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On Strongly Symmetric Rings

Research Article

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Abstract: A ring R is called strongly symmetric, if whenever polynomials $f(x), g(x), h(x)$ in $R[x]$ satisfy $f(x)g(x)h(x) = 0$, then $f(x)h(x)g(x) = 0$. It is proved that a ring R is strongly symmetric if and only if its polynomial ring $R[x]$ is strongly symmetric if and only if its Laurent polynomial ring $R[x, x^{-1}]$ is strongly symmetric. We also show that for a right Ore ring R with Q its classical right quotient ring, R is strongly symmetric if and only if Q is strongly symmetric. Finally we proved that, let R be an algebra over a commutative ring S, and D be the Dorroh extension of R by S. If R is strongly symmetric and S is a domain, then D is strongly symmetric.

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Keywords: Reduced rings, symmetric rings, strongly symmetric rings.

1. Introduction

Throughout this note all rings are associative with identity unless otherwise stated. According to Lambek [6], a ring R is called symmetric if $rst = 0$ implies $rts = 0$ for all $r, s, t \in R$; while Anderson and Camillo [3] took the term ZC_3 for this notion. Lambek proved that a ring R is symmetric if and only if $r_1 r_2 \cdots r_n = 0$, with n any positive integer, implies $r_{\sigma(1)} r_{\sigma(2)} \dots r_{\sigma(n)} = 0$ for any permutation σ of the set $\{1, 2, \dots, n\}$ and $r_i \in R$ [6, Proposition 1], Anderson and Camillo obtained this result independently in [3, Theorem I.1]. Given a ring $R, r_R(-)(l_R(-))$ is used for the right (left) annihilator in R. According to Cohn [8], a ring R is called reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Anderson and Camillo [3], observing the rings whose zero products commute, used the term ZC_2 for what is called reversible, and Krempa-Niewieczerzal [5] took the term C_0 for it. It is obvious that commutative rings are symmetric and symmetric rings are reversible; but reversible rings need not be symmetric and symmetric rings need not be commutative by the results of Anderson and Camillo [3, Examples I.5 and II.5] and Marks [4, Examples 5 and 7]. A ring is called reduced if it has no nonzero nilpotent elements. Reduced rings are symmetric by the result of Anderson and Camillo [3, Theorem I.3] , but there are many nonreduced commutative (so symmetric) rings. Another generalization of a reduced ring is an Armendariz ring. Rege and Chhawchharia [7] called a ring R Armendariz if whenever any polynomials $f(x) = a_0 + a_1 x + \cdots + a_m x^m, g(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each i and j. Huh and et al. [1, Example 3.1], showed that polynomial rings over symmetric rings need not be symmetric. In the paper, we consider these symmetric rings over which polynomial rings are symmetric and call them be strongly symmetric, i.e., a ring R is called strongly symmetric, if whenever polynomials $f(x), g(x), h(x)$ in $R[x]$ satisfy $f(x)g(x)h(x) = 0$, then $f(x)h(x)g(x) = 0$.

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2. Strongly Symmetric Rings

Definition 2.1. *A ring R is called strongly symmetric, if whenever polynomials $f(x), g(x), h(x)$ in $R[x]$ satisfy $f(x)g(x)h(x) = 0$, then $f(x)h(x)g(x) = 0$.*

Clearly, every strongly symmetric ring is symmetric. but the converse is not true [1, Example 3.1]. It is obvious that any reduced rings are strongly symmetric and symmetric.

Lemma 2.2. *The class of strongly symmetric rings is closed under subrings (not necessarily with identity) and direct products.*

Recall that an element u of a ring R is right regular if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, left regular elements can be defined. An element is regular if it is both left and right regular (and hence not a zero divisor).

