On Mildly B-Normal Spaces and Some Functions

Research Article

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Abstract: In this paper, by using Bg-closed sets we obtain a characterization of mildly B-normal spaces and use it to improve the preservation theorems of mildly B-normal spaces.

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1. Introduction and Preliminaries

The notion of mildly normal spaces was introduced by Singal and Singal [14]. Palaniappan and Rao [12] have defined and investigated the notion of regular g-closed sets as a generalization of g-closed sets due to Levine [6]. In this paper, by using regular Bg-closed sets we obtain a characterization of mildly B-normal simply extended topological spaces.

Throughout this paper, $(X, \tau(X))$, $(Y, \sigma(Y))$ and $(Z, \eta(Z))$ (briefly $X$, $Y$ and $Z$) will denote simply extended topological spaces.

Definition 1.1. A subset $A$ of a topological space $X$ is said to be

(1). regular open [5] if $A = \text{int}(\text{cl}(A))$;

(2). regular g-closed (briefly rg-closed) [12] if $\text{cl}(A) \subset U$ whenever $A \subset U$ and $U$ is a regular open set in $X$.

(3). generalized closed (briefly g-closed) [6] if $\text{cl}(A) \subset U$ whenever $A \subset U$ and $U$ is open in $X$.

(4). rg-open (resp. g-open, regular closed) if the complement of $A$ is rg-closed (resp. g-closed, regular open). The family of all regular open (resp. regular closed) sets of $X$ is denoted by $\text{RO}(X)$ (resp. $\text{RC}(X)$).

Definition 1.2 [15]. A topological space $X$ is said to be mildly normal if for every pair of disjoint $H, K \in \text{RC}(X)$, there exist disjoint open sets $U, V$ of $X$ such that $H \subset U$ and $K \subset V$.

Definition 1.3 [12]. A subset $A$ of $X$ is said to be quasi H-closed relative to $X$, if for every cover $\{V_\alpha : \alpha \in \nabla\}$ of $A$ by open sets of $X$, there exists a finite subset $\nabla_0$ of $\nabla$ such that $A \subset \cup\{\text{cl}(V_\alpha) : \alpha \in \nabla_0\}$.

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Definition 1.4 ([5]). A subset a of a space X is said to be α-regular if for each point of \( x \in A \) and each open set U of X containing x, there exists an open set G of X such that \( x \in G \subseteq \text{cl}(G) \subseteq U \).

Definition 1.5 ([13]). A subset a of a topological space X is said to be α-paracompact if every cover of A by open sets of X is defined by a cover of A which consists of open sets of X and is locally finite in X.

Definition 1.6 ([14]). A topological space X is said to be mildly-normal if for every pair of disjoint \( H, K \in \text{RC}(X) \), there exist disjoint open sets U, V of X such that \( H \subseteq U \) and \( K \subseteq V \).

Definition 1.7 ([10]). A function \( f : X \to Y \) is said to be almost g-continuous (resp. almost rg-continuous) if \( f^{-1}(R) \) is g-closed (resp. rg-closed) in X, for every \( R \in \text{RC}(Y) \).

Definition 1.8. A function \( f : X \to Y \) is said to be

1. g-continuous [3] (resp. rg-continuous [12]) if \( f^{-1}(F) \) is g-closed (resp. rg-closed) in X for every closed set F of Y;
2. R-map [4], rc-continuous [4] or regular irresolute [12] (resp. almost continuous [14]) if \( f^{-1}(V) \in \text{RO}(X) \) (resp. \( \tau(X) \)) for every \( V \in \text{RO}(Y) \);
3. completely continuous [1] or regular continuous [12] if \( f^{-1}(V) \in \text{RO}(X) \) for every open set V of Y.

Definition 1.9 ([10]). A topological space X is said to be regular-T\(_{1/2}\) if every rg-closed set of X is regular closed.

Definition 1.10 ([12]). A function \( f : X \to Y \) is said to be rg-irresolute if \( f^{-1}(F) \) is rg-closed in X for every rg-closed set F of Y.

