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Common Fixed Point of Three Contractive Type Mappings

Research Article

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Abstract: In this paper we prove common fixed point theorems for sequentially convergent Kannan and Chatterjea type mappings, which are generalization of many common fixed point theorems.

Keywords: Fixed point, Affine, Sequentially convergent, Compact.

1. Introduction

This paper aims to prove unique common fixed point theorems for three contractive type mappings on a complete metric space which extends the theorems of [1] and [2] for single mapping and [3, 4, 7] results for two mappings. Let (X, d) be a complete metric space, $T : X \to X$ be continuous, injective and sequentially convergent mapping and let S_1, S_2 be self maps of X. The authors of [7], 2016 proved that, if T, S_1, S_2 satisfies $d(TS_1x, TS_2y) \leq \alpha(d(Tx, TS_1x)+d(Ty, TS_2y))+\beta d(Tx, Ty)$, for all $x, y \in X$, where $\alpha > 0, \beta \geq 0$ such that $2\alpha + \beta < 1$ then S_1 and S_2 have a unique common fixed point. Also they proved, if T, S_1, S_2 satisfies $d(TS_1x, TS_2y) \leq \alpha(d(Tx, TS_2y) + d(Ty, TS_1x)) + \beta d(Tx, Ty)$, then S_1 and S_2 have a unique common fixed point. We establish the results for the existence of unique common fixed points for three contractive mappings T, S_1, S_2 by assuming that $d(Tx, S_1y) \leq d(x, y)$ (or) $d(Tx, S_2y) \leq d(x, y)$ for all $x, y \in X$. We also prove results showing the unique common fixed point for the self maps T, S_1, S_2 on a non-empty compact subset K of a metric space (X, d), by relaxing the condition of sequentially convergent on T. In a non-empty compact convex subset K of a Banach space X, we assume that T to be affine instead of sequentially convergent and $\|Tx - S_1y\| \leq \|x - y\|$ or $\|Tx - S_2y\| \leq \|x - y\|$ for all $x, y \in K$, for the common fixed point of T with S_1, S_2.

2. Preliminaries

Definition 2.1 ([6]). *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be sequentially convergent if for each sequence $\{y_n\}$ in X, the sequence $\{Ty_n\}$ converges $\Rightarrow \{y_n\}$ is convergent.*

Definition 2.2. *Let K be a non-empty subset of a Banach space X. A map $T : K \to K$ is said to be affine if $T((1-\lambda)x+\lambda y) = (1 - \lambda)Tx + \lambda Ty$ for all $x, y \in K$ and $\lambda \in (0, 1)$.*

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Theorem 2.3 ([7]). *Let (X,d) be a complete metric space, $T : X \to X$ be continuous, injective and sequentially convergent mapping and $S_1, S_2 : X \to X$. If there exist $\alpha > 0, \beta \geq 0$ such that $2\alpha + \beta < 1$ and*

$$d(TS_1x, TS_2y) \leq \alpha(d(Tx, TS_1x) + d(Ty, TS_2y)) + \beta d(Tx, Ty),$$

for all $x, y \in X$, then S_1 and S_2 have a unique common fixed point.

Theorem 2.4 ([7]). *Let (X,d) be a complete metric space, $T : X \to X$ be continuous, injective and sequentially convergent mapping and $S_1, S_2 : X \to X$. If there exist $\alpha > 0, \beta \geq 0$ so that $2\alpha + \beta < 1$ and*

$$d(TS_1x, TS_2y) \leq \alpha(d(Tx, TS_2y) + d(Ty, TS_1x)) + \beta d(Tx, Ty),$$

for all $x, y \in X$, then S_1 and S_2 have a unique common fixed point.

3. Main Results

Since $\alpha > 0$ in Theorem 2.3, The following theorem is not the corollary of the Theorem 2.3.