Proposition 2.3. *Let Δ be a multiplicatively closed subset of a ring R consisting of central regular elements. Then R is strongly symmetric ring if and only if so is $\Delta^{-1}R$.*

Proof. It is enouph to show that the necessity. Suppose that R is strongly symmetric. Let $\phi\varphi\psi = 0$, with $\phi = u^{-1}f(x), \varphi = v^{-1}g(x), \psi = w^{-1}h(x)$, $u, v, w \in \Delta$ and $f(x), g(x), h(x) \in R[x]$. Since Δ is contained in the center of R, we have $0 = \phi\varphi\psi = u^{-1}f(x)v^{-1}g(x)w^{-1}h(x) = (u^{-1}v^{-1}w^{-1})f(x)g(x)h(x) = (uvw)^{-1}f(x)g(x)h(x)$ and so $f(x)g(x)h(x) = 0$. But R is strongly symmetric by the condition, so $f(x)h(x)g(x) = 0$ and $\phi\psi\varphi = u^{-1}f(x)w^{-1}h(x)v^{-1}g(x) = (uwv)^{-1}f(x)h(x)g(x) = 0$. Hence $\Delta^{-1}R$ is strongly symmetric. □

The ring of Laurent polynomials in x, with coefficients in a ring R, consists of all formal sum $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers; denote it by $R[x; x^{-1}]$.

Corollary 2.4. *Let R be a ring. Then $R[x]$ is strongly symmetric rings if and only if $R[x; x^{-1}]$ is strongly symmetric.*

Proof. Let $\Delta = \{1, x, x^2, \ldots\}$. Then clearly Δ is a multiplicatively closed subset of $R[x]$. Since $R[x; x^{-1}] = \Delta^{-1}R[x]$, it follows that $R[x; x^{-1}]$ is strongly symmetric by the Proposition 2.3. □

Theorem 2.5. *A ring R is strongly symmetric if and only if $R[x]$ is strongly symmetric.*

Proof. ($\Leftarrow$) By Lemma 2.2.
($\Rightarrow$) Let $f(y) = f_0 + f_1y + \cdots + f_py^p, g(y) = g_0 + g_1y + \cdots + g_qy^q, h(y) = h_0 + h_1y + \cdots + h_ly^l \in R[x][y]$ satisfy $f(y)g(y)h(y) = 0$, where $f_i = \sum_{s=0}^{n_i} a_s^{(i)}x^s, g_j = \sum_{t=0}^{m_j} b_t^{(j)}x^t, h_k = \sum_{u=0}^{r_k} c_u^{(k)}x^u \in R[x]$ for $i = 0, 1, \ldots, p, j = 0, 1, \ldots, q, k = 0, 1, \ldots, l$. Let $w = deg(f_0) + deg(f_1) + \cdots + deg(f_p) + deg(g_0) + deg(g_1) + \cdots + deg(g_q) + deg(h_0) + deg(h_1) + \cdots + deg(h_l)$, where degree is as polynomials in x and the degree of the zero polynomial is taken to be 0. Then $f(x^w) = f_0 + f_1x^w + \cdots + f_px^{pw}, g(x^w) = g_0 + g_1x^w + \cdots + g_qx^{qw}, h(x^w) = h_0 + h_1x^w + \cdots + h_lx^{lw} \in R[x]$ and the set of coefficients of $f_i^{,}s, g_j^{,}s$ (resp. $h_k^{,}s$) equals the set of coefficients of $f(x^w), g(x^w)$ (resp. $h(x^w)$). Since $f(y)g(y)h(y) = 0$ and x commutes with elements of R, we have that $f(x^w)g(x^w)h(x^w) = 0$, thus $f(x^w)h(x^w)g(x^w) = 0 = f(y)h(y)g(y)$ since R is strongly symmetric, which implies $R[x]$ is strongly symmetric. □

Corollary 2.6. *Let R be a strongly symmetric ring and $\{x_\alpha\}$ any set of commuting indeterminates over R. Then any subring of $R[\{x_\alpha\}]$ is strongly symmetric*

Proof. Let $f(y), g(y), h(y) \in R[\{x_\alpha\}]$ with $f(y)g(y)h(y) = 0$. Then

$$f(y), g(y), h(y) \in R[\{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}\}][y]$$

for some finite subset $\{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}\} \subseteq \{x_\alpha\}$. The ring $R[\{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}\}][y]$, by induction, is strongly symmetric, so we have that $f(y)h(y)g(y) = 0$. Hence $R[\{x_\alpha\}]$ is strongly symmetric and thus so is any subring of $R[\{x_\alpha\}]$. □

Let R be a ring. Suppose that $Z(R)$ contains an infinite subring whose nonzero element are regular in R, where $Z(R)$ denotes the set of all central elements of R, if R is symmetric, then R is strongly symmetric by [1, Proposition 3.3]. Another example of a strongly symmetric ring is given in the following which also shows that strongly symmetric rings are not reduced in general.

