Banach space using direct method.

Theorem 4.1. *If $f:E\to F$ be a function satisfying the functional inequality*

$$\|Df(x,y,z)\|\le A(x,y,z) \tag{15}$$

where $A:E^3\to[0,\infty)$ be a function such that

$$\lim_{m\to\infty}2^{mj}A\left(2^{\frac{mj}{2}}x,2^{\frac{mj}{2}}y,2^{\frac{mj}{2}}\right)=0 \tag{16}$$

for all $x,y,z\in E$. Then there exists a unique radical reciprocal quadratic function $Q:E\to F$ satisfying the functional equation (8) and the inequality

$$\|f(x)-Q(x)\|\le\frac{2}{3}\sum_{i=\frac{i-j}{2}}^{\infty}2^{ij}A\left(2^{\frac{ij}{2}}x,2^{\frac{ij}{2}}x,2^{\frac{ij}{2}}x\right) \tag{17}$$

for all $x\in E$, where $j=\pm1$. The function $Q(x)$ is defined as

$$Q(x)=\lim_{m\to\infty}2^{mj}f\left(2^{\frac{mj}{2}}x\right),\ \textit{for all } x\in E. \tag{18}$$

Proof. Replacing (x,y,z) by (x,x,x) in (15), we get

$$\left\|3f\left(\sqrt{2}\,x\right)-3\left(\frac{f(x)}{2}\right)\right\|\le A(x,x,x) \tag{19}$$

for all $x \in E$. It follows from (19), we have

$$\left\| 2f\left(\sqrt{2}\,x\right) - f(x)\right\| \leq \frac{2}{3} A(x,x,x) \tag{20}$$

for all $x \in E$. Setting x by $\sqrt{2}\,x$ and multiply by 2 in (20), we arrive

$$\left\| 2^2 f\left((\sqrt{2})^2\,x\right) - 2f(\sqrt{2}\,x)\right\| \leq \frac{4}{3} A\left(\sqrt{2}\,x, \sqrt{2}\,x, \sqrt{2}\,x\right) \tag{21}$$

for all $x \in E$. With the use of triangle inequality it follows from (20) and (21), we obtain

$$\left\| 2^2 f\left((\sqrt{2})^2\,x\right) - f(x)\right\| \leq \frac{2}{3}\left[A(x,x,x) + 2A\left(\sqrt{2}\,x, \sqrt{2}\,x, \sqrt{2}\,x\right)\right] \tag{22}$$

for all $x \in E$. Generalizing, for any positive integer m, one can reach

$$\left\| 2^m f\left(2^{\frac{m}{2}}x\right) - f(x)\right\| \leq \frac{2}{3}\sum_{i=0}^{m-1} 2^i A\left(2^{\frac{i}{2}}x, 2^{\frac{i}{2}}x, 2^{\frac{i}{2}}x\right) \tag{23}$$

for all $x \in E$. Thus the sequence $\{2^m f\left(2^{\frac{m}{2}}x\right)\}$ is a Cauchy sequence F. Indeed, replacing x by $2^{\frac{n}{2}}x$ and multiply by 2^n in (24) and using (16), we have

$$\begin{aligned}
\left\| 2^{m+n} f\left(2^{\frac{m+n}{2}}x\right) - 2^n f(2^{\frac{n}{2}}x)\right\| &= 2^n \left\| 2^m f\left(2^{\frac{m}{2}} \cdot 2^{\frac{n}{2}}x\right) - f(2^{\frac{n}{2}}x)\right\| \\
&\leq \frac{2}{3}\sum_{i=0}^{m-1} 2^{i+n} A\left(2^{\frac{i+n}{2}}x, 2^{\frac{i+n}{2}}x, 2^{\frac{i+n}{2}}x\right) \\
&\rightarrow \quad 0 \quad as \quad n \quad \rightarrow \quad \infty
\end{aligned} \tag{24}$$