Definition 1.11. A function \( f : X \to Y \) is said to be

1. regular closed [12] (resp. g-closed [8], rg-closed [10]) if \( f(F) \) is regular closed (resp. g-closed, rg-closed [10]) in Y for every closed set F of X;
2. rc-preserving [10] (resp. almost closed [14], almost g-closed [10], almost rg-closed [10]) if \( f(F) \) is regular closed (resp. closed, g-closed, rg-closed) in Y for every \( F \in \text{RC}(X) \).

Remark 1.12 ([11]). In among others, it is shown that a compact set of a regular space is rg-closed.

Definition 1.13 ([7]). Levine in 1964 defined \( \tau(B) = \{ O \cup (\bar{O} \cap B) : O, \bar{O} \in \tau \} \) and called it simple extension of \( \tau \) by \( B \), where \( B \notin \tau \). The sets in \( \tau(B) \) are called B-open sets. and the complement of B-open set is called B-closed.

Definition 1.14 ([7]). Let S be a subset of a simply extended topological space X. Then

1. The B-closure of S, denoted by Bcl(S), is defined as \( \cap \{ F : S \subseteq F \text{ and } F \text{ is B-closed} \} \);"
2. Regular Bg-closed Sets

Definition 2.1. A subset $A$ is said to be regular $B$-open (resp. regular $B$-closed) if $A = \text{Bint}(\text{Bcl}(A))$ (resp. $A = \text{Bcl}(\text{Bint}(A))$). The family of regular $B$-open (resp. regular $B$-closed) sets of a simply extended topological space $X$ is denoted by $\text{BRO}(X)$ (resp. $\text{BRC}(X)$).

Definition 2.2. A subset $A$ of a simply extended topological space $X$ is said to be

(1). regular Bg-closed (briefly rBg-closed) if \( Bcl(A) \subset U \) whenever \( A \subset U \) and \( U \in \text{BRO}(X) \).

(2). B-generalized closed (briefly Bg-closed) if \( Bcl(A) \subset U \) whenever \( A \subset U \) and $U$ is $B$-open in $X$.

(3). rBg-open (resp. Bg-open) if the complement of $A$ is rBg-closed (resp. Bg-closed).

Result 2.3. We have the following implications for properties of subsets:

\[
\text{regular } B\text{-closed} \Rightarrow \text{B-closed} \Rightarrow \text{Bg-closed} \Rightarrow \text{rBg-closed}.
\]

where none of these implications is reversible as shown by Examples (below).

Example 2.4. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset\}$ and $B = \{b, c\}$ then $\tau(B) = \{\emptyset, X, \{b, c\}\}$. Then

(1). $\{a, b\}$ is Bg-closed but not B-closed.

(2). $\{b\}$ is rBg-closed but not Bg-closed.

Example 2.5. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}\}$ and $B = \{b\}$ then $\tau(B) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then $\{c\}$ is

B-closed but not regular B-closed.

3. Characterization of Mildly B-normal Spaces

Definition 3.1. A simply extended topological space $X$ is said to be mildly B-normal if for every pair of disjoint $H, K \in \text{BRC}(X)$, there exist disjoint $B$-open sets $U, V$ of $X$ such that $H \subset U$ and $K \subset V$.

Lemma 3.2. A subset $A$ of a simply extended topological space $X$ is rBg-open if and only if \( F \subset \text{Bint}(A) \) whenever \( F \in \text{BRC}(X) \) and \( F \subset A \).

Theorem 3.3. The following are equivalent for a simply extended topological space $X$.

(1). $X$ is mildly B-normal;

(2). for any disjoint $H, K \in \text{BRC}(X)$, there exist disjoint $B$-open sets $U, V$ of $X$ such that $H \subset U$ and $K \subset V$;

(3). for any disjoint $H, K \in \text{BRC}(X)$, there exist disjoint rBg-open sets $U, V$ such that $H \subset U$ and $K \subset V$;

(4). for any disjoint $H \in \text{BRC}(X)$ and any $V \in \text{BRO}(X)$ containing $H$, there exists a rBg-open set $U$ of $X$ such that $H \subset U \subset \text{Bcl}(U) \subset V$.