Theorem 3.1. *Let (X,d) be a complete metric space and let $T : X \to X$ be continuous, injective, sequentially convergent mapping. Let $S_1, S_2 : X \to X$ be self maps such that $d(TS_1x, TS_2y) \leq ad(Tx, TS_1x) + bd(Ty, TS_2y) + cd(Tx, Ty)$, where $a, b, c \in [0, 1)$ with $a + b + c < 1$ and $d(Tx, S_1y) \leq d(x, y)$ (or) $d(Tx, S_2y) \leq d(x, y)$ for all $x, y \in X$, then T, S_1 and S_2 have a unique common fixed point.*

Proof. Let $x_0 \in X$, Define x_n by $x_{2n+1} = S_1x_{2n}, x_{2n+2} = S_2x_{2n+1}$ for $n = 0, 1, 2, ...$
Let n be even.

$$\begin{aligned}
d(Tx_n, Tx_{n+1}) &= d(TS_2x_{n-1}, TS_1x_n) \\
&\leq ad(Tx_{n-1}, TS_2x_{n-1}) + bd(Tx_n, TS_1x_n) + cd(Tx_{n-1}, Tx_n) \\
&= ad(Tx_{n-1}, Tx_n) + bd(Tx_n, Tx_{n+1}) + cd(Tx_{n-1}, Tx_n) \\
d(Tx_n, Tx_{n+1}) &\leq \frac{a+c}{1-b} \; d(Tx_{n-1}, Tx_n)
\end{aligned}$$

Since $a + b + c < 1$, hence $\{Tx_n\}$ is a Cauchy sequence in X. Therefore $\{Tx_n\}$ is convergent in X. Since T is sequentially convergent, $\{x_n\}$ is convergent. (i.e) $\exists \; x \in X$ such that $x_n \to x$ as $n \to \infty$. Since T is continuous, $Tx_n \to Tx$ as $n \to \infty$. Now

$$\begin{aligned}
d(Tx, TS_1x) &\leq d(Tx, Tx_{2n}) + d(Tx_{2n}, TS_1x) \\
&= d(Tx, Tx_{2n}) + d(TS_2x_{2n-1}, TS_1x) \\
&\leq d(Tx, Tx_{2n}) + ad(Tx_{2n-1}, TS_2x_{2n-1}) + bd(Tx, TS_1x) + cd(Tx_{2n-1}, Tx) \\
&= d(Tx, Tx_{2n}) + ad(Tx_{2n-1}, Tx_{2n}) + bd(Tx, TS_1x) + cd(Tx_{2n-1}, Tx) \\
&\to bd(Tx, TS_1x) \text{ as } n \to \infty.
\end{aligned}$$

$\therefore Tx = TS_1x$, Since T is injective, $x = S_1x$. Similarly $x = S_2x$. Hence $x = S_1x = S_2x$ and

$$\begin{aligned} d(x,Tx) &\le d(x,x_{2n}) + d(x_{2n},Tx) \\ &= d(x,x_{2n}) + d(S_2x_{2n-1},Tx) \\ &\le d(x,x_{2n}) + d(x_{2n-1},x) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

$\therefore Tx = x$. Hence T, S_1, S_2 have a common fixed point.

Uniqueness: Suppose $\exists\ y \in X$ such that $S_1y = S_2y = Ty = y$. Now

$$\begin{aligned} d(Tx,Ty) &= d(TS_1x,TS_2y) \\ &\le ad(Tx,TS_1x) + bd(Ty,TS_2y) + cd(Tx,Ty) \\ &= ad(Tx,Tx) + bd(Ty,Ty) + cd(Tx,Ty) \end{aligned}$$

Since $c < 1, Tx = Ty$ and hence $x = y$. □

Corollary 3.2. *Let (X,d) be a complete metric space and let $T : X \to X$ be continuous, injective, sequentially convergent mapping. Let $S_1, S_2 : X \to X$ be self maps such that $d(TS_1x,TS_2y) \le \alpha d(Tx,Ty)$, where $\alpha \in [0,1)$ and $d(Tx,S_1y) \le d(x,y)$ (or) $d(Tx,S_2y) \le d(x,y)$ for all $x,y \in X$, then T, S_1 and S_2 have a unique common fixed point.*

Proof. The proof of the corollary follows from the above theorem by putting $a = b = 0$ and $c = \alpha$. □