for all $x \in E$. Since F is complete, there exists a mapping $Q(x)$ such that

$$Q(x) = \lim_{m\to\infty} 2^m f\left(2^{\frac{m}{2}}x\right)$$

for all $x \in E$. Replacing (x,y,z) by $(2^{\frac{m}{2}}x, 2^{\frac{m}{2}}y, 2^{\frac{m}{2}}z)$ in (15) and multiply by 2^m, we arrive

$$\begin{aligned}
2^m \Bigg\| f\left(\sqrt{2^{\frac{m}{2}}(x^2+y^2)}\right) + f\left(\sqrt{2^{\frac{m}{2}}(y^2+z^2)}\right) + f\left(\sqrt{2^{\frac{m}{2}}(z^2+x^2)}\right) \\
- \frac{f(2^{\frac{m}{2}}x)f(2^{\frac{m}{2}}y)}{f(2^{\frac{m}{2}}x)+f(2^{\frac{m}{2}}y)} - \frac{f(2^{\frac{m}{2}}y)f(2^{\frac{m}{2}}z)}{f(2^{\frac{m}{2}}y)+f(2^{\frac{m}{2}}z)} - \frac{f(2^{\frac{m}{2}}z)f(2^{\frac{m}{2}}x)}{f(2^{\frac{m}{2}}z)+f(2^{\frac{m}{2}}x)}\Bigg\| \leq 2^m A(2^{\frac{m}{2}}x, 2^{\frac{m}{2}}y, 2^{\frac{m}{2}}z)
\end{aligned}$$

for all $x,y,z \in E$. Letting m tends to infinity in the above inequality we see that $Q(x)$ satisfies the radical reciprocal functional equation (8) for all $x,y,z \in E$. To prove $Q(x)$ is unique, let $Q'(x)$ be another radical reciprocal quadratic functional equation satisfying (8) and (17) such that $Q(2^{\frac{m}{2}}x) = 2^m Q(x)$ and $Q'(2^{\frac{m}{2}}x) = 2^m Q'(x)$ for all $x \in E$. Now,

$$\begin{aligned}
\left\| Q(x) - Q'(x)\right\| &= \frac{1}{2^m}\left\| Q(2^{\frac{m}{2}}x) - Q'(2^{\frac{m}{2}}x)\right\| \\
&\leq \frac{1}{2^m}\left\{\left\| Q(2^{\frac{m}{2}}x) - f(2^{\frac{m}{2}}x)\right\| + \left\| f(2^{\frac{m}{2}}x) - Q'(2^{\frac{m}{2}}x)\right\|\right\} \\
&\leq \frac{4}{3}\sum_{i=0}^{\infty} 2^i A\left(2^{\frac{i+m}{2}}x, 2^{\frac{i+m}{2}}x, 2^{\frac{i+m}{2}}x\right) \\
&\rightarrow \quad 0 \quad as \quad n \quad \rightarrow \quad \infty
\end{aligned}$$

for all $x \in E$. Thus $Q(x)$ is unique. Hence theorem holds for $j = 1$. Setting x by $\frac{x}{\sqrt{2}}$ in (19), we get

$$\left\|3f\left(x\right) - \frac{3}{2}f\left(\frac{x}{\sqrt{2}}\right)\right\| \leq A\left(\frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}}\right) \tag{25}$$

for all $x \in E$. It follows from (25), we have

$$\left\|f\left(x\right) - \frac{1}{2}f\left(\frac{x}{\sqrt{2}}\right)\right\| \leq \frac{1}{3}A\left(\frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}}\right) \tag{26}$$

for all $x \in E$. Replacing x by $\frac{x}{\sqrt{2}}$ and divided by $\frac{1}{2}$ in (26), we obtain

$$\left\|\frac{1}{2}f\left(\frac{x}{\sqrt{2}}\right) - \frac{1}{2^2}f\left(\frac{x}{(\sqrt{2})^2}\right)\right\| \leq \frac{1}{3\cdot 2}A\left(\frac{x}{(\sqrt{2})^2}, \frac{x}{(\sqrt{2})^2}, \frac{x}{(\sqrt{2})^2}\right) \tag{27}$$

for all $x \in E$. The rest of the proof is similar to that of case $j = 1$. Thus the theorem holds for $j = -1$. Hence the proof is complete. □