Proof. It is obvious that (1) implies (2) and (2) implies (3).

(3) $\Rightarrow$ (4) Let $H \in \text{BRC}(X)$ and $H \subset V \in \text{BRO}(X)$. There exist disjoint rBg-open sets $U, W$ such that $H \subset U$ and $X - V \subset W$. By Lemma 3.2, we have $X - V \subset \text{Bint}(W)$ and $U \cap \text{Bint}(W) = \emptyset$. Therefore, we obtain $\text{Bcl}(U) \cap \text{Bint}(W) = \emptyset$ and hence $H \subset U \subset \text{Bcl}(U) \subset X - \text{Bint}(W) \subset V$.

(4) $\Rightarrow$ (1) Let $H, K$ be disjoint regular B-closed sets of $X$. Then $H \subset X - K \in \text{BRO}(X)$ and there exists a rBg-open set $G$ of $X$ such that $H \subset G \subset \text{Bcl}(G) \subset X - K$. Put $U = \text{Bint}(G)$ and $V = X - \text{Bcl}(G)$. Then $U$ and $V$ are disjoint $B$-open sets of $X$ such that $H \subset U$ and $K \subset V$. Therefore, $X$ is mildly B-normal.
4. Some Functions

**Definition 4.1.** A function \( f : X \to Y \) is said to be almost Bg-continuous (resp. almost rBg-continuous) if \( f^{-1}(R) \) is Bg-closed (resp. rBg-closed), for every \( R \in BRC(Y) \).

**Definition 4.2.** A function \( f : X \to Y \) is said to be

1. Bg-continuous (resp. rBg-continuous) if \( f^{-1}(F) \) is Bg-closed (resp. rBg-closed) for every B-closed set \( F \) of \( Y \);
2. BR-map (resp. almost B-continuous) if \( f^{-1}(V) \in BRO(X) \) (resp. \( \tau(B)(X) \)) for every \( V \in BRO(Y) \);
3. completely B-continuous if \( f^{-1}(V) \in BRO(X) \) for every B-open set \( V \) of \( Y \).

From the definitions stated above, we obtain the following diagram:

\[
\begin{array}{c}
\text{complete B-continuity} \quad \rightarrow \quad \text{BR-map} \\
\downarrow \quad \quad \downarrow \\
\text{B-continuity} \quad \quad \rightarrow \quad \text{almost B-continuity} \\
\downarrow \quad \quad \downarrow \\
\text{Bg-continuity} \quad \quad \rightarrow \quad \text{almost Bg-continuity} \\
\downarrow \quad \quad \downarrow \\
rBg-continuity \quad \quad \rightarrow \quad \text{almost rBg-continuity}
\end{array}
\]

**Remark 4.3.** None of the implications in Diagram I is reversible as shown by the following Examples.

**Example 4.4.**

1. Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, X\} \) and \( B_X = \{a\} \) then \( \tau(B_X) = \{\phi, X, \{a\}\} \). Let \( \sigma = \{\phi, Y\} \) and \( B_Y = \{a, b\} \) then \( \sigma(B_Y) = \{\phi, Y, \{a, b\}\} \). Let \( f : (X, \tau(B_X)) \to (Y, \sigma(B_X)) \) be an identity map. Then \( f \) is BR-map (resp. almost B-continuous) but not completely B-continuous (resp. B-continuous).

2. Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, X, \{a\}\} \) and \( B_X = \{a, b\} \) then \( \tau(B_X) = \{\phi, X, \{a\}\} \). Let \( \sigma = \{\phi, Y\} \) and \( B_Y = \{a\} \) then \( \sigma(B_Y) = \{\phi, Y, \{a\}\} \). Let \( f : (X, \tau(B_X)) \to (Y, \sigma(B_X)) \) be an identity map. Then \( f \) is almost Bg-continuous but not Bg-continuous.

**Example 4.5.**

1. Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, X\} \) and \( B_X = \{a\} \) then \( \tau(B_X) = \{\phi, X, \{a\}\} \). Let \( \sigma = \{\phi, Y\} \) and \( B_Y = \{a\} \) then \( \sigma(B_Y) = \{\phi, Y, \{a\}\} \). Let \( f : (X, \tau(B_X)) \to (Y, \sigma(B_X)) \) be an identity map. Then \( f \) is B-continuous but not completely B-continuous.

2. Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, X, \{a\}\} \) and \( B_X = \{b\} \) then \( \tau(B_X) = \{\phi, X, \{a\}, \{b\}\} \). Let \( \sigma = \{\phi, Y, \{a, b\}\} \) and \( B_Y = \{b\} \) then \( \sigma(B_Y) = \{\phi, Y, \{a, b\}\} \). Let \( f : (X, \tau(B_X)) \to (Y, \sigma(B_X)) \) be an identity map. Then \( f \) is almost B-continuous.

**Example 4.6.** Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, X, \{a\}\} \) and \( B_X = \{a, b\} \) then \( \tau(B_X) = \{\phi, X, \{a\}, \{a, b\}\} \). Let \( \sigma = \{\phi, Y, \{a\}\} \) and \( B_Y = \{b\} \) then \( \sigma(B_Y) = \{\phi, Y, \{a\}, \{b\}\} \). Let \( f : (X, \tau(B_X)) \to (Y, \sigma(B_X)) \) be an identity map. Then \( f \) is Bg-continuous (resp. almost B-continuous) but not B-continuous (resp. almost Bg-continuous).
Example 4.7. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$ and $B_X = \{a, b\}$ then $\tau(B_X) = \{\phi, X, \{a\}, \{a, b\}\}$. Let $\sigma = \{\phi, Y, \{a\}\}$ and $B_Y = \{b\}$ then $\sigma(B_Y) = \{\phi, Y, \{a\}, \{a, c\}\}$. Let $f : (X, \tau(B_X)) \to (Y, \sigma(B_X))$ be an identity map. Then $f$ is rBg-continuous but not Bg-continuous.

Example 4.8. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$ and $B_X = \{c\}$ then $\tau(B_X) = \{\phi, X, \{a\}, \{a, c\}\}$. Let $\sigma = \{\phi, Y, \{a\}\}$ and $B_Y = \{b\}$ then $\sigma(B_Y) = \{\phi, Y, \{a\}, \{b, \{a, b\}\}\}$. Let $f : (X, \tau(B_X)) \to (Y, \sigma(B_X))$ be an identity map. Then $f$ is almost rBg-continuous but neither almost Bg-continuous nor rBg-continuous.

Definition 4.9. A simply extended topological space $X$ is said to be regular B-$T_{1/2}$ if every rBg-closed set of $X$ is regular B-closed.

Proposition 4.10. If a function $f : X \to Y$ is rBg-continuous and $X$ is regular B-$T_{1/2}$, then $f$ is completely B-continuous.

Proof. Let $F$ be any B-closed set of $Y$. Since $f$ is rBg-continuous, $f^{-1}(F)$ is rBg-closed in $X$ and hence $f^{-1}(F) \in BRC(X)$. Therefore, $f$ is completely B-continuous.

Definition 4.11. A function $f : X \to Y$ is said to be rBg-irresolute if $f^{-1}(F)$ is rBg-closed in $X$ for every rBg-closed set $F$ of $Y$. Every rBg-irresolute function is rBg-continuous but not conversely as shown by the following Example.

Example 4.12. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$ and $B_X = \{a, b\}$ then $\tau(B_X) = \{\phi, X, \{a\}, \{a, b\}\}$. Let $\sigma = \{\phi, Y, \{a\}\}$ and $B_Y = \{a\}$ then $\sigma(B_Y) = \{\phi, Y, \{a\}\}$. Let $f : (X, \tau(B_X)) \to (Y, \sigma(B_X))$ be an identity map. Then $f$ is B-continuous and Bg-continuous but not rBg-irresolute.

Corollary 4.13. If $f : X \to Y$ is rBg-irresolute and $X$ is regular B-$T_{1/2}$, then $f$ is BR-map.