Proposition 2.7. *Let R be a ring and n any positive integer. If R is reduced, then $R[x]/(x^n)$ is a strongly symmetric ring, where (x^n) is the ideal generated by x^n.*

Proof. It is obvious that $R[x]/(x^n)$ is strongly symmetric since $R[x]/(x^n)$ is both symmetric [1, Theorem 2.3] and Armendariz [2, Theorem 5] □

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M, write $T(R, M)$ is the ring $R \bigoplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

Note that $T(R, M)$ is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Corollary 2.8. *Let R be a ring and $T = R \bigoplus M$ be the trivial extension of R by R. If R is reduced, then T is strongly symmetric.*

Proof. $T \cong R[x]/(x^2)$ is strongly symmetric by Proposition 2.7. □

Proposition 2.9. *Let R be a subdirect sum of strongly symmetric rings. Then R is strongly symmetric.*

Proof. Let $I_\lambda(\lambda \in \Lambda)$ be ideals of R such that R/I_λ is strongly symmetric and $\cap_{\lambda \in \Lambda} I_\lambda = 0$. Suppose that $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j, h(x) = \sum_{k=0}^{r} c_k x^r \in R[x]$ satisfy $f(x)g(x)h(x) = 0$. Then $\bar{f}(x)\bar{h}(x)\bar{g}(x) = 0$ in $(R/I_\lambda)[x]$ for each $\lambda \in \Lambda$since R/I_λ is strongly symmetric. So $\sum_{i+j+k=l} a_i c_k b_j \in I_\lambda$ for $l = 0, 1, \ldots, n + m + r$ and any $\lambda \in \Lambda$ which implies that $\sum_{i+j+k=l} a_i c_k b_j = 0$ for $l = 0, 1, \ldots, n + m + r$ since $\cap_{\lambda \in \Lambda} I_\lambda = 0$, and we obtain $f(x)h(x)g(x) = 0$. □

Proposition 2.10. *Let R be a ring and I be a proper ideal of R. If R/I is strongly symmetric and I is reduced (as a ring without identity) then R is strongly symmetric.*

Proof. Let $f(x), g(x), h(x) \in R[x]$ satisfy $f(x)g(x)h(x) = 0$. Then $g(x)h(x)f(x) = 0$ and $(f(x)h(x)g(x))(h(x)f(x)h(x)g(x)) = 0 \Rightarrow (h(x)f(x)h(x)g(x))(f(x)h(x)g(x)) = 0$; $0 = h(x)f(x)h(x)g(x)f(x)h(x)g(x) = (h(x)f(x)h(x)g(x)f(x)h(x)g(x))f(x) = h(x)(f(x)h(x)g(x)f(x)h(x)g(x)f(x)) \Rightarrow (f(x)h(x)g(x)f(x)h(x)g(x)f(x))h(x) = 0$ and so we have $0 = (f(x)h(x)g(x)f(x)h(x)g(x)f(x)h(x))g(x) = (f(x)h(x)g(x))^3$. Thus $f(x)h(x)g(x) = 0$ since $f(x)h(x)g(x) \in I[x]$ and $I[x]$ is reduced. Therefore R is strongly symmetric. □

Theorem 2.11. *Let R be a right Ore ring and Q be the classical right quotient ring of R. Then R is strongly symmetric if and only if so is Q.*

Proof. It suffices to obtain the necessity by Lemma 2.2. Suppose that R is strongly symmetric and let $\phi\varphi\psi = 0$ for $\phi = f(x)u^{-1}, \varphi = g(x)v^{-1}$ and $\psi = h(x)w^{-1}$ in Q. There exist $g_1(x), u_1 \in R[x]$ with u_1 regular such that $g_1(x)u_1 = ug_1(x)$ and $u^{-1}g(x) = g_1(x)u^{-1}$, so we have $0 = \phi\varphi\psi = f(x)u^{-1}g(x)v^{-1}h(x)w^{-1} = f(x)g_1(x)u_1^{-1}v^{-1}h(x)w^{-1}$. Next there exist $h_1(x), v_1 \in R$ with v_1 regular such that $h(x)v_1 = vh_1(x)$ and $v^{-1}h(x) = h_1(x)v_1^{-1}$ so we have $0 = \phi\varphi\psi = f(x)g_1(x)u_1^{-1}h_1(x)v_1^{-1}w^{-1}$.