Theorem 3.3. *Let K be a non-empty compact subset of a metric space (X,d). Let $T : K \to K$ be continuous, injective mapping and let S_1, S_2 be self mappings of K. If there exist $a \in [0,1)$ and $b \ge 0$ such that $2a + b \le 1$ and $d(TS_1x,TS_2y) \le a[d(Tx,TS_1x) + d(Ty,TS_2y)] + bd(Tx,Ty)$, and $d(Tx,S_1y) \le d(x,y)$ (or) $d(Tx,S_2y) \le d(x,y)$ for all $x,y \in K$, then T, S_1 and S_2 have a common fixed point. Further if $b < 1$ then T, S_1 and S_2 have a unique common fixed point.*

Proof. For each $n \in N$, let $x_{2n+1} = S_1x_{2n}, x_{2n+2} = S_2x_{2n+1}$. Then the sequence $\{x_n\} \subseteq K$. Since K is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$ as $k \to \infty$. Therefore $Tx_{n_k} \to Tx$. Now

$$\begin{aligned} d(Tx,TS_1x) &\le d(Tx,Tx_{2n_k}) + d(Tx_{2n_k},TS_1x) \\ &= d(Tx,Tx_{2n_k}) + d(TS_2x_{2n_k-1},TS_1x) \\ &\le d(Tx,Tx_{2n_k}) + a[d(Tx_{2n_k-1},TS_2x_{2n_k-1}) + d(Tx,TS_1x)] + bd(Tx_{2n_k-1},Tx) \\ &= d(Tx,Tx_{2n_k}) + a[d(Tx_{2n_k-1},Tx_{2n_k}) + d(Tx,TS_1x)] + bd(Tx_{2n_k-1},Tx) \\ &\to ad(Tx,TS_1x) \text{ as } k \to \infty. \end{aligned}$$

$\therefore Tx = TS_1x$, Since T is injective, $x = S_1x$. Similarly $x = S_2x$. Hence $x = S_1x = S_2x$.

$$\begin{aligned} d(x,Tx) &\le d(x,x_{2n}) + d(x_{2n},Tx) \\ &= d(x,x_{2n}) + d(S_2x_{2n-1},Tx) \\ &\le d(x,x_{2n}) + d(x_{2n-1},x) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

$\therefore Tx = x$. Hence T, S_1, S_2 have a common fixed point.

Uniqueness: Let y be an element in K such that $S_1y = S_2y = Ty = y$. Now

$$\begin{aligned} d(Tx,Ty) &= d(TS_1x, TS_2y) \\ &\leq a[d(Tx,TS_1x) + d(Ty,TS_2y)] + bd(Tx,Ty) \\ &= a[d(Tx,Tx) + d(Ty,Ty)] + bd(Tx,Ty) \end{aligned}$$

If $b < 1, Tx = Ty$ and hence $x = y$. □

Since $\alpha > 0$ in Theorem 2.4, The following theorem is not the corollary of the Theorem 2.4.

Theorem 3.4. *Let (X, d) be a complete metric space and let $T : X \to X$ be continuous, injective, sequentially convergent mapping. Let $S_1, S_2 : X \to X$ be self maps such that $d(TS_1x, TS_2y) \leq ad(Tx, TS_2y) + bd(Ty, TS_1x) + cd(Tx, Ty)$, where $a, b, c \in [0, 1)$ with $2a + b + c < 1$ and $d(Tx, S_1y) \leq d(x, y)$ (or) $d(Tx, S_2y) \leq d(x, y)$ for all $x, y \in X$, then T, S_1 and S_2 have a unique common fixed point.*

Proof. Let $x_0 \in X$, Define x_n by $x_{2n+1} = S_1x_{2n}, x_{2n+2} = S_2x_{2n+1}$ for $n = 0, 1, 2, \ldots$. Let n be even.