Definition 4.14. A function $f : X \to Y$ is said to be

1. regular B-closed (resp. Bg-closed, rBg-closed) if $f(F)$ is regular B-closed (resp. Bg-closed, rBg-closed) in $Y$ for every B-closed set $F$ of $X$;

2. rBc-preserving (resp. almost B-closed, almost Bg-closed, almost rBg-closed) if $f(F)$ is regular B-closed (resp. B-closed, Bg-closed, rBg-closed) in $Y$ for every $F \in BRC(X)$.

From the definitions stated above, we obtain the following diagram:

- regular B-closed $\rightarrow$ rBc-preserving
- $\downarrow$ $\downarrow$
- B-closed $\rightarrow$ almost B-closed
- $\downarrow$ $\downarrow$
- Bg-closed $\rightarrow$ almost Bg-closed
- $\downarrow$ $\downarrow$
- rBg-closed $\rightarrow$ almost rBg-closed

Remark 4.15. None of the implications in Diagram II is reversible.

Example 4.16. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$ and $B_X = \{b\}$ then $\tau(B_X) = \{\phi, X, \{a\}, \{b\}\}$. Let $\sigma = \{\phi, Y, \{a\}\}$ and $B_Y = \{b\}$ then $\sigma(B_Y) = \{\phi, Y, \{a\}, \{b\}\}$. Let $f : (X, \tau(B_X)) \to (Y, \sigma(B_X))$ be an identity map. Then $f$ is

1. rBc-preserving but not regular B-closed.
(2). regular B-closed but not B-closed.

Example 4.17.

(1). Let $X = \{a, b, c\}$, $\tau = \{\phi, X\}$ and $B_X = \{a\}$ then $\tau(B_X) = \{\phi, X, \{a\}\}$. Let $\sigma = \{\phi, Y\}$ and $B_Y = \{a, b\}$ then $\sigma(B_Y) = \{\phi, Y, \{a, b\}\}$. Let $f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_Y))$ be an identity map. Then f is B-closed but not B-closed.

(2). Let $X = \{a, b, c\}$, $\tau = \{\phi, X\}$ and $B_X = \{b, c\}$ then $\tau(B_X) = \{\phi, X, \{b, c\}\}$. Let $\sigma = \{\phi, Y\}$ and $B_Y = \{a, b\}$ then $\sigma(B_Y) = \{\phi, Y, \{a, b\}\}$. Let $f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_Y))$ be an identity map. Then f is Bg-closed but not B-closed.

(3). Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$ and $B_X = \{b\}$ then $\tau(B_X) = \{\phi, X, \{a\}, \{b\}\}$. Let $\sigma = \{\phi, Y, \{a\}, \{b\}\}$ and $B_Y = \{b\}$ then $\sigma(B_Y) = \{\phi, Y, \{a\}, \{b\}\}$. Let $f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_Y))$ be an identity map. Then f is almost Bg-closed (resp. Bg-closed, Bg-closed) but not almost B-closed (resp. almost Bg-closed, rBg-closed).

(4). Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$ and $B_X = \{b\}$ then $\tau(B_X) = \{\phi, X, \{a\}, \{b\}\}$. Let $\sigma = \{\phi, Y, \{c\}, \{b\}\}$ and $B_Y = \{b\}$ then $\sigma(B_Y) = \{\phi, Y, \{c\}, \{b\}\}$. Let $f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_Y))$ be an identity map. Then f is almost Bg-closed (resp. Bg-closed, Bg-closed) but not almost B-closed (resp. almost Bg-closed, rBg-closed).

Proposition 4.18. Let X and Y be simply extended topological spaces. Let $f : X \rightarrow Y$ be a function. Then

(1). if f is rBg-continuous rBc-preserving, then it is rBg-irresolute;

(2). if f is an BR-map and rBg-closed, then f(A) is rBg-closed in Y for every rBg-closed set A of X.

Proof.

(1). Let A be any rBg-closed set of Y and $U \in BRO(X)$ containing $f^{-1}(A)$. Put $V = Y - f(X - U)$, then we have $A \subset V$, $f^{-1}(V) \subset U$ and $V \in BRO(Y)$ since f is rBc-preserving. Hence we obtain $Bcl(A) \subset V$ and hence $f^{-1}(Bcl(A)) \subset U$. By the rBg-continuity of f, we have $Bcl(f^{-1}(A)) \subset Bcl(f^{-1}(Bcl(A))) \subset U$. This shows that $f^{-1}(A)$ is rBg-closed in X. Therefore, f is rBg-irresolute.