Example 4.2. *Let $f : E \to F$ be a mapping fulfilling the inequality $\|Df(x, y, z)\| \leq \eta$, where $\eta > 0$ and for all $x, y, z \in E$. Then there exists a unique radical reciprocal quadratic function satisfying the functional equation (8) and*

$$\|f(x) - Q(x)\| \leq \left|\frac{2\eta}{3}\right|$$

for all $x \in E$.

Corollary 4.3. *Let $f : E \to F$ be a mapping fulfilling the inequality $\|Df(x, y, z)\| \leq \eta(||x||^a + ||y||^a + ||z||^a)$, where $\eta > 0, a > 0$ and for all $x, y, z \in E$. Then there exists a unique radical reciprocal quadratic function satisfying the functional equation (8) and*

$$\|f(x) - Q(x)\| \leq \frac{2\eta||x||^a}{\left|1 - 2^{\frac{a}{2}+1}\right|}, \quad a \neq 2$$

for all $x \in E$.

Corollary 4.4. *Let $f : E \to F$ be a mapping fulfilling the inequality $\|Df(x, y, z)\| \leq \eta(||x||^a + ||y||^b + ||z||^c)$, where $\eta > 0, a, b, c > 0$ and for all $x, y, z \in E$. Then there exists a unique radical reciprocal quadratic function satisfying the functional equation (8) and*

$$\|f(x) - Q(x)\| \leq \frac{2\eta||x||^a}{3\left|1 - 2^{\frac{a}{2}+1}\right|} + \frac{2\eta||x||^b}{3\left|1 - 2^{\frac{b}{2}+1}\right|} + \frac{2\eta||x||^c}{\left|1 - 2^{\frac{c}{2}+1}\right|}, \quad a, b, c \neq 2$$

for all $x \in E$.

Corollary 4.5. *Let $f : E \to F$ be a mapping fulfilling the inequality $\|Df(x, y, z)\| \leq \eta||x||^a||y||^a||z||^a$, where $\eta > 0, a > 0$ and for all $x, y, z \in E$. Then there exists a unique radical reciprocal quadratic function satisfying the functional equation (8) and*

$$\|f(x) - Q(x)\| \leq \frac{2\eta||x||^{3a}}{3\left|1 - 2^{\frac{3a}{2}+1}\right|}, \quad 3a \neq 2$$

for all $x \in E$.

Corollary 4.6. *Let $f : E \to F$ be a mapping fulfilling the inequality $\|Df(x, y, z)\| \leq \eta||x||^a||y||^b||z||^c$, where $\eta > 0, a, b, c > 0$ and for all $x, y, z \in E$. Then there exists a unique radical reciprocal quadratic function satisfying the functional equation (8) and*

$$\|f(x) - Q(x)\| \leq \frac{2\eta||x||^{a+b+c}}{3\left|1 - 2^{\frac{a+b+c}{2}+1}\right|}, \quad a + b + c \neq 2$$

for all $x \in E$.

Corollary 4.7. *Let $f : E \to F$ be a mapping fulfilling the inequality*

$$\|Df(x, y, z)\| \leq \eta(||x||^{3a} + ||y||^{3a} + ||z||^{3a} + ||x||^a||y||^a||z||^a),$$

where $\eta > 0, a > 0$ and for all $x, y, z \in E$. Then there exists a unique radical reciprocal quadratic function satisfying the functional equation (8) and

$$\|f(x) - Q(x)\| \leq \frac{8\eta||x||^{3a}}{3\left|1 - 2^{\frac{3a}{2}+1}\right|}, \quad 3a \neq 2$$

for all $x \in E$.