Also there exist $h_2(x), u_2 \in R[x]$ with u_1 regular such that $h_1(x)u_2 = u_1h_2(x)$ and $u_1^{-1}h_1(x) = h_2(x)u_2^{-1}$. So we have $0 = \phi\varphi\psi = f(x)u^{-1}g(x)v^{-1}h(x)w^{-1} = f(x)g_1(x)h_2(x)u_2^{-1}v_1^{-1}w^{-1}$. Hence we get $f(x)g_1(x)h_2(x) = 0$. In the following computation we use the condition that R is strongly symmetric; $f(x)g_1(x)h_2(x) = 0, f(x)g_1(x)h_2(x)u = 0$ and $0 = f(x)ug_1(x)h_2(x) = f(x)g(x)u_1h_2(x) \Rightarrow 0 = f(x)g(x)h_2(x)u_1$ implies $f(x)g(x)h_2(x) = 0 \Rightarrow 0 = f(x)g(x)h_2(x)u_1 = f(x)g(x)u_1h_2(x) = f(x)g(x)h_1(x)u_2 \Rightarrow f(x)g(x)h_1(x) = 0, 0 = f(x)g(x)h_1(x)v = f(x)g(x)vh_1(x) = f(x)g(x)h(x)v_1 \Rightarrow f(x)g(x)h(x) = 0$ and we get $f(x)h(x)g(x) = 0$.

Similarly there exist $h_3(x), u_3, g_3(x), w_3, g_4(x), u_4 \in R$ with u_3, w_3, u_4 regular such that $h(x)u_3 = uh_3(x), g(x)w_3 = wg_3(x), g_3(x)u_4 = u_3g_4(x)$ and $\phi\varphi\psi = f(x)u^{-1}h(x)w^{-1}g(x)v^{-1} = f(x)h_3(x)u_3^{-1}w^{-1}g(x)v^{-1} = f(x)h_3(x)u_3^{-1}g_3(x)w_3^{-1}v^{-1} = f(x)h_3(x)g_4(x)u_4^{-1}w_3^{-1}v^{-1}$.

Consequently we obtain the following computation $f(x)h(x)g(x) = 0$, $0 = f(x)h(x)g(x)u_3 = f(x)h(x)u_3g(x) = f(x)uh_3(x)g(x) = f(x)h_3(x)g(x)u \Rightarrow f(x)h_3(x)g(x) = 0$, $0 = f(x)h_3(x)g(x) = f(x)h_3(x)g(x)w_3 = f(x)h_3(x)wg_3(x) = f(x)h_3(x)g_3(x)w \Rightarrow f(x)h_3(x)g_3(x) = 0$, $0 = f(x)h_3(x)g_3(x) = f(x)h_3(x)g_3(x)u_4 = f(x)h_3(x)u_3g_4(x) = f(x)h_3(x)g_4(x)u_3 \Rightarrow f(x)h_3(x)g_4(x) = 0$. Therefore $\phi\psi\varphi = f(x)h_3(x)g_4(x)u_4^{-1}w_3^{-1}v^{-1} = 0$, proving that Q is strongly symmetric. □

Proposition 2.12. *For an abelian ring R. The following statements are equivalent:*

(1). R is strongly symmetric rings.

(2). eR and $(1-e)R$ are strongly symmetric rings.

Proof. (1)⇔(2) is straightforward since subrings and direct products of strongly symmetric rings are strongly symmetric. □

Let R be an algebra over a commutative ring S. The Dorroh extension of R by S is the ring $R \times S$ with operations $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$

Proposition 2.13. *Let R be an algebra over a commutative ring S, and D be the Dorroh extension of R by S. If R is strongly symmetric and S is a domain, then D is strongly symmetric.*

Proof. Let $(f_1(x), g_1(x)), (f_2(x), g_2(x)), (f_3(x), g_3(x)) \in D$ with $(f_1(x), g_1(x))(f_2(x), g_2(x))(f_3(x), g_3(x)) = 0$. Then $(f_1(x)f_2(x)f_3(x) + g_1(x)f_2(x)f_3(x) + g_2(x)f_1(x)f_3(x) + g_3(x)f_1(x)f_2(x) + g_1(x)g_2(x)f_3(x) + g_1(x)g_3(x)f_2(x) + g_2(x)g_3(x)f_1(x), g_1(x)g_2(x)g_3(x)) = 0$, so we have $f_1(x)f_2(x)f_3(x) + g_1(x)f_2(x)f_3(x) + g_2(x)f_1(x)f_3(x) + g_3(x)f_1(x)f_2(x) + g_1(x)g_2(x)f_3(x) + g_1(x)g_3(x)f_2(x) + g_2(x)g_3(x)f_1(x) = 0$ and $g_1(x)g_2(x)g_3(x) = 0$. Since S is a domain, $g_1(x) = 0, g_2(x) = 0$ or $g_3(x) = 0$.