(2). Let A be any rBg-closed set of X and $V \in BRO(X)$ containing f(A). Since f is an BR-map, $f^{-1}(V) \in BRO(X)$ and $A \subset f^{-1}(V)$. Therefore, we have $Bcl(A) \subset f^{-1}(V)$ and hence $f(Bcl(A)) \subset V$. Since f is rBg-closed, f(Bcl(A)) is rBg-closed in Y and hence we obtain $Bcl(f(A)) \subset Bcl(f(Bcl(A))) \subset U$. This shows that f(A) is rBg-closed in Y.

Corollary 4.19. Let X and Y be simply extended topological spaces. Let $f : X \rightarrow Y$ be a function. Then

(1). if f is B-continuous regular B-closed, $f^{-1}(A)$ is rBg-closed in X for every rBg-closed set A of Y;

(2). if f is BR-map and B-closed, f(A) is rBg-closed in Y for every rBg-closed set A if X.

Proposition 4.20. Let X and Y be simply extended topological spaces. A surjection $f : X \rightarrow Y$ is almost rBg-closed (resp. almost Bg-closed) if and only if for each subset $S$ of Y and each $U \in BRO(X)$ containing $f^{-1}(S)$ there exists an rBg-open (resp. Bg-open) set $V$ of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof. We prove only the first case, the proof of the second being entirely analogous.

Necessity : Suppose that f is almost rBg-closed. Let S be a subset of Y and $U \in BRO(X)$ containing $f^{-1}(S)$. Put $V = Y - f(X - U)$, then V is an rBg-open set of Y such that $S \subset V$ and $f^{-1}(V) \subset U$. 

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Sufficiency : Let \( F \) be any regular \( B \)-closed set of \( X \). Then \( f^{-1}(Y - f(F)) \subset X - F \) and \( X - F \in \text{BRO}(X) \). There exists an \( \text{rBg} \)-open set \( V \) of \( Y \) such that \( Y - f(F) \subset V \) and \( f^{-1}(V) \subset X - F \). Therefore, we have \( f(F) \subset Y - V \) and \( F \subset f^{-1}(Y - V) \). Hence, we obtain \( f(F) = Y - V \) and \( f(F) \) is \( \text{rBg} \)-closed in \( Y \). This shows that \( f \) is almost \( \text{rBg} \)-closed.

\[ \square \]

5. Preservation Theorems

In this section we investigate preservation theorems concerning mildly \( B \)-normal spaces

**Theorem 5.1.** Let \( X \) and \( Y \) be simply extended topological spaces. If \( f : X \to Y \) is an almost \( \text{rBg} \)-continuous \( \text{rBc} \)-preserving (resp. almost \( B \)-closed) injection and \( Y \) is mildly \( B \)-normal (resp. \( B \)-normal), then \( X \) is mildly \( B \)-normal.

**Proof.** Let \( A \) and \( C \) be any disjoint regular \( B \)-closed sets of \( Y \). Then \( f^{-1}(A) \) and \( f^{-1}(C) \) are disjoint regular \( B \)-closed (resp. \( B \)-closed) sets of \( Y \). Since \( Y \) is mildly \( B \)-normal, there exist disjoint \( B \)-open sets \( U \) and \( V \) of \( Y \) such that \( f^{-1}(A) \subset U \) and \( f^{-1}(C) \subset V \). Now, put \( G = \text{Bint}(\text{Bcl}(U)) \) and \( H = \text{Bint}(\text{Bcl}(V)) \), then \( G \) and \( H \) are disjoint regular \( B \)-open sets such that \( f^{-1}(A) \subset G \) and \( f^{-1}(C) \subset H \). Since \( f \) is almost \( \text{rBg} \)-continuous, \( f^{-1}(G) \) and \( f^{-1}(H) \) are disjoint \( \text{rBg} \)-open sets containing \( A \) and \( C \), respectively. It follows from Theorem 3.3 that \( X \) is mildly \( B \)-normal.

\[ \square \]

**Theorem 5.2.** Let \( X \) and \( Y \) be simply extended topological spaces. If \( f : X \to Y \) is a completely \( B \)-continuous almost \( \text{Brg} \)-closed surjection and \( X \) is mildly \( B \)-normal, then \( Y \) is \( B \)-normal.