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(TS_2x_{n-1}, TS_1x_n) \\ &\leq ad(Tx_{n-1}, TS_1x_n) + bd(Tx_n, TS_2x_{n-1}) + cd(Tx_{n-1}, Tx_n) \\ &= ad(Tx_{n-1}, Tx_{n+1}) + bd(Tx_n, Tx_n) + cd(Tx_{n-1}, Tx_n) \\ &\leq a[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})] + cd(Tx_{n-1}, Tx_n) \\ d(Tx_n, Tx_{n+1}) &\leq \frac{a+c}{1-a} \; d(Tx_{n-1}, Tx_n) \end{aligned}$$

Since $2a + b + c < 1$, hence $\{Tx_n\}$ is a Cauchy sequence in X. Therefore $\{Tx_n\}$ is convergent in X. Since T is sequentially convergent, $\{x_n\}$ is convergent. (i.e) $\exists \; x \in X$ such that $x_n \to x$ as $n \to \infty$. Since T is continuous, $Tx_n \to Tx$ as $n \to \infty$. Now

$$\begin{aligned} d(Tx, TS_1x) &\leq d(Tx, Tx_{2n}) + d(Tx_{2n}, TS_1x) \\ &= d(Tx, Tx_{2n}) + d(TS_2x_{2n-1}, TS_1x) \\ &\leq d(Tx, Tx_{2n}) + ad(Tx_{2n-1}, TS_1x) + bd(Tx, TS_2x_{2n-1}) + cd(Tx_{2n-1}, Tx) \\ &\to ad(Tx, TS_1x) \text{ as } n \to \infty. \end{aligned}$$

Since $a < 1, d(Tx, TS_1x) = 0$ implies $Tx = TS_1x, x = S_1x$, similarly $x = S_2x$ and

$$\begin{aligned} d(x, Tx) &\leq d(x, x_{2n}) + d(x_{2n}, Tx) \\ &= d(x, x_{2n}) + d(S_2x_{2n-1}, Tx) \\ &\leq d(x, x_{2n}) + d(x_{2n-1}, x) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

$\therefore Tx = x$. Hence T, S_1, S_2 have a common fixed point.

Uniqueness: Suppose $\exists\ y \in X$ such that $S_1 y = S_2 y = Ty = y$. Now

$$\begin{aligned} d(Tx, Ty) &= d(TS_1x, TS_2y) \\ &\leq ad(Tx, TS_2y) + bd(Ty, TS_1x) + cd(Tx, Ty) \\ &= ad(Tx, Ty) + bd(Ty, Tx) + cd(Tx, Ty) \end{aligned}$$

Since $a + b + c < 1, Tx = Ty$ and hence $x = y$. □

Corollary 3.5. *Let (X, d) be a complete metric space and let $T : X \to X$ be continuous, injective, sequentially convergent mapping. Let $S_1, S_2 : X \to X$ be self maps such that $d(TS_1x, TS_2y) \leq \alpha d(Tx, Ty)$, where $\alpha \in [0, 1)$ and $d(Tx, S_1y) \leq d(x, y)$ (or) $d(Tx, S_2y) \leq d(x, y)$ for all $x, y \in X$, then T, S_1 and S_2 have a unique common fixed point.*

Proof. The proof of the corollary follows from the above theorem by putting $a = b = 0$ and $c = \alpha$. □

Theorem 3.6. *Let K be a non-empty compact subset of a metric space (X, d). Let $T : K \to K$ be continuous, injective mapping and let S_1, S_2 be self mappings of K. If there exist $a \in [0, 1)$ and $b \geq 0$ such that $2a + b \leq 1$ and $d(TS_1x, TS_2y) \leq a[d(Tx, TS_2y) + d(Ty, TS_1x)] + bd(Tx, Ty)$, and $d(Tx, S_1y) \leq d(x, y)$ (or) $d(Tx, S_2y) \leq d(x, y)$ for all $x, y \in K$, then T, S_1 and S_2 have a common fixed point. Further if $2a + b < 1$ then T, S_1 and S_2 have a unique common fixed point.*

Proof. For each $n \in N$, let $x_{2n+1} = S_1x_{2n}, x_{2n+2} = S_2x_{2n+1}$. Then the sequence $\{x_n\} \subseteq K$. Since K is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$ as $k \to \infty$. Therefore $Tx_{n_k} \to Tx$. Now

$$\begin{aligned} d(Tx, TS_1x) &\leq d(Tx, Tx_{2n_k}) + d(Tx_{2n_k}, TS_1x) \\ &= d(Tx, Tx_{2n_k}) + d(TS_2x_{2n_k-1}, TS_1x) \\ &\leq d(Tx, Tx_{2n_k}) + a[d(Tx_{2n_k-1}, TS_1x) + d(Tx, TS_2x_{2n_k-1})] + bd(Tx_{2n_k-1}, Tx) \\ &= d(Tx, Tx_{2n_k}) + a[d(Tx_{2n_k-1}, TS_1x) + d(Tx, Tx_{2n_k})] + bd(Tx_{2n_k-1}, Tx) \\ &\to ad(Tx, TS_1x) \text{ as } k \to \infty. \end{aligned}$$