In the following computations we use freely the condition that R is strongly symmetric. Say $g_1(x) = 0$ then $f_1(x)f_2(x)f_3(x) + g_2(x)f_1(x)f_3(x) + g_3(x)f_1(x)f_2(x) + g_2(x)g_3(x)f_1(x) = 0$ and so we have $0 = f_1(x)f_2(x)f_3(x) + g_2(x)f_1(x)f_3(x) + g_3(x)f_1(x)f_2(x) + g_2(x)g_3(x)f_1(x) = f_1(x)(f_2(x) + g_2(x))(f_3(x) + g_3(x)) = f_1(x)(f_3(x) + g_3(x))(f_2(x) + g_2(x)) = f_1(x)f_3(x)f_2(x) + f_1(x)f_3(x)g_2(x) + f_1(x)g_3(x)f_2(x) + f_1(x)g_3(x)g_2(x) = f_1(x)f_3(x)f_2(x) + f_1(x)f_3(x)g_2(x) + f_1(x)g_3(x)f_2(x) + f_1(x)g_3(x)g_2(x) + g_1(x)f_3(x)f_2(x) + g_1(x)f_3(x)g_2(x) + g_2(x)g_3(x)f_2(x)$.

Say $g_2(x) = 0$, then $f_1(x)f_2(x)f_3(x) + g_1(x)f_2(x)f_3(x) + f_1(x)f_2(x)g_3(x) + g_1(x)f_2(x)g_3(x) = 0$ and so we have $0 = f_1(x)f_2(x)f_3(x) + g_1(x)f_2(x)f_3(x) + f_1(x)f_2(x)g_3(x) + g_1(x)f_2(x)g_3(x) = (f_1(x) + g_1(x))f_2(x)(f_3(x) + g_3(x)) = (f_1(x) + g_1(x))(f_3(x) + g_3(x))f_2(x) = f_1(x)f_3(x)f_2(x) + f_1(x)g_3(x)f_2(x) + g_1(x)f_3(x)f_2(x) + g_1(x)g_3(x)f_2(x) = f_1(x)f_3(x)f_2(x) + f_1(x)g_3(x)f_2(x) + g_1(x)f_3(x)f_2(x) + g_1(x)g_3(x)f_2(x) + f_1(x)f_3(x)g_2(x) + f_1(x)g_3(x)g_2(x) + g_1(x)f_3(x)g_2(x)$.
Say $g_3(x) = 0$, then $f_1(x)f_2(x)f_3(x) + g_1(x)f_2(x)f_3(x) + f_1(x)g_2(x)f_3(x) + g_1(x)g_2(x)f_3(x) = 0$ and so we have $0 = f_1(x)f_2(x)f_3(x) + g_1(x)f_2(x)f_3(x) + f_1(x)g_2(x)f_3(x) + g_1(x)g_2(x)f_3(x) = (f_1(x) + g_1(x))(f_2(x) + g_2(x))f_3(x) = (f_1(x) + g_1(x))f_3(x)(f_2(x) + g_2(x)) = f_1(x)f_3(x)f_2(x) + f_1(x)f_3(x)g_2(x) + g_1(x)f_3(x)f_2(x) + g_1(x)f_3(x)g_2(x) = f_1(x)f_3(x)f_2(x) + f_1(x)f_3(x)g_2(x) + g_1(x)f_3(x)f_2(x) + g_1(x)f_3(x)g_2(x) + f_1(x)g_3(x)f_2(x) + g_1(x)g_3(x)f_2(x) + f_1(x)g_3(x)g_2(x)$. Therefore we obtain $(f_1(x), g_1(x))(f_3(x), g_3(x))(f_2(x), g_2(x)) = 0$ in any case, proving that D is strongly symmetric. □

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