**Proof.** Let \( A \) and \( C \) be any disjoint \( B \)-closed sets of \( Y \). Then \( f^{-1}(A) \) and \( f^{-1}(C) \) are disjoint regular \( B \)-closed sets of \( X \). Since \( X \) is mildly \( B \)-normal, there exist disjoint \( B \)-open sets \( U \) and \( V \) of \( X \) such that \( f^{-1}(A) \subset U \) and \( f^{-1}(C) \subset V \). Let \( G = \text{Bint}(\text{Bcl}(U)) \) and \( H = \text{Bint}(\text{Bcl}(V)) \), then \( G \) and \( H \) are disjoint regular \( B \)-open sets such that \( f^{-1}(A) \subset G \) and \( f^{-1}(C) \subset H \). By Proposition 4.20, there exists \( \text{Brg} \)-open sets \( K \) and \( L \) of \( Y \) such that \( A \subset K \), \( C \subset L \), \( f^{-1}(K) \subset G \) and \( f^{-1}(L) \subset H \). Since \( G \) and \( H \) are disjoint, so are \( K \) and \( L \). Since \( K \) and \( L \) are \( \text{Brg} \)-open, we obtain \( A \subset \text{Bint}(K) \), \( C \subset \text{Bint}(L) \) and \( \text{Bint}(K) \cap \text{Bint}(L) = \emptyset \). This shows that \( Y \) is \( B \)-normal.

\[ \square \]

**Corollary 5.3.** Let \( X \) and \( Y \) be simply extended topological spaces. If \( f : X \to Y \) is a completely \( B \)-continuous \( B \)-closed surjection and \( X \) is mildly \( B \)-normal, then \( Y \) is \( B \)-normal.

**Theorem 5.4.** Let \( X \) and \( Y \) be simply extended topological spaces. Let \( f : X \to Y \) be an \( \text{BR} \)-map (resp. almost \( B \)-continuous) and almost \( \text{rBrg} \)-closed surjection. If \( X \) is mildly \( B \)-normal (resp. \( B \)-normal), then \( Y \) is mildly \( B \)-normal.

**Proof.** Let \( A \) and \( C \) be any disjoint regular \( B \)-closed sets of \( Y \). Then \( f^{-1}(A) \) and \( f^{-1}(C) \) are disjoint regular \( B \)-closed (resp. \( B \)-closed) sets of \( X \). Since \( X \) is mildly \( B \)-normal (resp. \( B \)-normal), there exist disjoint \( B \)-open sets \( U \) and \( V \) of \( X \) such that \( f^{-1}(A) \subset U \) and \( f^{-1}(C) \subset V \). Put \( G = \text{Bint}(\text{Bcl}(U)) \) and \( H = \text{Bint}(\text{Bcl}(V)) \), then \( G \) and \( H \) are disjoint regular \( B \)-open sets of \( X \) such that \( f^{-1}(A) \subset G \) and \( f^{-1}(C) \subset H \). By Proposition 4.20, there exists \( \text{rBrg} \)-open sets \( K \) and \( L \) of \( Y \) such that \( A \subset K \), \( C \subset L \), \( f^{-1}(K) \subset G \) and \( f^{-1}(L) \subset H \). Since \( G \) and \( H \) are disjoint, so are \( K \) and \( L \). It follows from Theorem 3.3 that \( Y \) is mildly \( B \)-normal.

\[ \square \]

**Corollary 5.5.** Let \( X \) and \( Y \) be simply extended topological spaces. If \( f : X \to Y \) is an almost \( B \)-continuous almost \( B \)-closed surjection and \( X \) is \( B \)-normal, then \( Y \) is mildly \( B \)-normal.
References