Corollary 4.8. *Let $f : E \to F$ be a mapping fulfilling the inequality*

$$\|Df(x, y, z)\| \leq \eta(||x||^{a+b+c} + ||y||^{a+b+c} + ||z||^{a+b+c} + ||x||^a||y||^b||z||^c),$$

where $\eta > 0, a, b, c > 0$ and for all $x, y, z \in E$. Then there exists a unique radical reciprocal quadratic function satisfying the functional equation (8) and

$$\|f(x) - Q(x)\| \leq \frac{8\eta||x||^{a+b+c}}{3\left|1 - 2^{\frac{a+b+c}{2}+1}\right|}, \quad a + b + c \neq 2$$

for all $x \in E$.

References

[1] J.Aczel and J.Dhombres, *Functional Equations in Several Variables*, Cambridge Univ, Press, (1989).

[2] T.Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, 2(1950), 64-66.

[3] E.Castillo, A.Iglesias and R.Ruiz-coho, *Functional Equations in Applied Sciences*, Elsevier, B.V.Amslerdam, (2005).

[4] P.W.Cholewa, *Remarks on the stability of functional equations*, Aequationes Math., 27(1984), 76-86.

[5] S.Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, (2002).

[6] H.Eves, *Means Appearing in Geometric Figures*, Math. Mag., 76(2003), 292-294.

[7] G.L.Forti, *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math., 50(1995), 143-190.

[8] K.Freeman, *Ancilla to the Pre-Socratic Philosophers*, Sacred-texts.com; http://www.sacred-texts.com/cla/app/app42.htm, (1948).

[9] Z.Gajda, *On the stability of additive mappings*, Inter. J. Math. Math. Sci., 14(1991), 431-434.

[10] P.Gavruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., 184(1994), 431-436.

[11] R.Ger, *Abstract Pythagorean theorem and corresponding functional equations*, Tatra Mt. Math. Publ., 55(2013), 67-75.

[12] A.Grabiec, *The generalized Hyers-Ulam stability of a class of functional equations*, Publ. Math. Debrecen, 48(1996), 217-235.

[13] D.H.Hyers, *On the stability of the linear functional equation*, Proc.Nat. Acad.Sci., U.S.A., 27(1941), 222-224.

[14] D.H.Hyers, G.Isac and Th.M.Rassias, *Stability of functional equations in several variables*, Birkhauser, Basel, (1998).

[15] G.Isac and Th.M.Rassias, *Stability of ψ additive mappings: applications to nonlinear analysis*, Int. J. Math. Math. Sci., 19(2)(1996), 219-228.

[16] S.M.Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, (2001).

[17] Pl.Kannappan, *Functional Equations and Inequalities with Applications*, Springer Monographs in Mathematics, (2009).

[18] L.Maligranda, *A result of Tosio Aoki about a generalization of Hyers-Ulam stability of additive functions-a question of priority*, Aequ. Math., 75(2008), 289-296.

[19] P.Nakmahachalasint, *Hyers-Ulam-Rassias and Ulam-Gavruta-Rassias stabilities of an Additive Functional Equation in Several variables*, Int. J. Math. and Math. Sc., (2007), Article ID 13437.

[20] P.Narasimman, K.Ravi and Sandra Pinelas, *Stability of Pythagorean Mean Functional Equation*, Global Journal of Mathematics, 4(1)(2015), 398-411.

[21] J.M.Rassias, *On approximately of approximately linear mappings by linear mappings*, J. Funct. Anal. USA, 46(1982), 126-130.

[22] J.M. Rassias, *Solution of problem of Ulam*, J. Approx. Th. USA, 57(1989), 268-273.

[23] Th.M.Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., 72(1978), 297-300.

[24] Th.M.Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Acedamic Publishers, Dordrecht, Bostan London, (2003).

[25] K.Ravi, M.Arunkumar and J.M.Rassias, *On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation*, International Journal of Mathematical Sciences, 3(8)(2008), 36-47.

[26] S.M.Ulam, *Problems in Modern Mathematics*, Science Editions, Wiley, NewYork, (1964).