$\therefore Tx = TS_1x$, Since T is injective, $x = S_1x$. Similarly $x = S_2x$. Hence $x = S_1x = S_2x$.

$$\begin{aligned} d(x, Tx) &\leq d(x, x_{2n+1}) + d(x_{2n+1}, Tx) \\ &= d(x, x_{2n+1}) + d(S_1x_{2n}, Tx) \\ &\leq d(x, x_{2n+1}) + d(x_{2n}, x) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

$\therefore Tx = x$. Hence T, S_1, S_2 have a common fixed point.

Uniqueness: Let y be an element in K such that $S_1y = S_2y = Ty = y$. Now

$$\begin{aligned} d(Tx, Ty) &= d(TS_1x, TS_2y) \\ &\leq a[d(Tx, TS_2y) + d(Ty, TS_1x)] + bd(Tx, Ty) \\ &= a[d(Tx, Ty) + d(Ty, Tx)] + bd(Tx, Ty) \\ (1 - (2a + b))d(Tx, Ty) &\leq 0 \end{aligned}$$

If $2a + b < 1, Tx = Ty$ and hence $x = y$. □

Theorem 3.7. *Let K be a non-empty compact convex subset of a Banach space X. Let $T : K \to K$ be continuous, injective, affine and S_1, S_2 be self mappings of K. If there exist $\alpha \in [0,1)$ such that $\|TS_1x - TS_2y\| \leq \alpha\|Tx - Ty\|$ and $\|Tx - S_1y\| \leq \|x - y\|$ or $\|Tx - S_2y\| \leq \|x - y\|$ for all $x, y \in K$. Then T, S_1, S_2 have a common fixed point.*

Proof. Let $x_0 \in K, \alpha_n \in (0,1)$ such that $\alpha_n \to 1$, as $n \to \infty$. Define $S_{1n}, S_{2n} : K \to K$ by $S_{1n}(x) = (1-\alpha_n)x_0 + \alpha_n S_1 x, S_{2n}(x) = (1-\alpha_n)x_0 + \alpha_n S_2 x$. Then $||TS_{1n}x - TS_{2n}y|| = \alpha_n\|TS_1x - TS_2y\| \leq \alpha_n\alpha\|Tx - Ty\|$. Then by Corollary 3.2, S_{1n}, S_{2n} have a common fixed point. Let $S_{1n}(x_n) = S_{2n}(x_n) = x_n, \forall n$. Since X is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$ as $k \to \infty$. Therefore $Tx_{n_k} \to Tx$. Now, $x_{n_k} = S_{1n_k}x_{n_k} = (1-\alpha_{n_k})x_0 + \alpha_{n_k}S_1x_{n_k}$, $S_1x_{n_k} \to x$ as $k \to \infty$. Similarly $S_2x_{n_k} \to x$

$$\begin{aligned}\|Tx - TS_1x\| &\leq \|Tx - TS_2x_{n_k}\| + \|TS_2x_{n_k} - TS_1x\| \\ &\leq \|Tx - TS_2x_{n_k}\| + \alpha\|Tx_{n_k} - Tx\| \\ &\to 0 \text{ as } k \to \infty.\end{aligned}$$

Hence $\|Tx - TS_1x\| = 0$. Since T is injective, $x = S_1x$. Similarly $x = S_2x$. Now

$$\begin{aligned}\|x - Tx\| &\leq \|x - S_1x_{n_k}\| + \|S_1x_{n_k} - Tx\| \\ &= \|x - S_1x_{n_k}\| + \|x_{n_k} - x\| \\ &\to 0 \text{ as } k \to \infty.\end{aligned}$$

Hence $x = Tx$. Thus T, S_1, S_2 have a common fixed point. □

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