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A Note on Extremal Disconnectedness

Research Article

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Abstract: In this paper, we show that in an extremally disconnected topological space, there exists a maximal family of subsets of the space containing the family of all semi-open sets, where the closure operator is distributive over the intersection of every two members of the family. A characterization of extremal disconnectedness in terms of open filters is also obtained.

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Keywords: Semi-open set, extremally disconnected space, open filter, open ultrafilter.

1. Introduction

The class of extremely disconnected (e.d.) topological spaces forms an important part of the class of all topological spaces. Gleason in [3] has shown that extremely disconnected topological spaces are precisely the projective spaces in the category of compact topological spaces and continuous maps. There are many equivalent definitions of extremal disconnectedness ([4–6, 8, 14]). Most of these results involve either open sets or closed sets or both open and closed sets. Some weak forms of open sets (closed sets) like pre-open (pre-closed), semi-open (semi-closed) and α-open (α-closed) exist in literature. For details, the reader is referred to [1, 2, 9–13, 15]. Some equivalent forms of extremal disconnectedness are known in the form of families of sets, where the closure operator is distributive over the intersection of every two members of the family. In [8], the concept of "rounding" for any open filter on a topological space is introduced to characterize extremal disconnectedness.

In this paper, we show that closure operator is distributive over the intersection of every two semi-open sets of an e.d. topological space. In this situation, there arises a natural question of finding a family (possibly the largest) of subsets of an e.d. topological space containing the family of all semi-open sets, where the closure operator is distributive over the intersection of every two members of the family. It is shown that such largest family does not exist. However in an e.d. topological space, there exists a maximal family of subsets of the space containing the family of all semi-open sets, where the closure operator is distributive over the intersection of every two members of the family. Finally, the necessary and sufficient conditions for a family of subsets of the space containing the family of all semi-open sets, where the closure operator is distributive over the intersection of every two members of the family to be largest, are obtained. A new characterization of extremal disconnectedness is also obtained using the concept of 'rounding' for any open filter.

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2. Notation and Definitions

For completeness, we have included some of the standard notation and definitions. Herein by a space we mean a topological space. Let X be a space. For $A \subset X$, $int_X(A)$ and $cl_X(A)$ denote interior and closure of A respectively. A subset A of a space X is called semi-open ([9]) (pre-open ([2]) if $A \subset cl_X(int_X(A))$ $(A \subset int_X(cl_X(A)))$. A set whose complement is semi-open (pre-open) is called semi-closed (pre-closed). An open filter F on X is a prime open filter if for open sets A, B of X, $A \cup B \in F$ only if $A \in F$ or $B \in F$. An open ultrafilter on X is a maximal open filter. For an open filter F on X, the open filter $rF = \{A \subset X :$ A is open in X and $int_X(cl_X(A)) \in F\}$ is called the rounding of F, and F is said to be round if $rF = F$. For $x \in X$, ox denotes the open filter $\{G \subset X :$ G is open in X and $x \in X\}$. odX is used to denote the open filter $\{G \subset X :$ G is open in X and $cl_X(G) = X\}$.

Following [12], a subset A of a space X is said to be n-regularly nowhere dense (for n an integer greater than 1) if there exists open sets $A_1, A_2, \ldots, A_n$ in X such that $A \subset \cap\{cl_X(A_i) : i = 1, 2, \ldots, n\}$ and $\cap\{A_i : i = 1, 2, \ldots, n\} = \phi$. Let R denotes the open filter $\{X - A :$ A is n-regularly nowhere dense for some $n > 1\}$.

3. Extremal Disconnectedness and Semi-open Sets

We shall take following as the definition of an extremally disconnected space.

Definition 3.1. *A space X is called an extremally disconnected (e.d.) space ([7]) if closure of every open set of X is open in X.*

Lemma 3.2. *Let X be an e.d. space. Let A and B be two semi-open sets in X. Then* $cl_X(A \cap B) = cl_X(A) \cap cl_X(B)$.

Proof. As A and B are semi-open sets in X, $cl_X(A) \cap cl_X(B) = cl_X(int_X(A)) \cap cl_X(int_X(B))$. Since X is e.d., $cl_X(int_X(A)) \cap cl_X(int_X(B)) = cl_X(int_X(A) \cap int_X(B))$. This implies that $cl_X(A \cap B) = cl_X(A) \cap cl_X(B)$. ☐

Lemma 3.3. *Let ς be a family of subsets of a space X containing the family of all open sets of X, where the closure operator is distributive over the intersection of every two members of ς. Then for every $A \in \varsigma$, $cl_X(A)$ is open in X.*

Proof. Let $A \in \varsigma$, $X - cl_X(A) \in \varsigma$ by the given condition. $cl_X(A) \cap cl_X(X - cl_X(A)) = cl_X(A \cap (X - cl_X(A))) = cl_X(\phi) = \phi$ again by the given condition. This implies that $cl_X(A) \subset int_X(cl_X(A))$; so $cl_X(A)$ is open. Hence the result follows. ☐

Lemma 3.4. *Let X be an e.d. space. Let A be semi-open and B pre-open in X.* $cl_X(A \cap B) = cl_X(A) \cap cl_X(B)$.

Proof. As A is semi-open and B pre-open in X, $cl_X(A) \cap cl_X(B) = cl_X(int_X(A)) \cap cl_X(int_X(cl_X(B)))$. Since X is e.d., $cl_X(int_X(A)) \cap cl_X(int_X(cl_X(B))) = cl_X(int_X(A) \cap int_X(cl_X(B)))$. This implies that $cl_X(A) \cap cl_X(B) = cl_X(A \cap B) as int_X(A) \cap cl_X(B) \subset cl_X(int_X(A) \cap B)$. ☐

Remark 3.5. *It is known that in an e.d. space, the family of all open sets is a family where closure operator is distributive over the intersection of every two members of the family ([8]). Lemma 3.2 makes that family considerably larger by replacing open sets by semi-open sets as every open set is semi-open. Since each α-open set is semi-open and every regular closed set is semi-open, the family of semi-open sets become quite a large family. But still there is a scope to determine a larger family (possibly the largest) of subsets of an e.d. space where the closure operator is distributive over the intersection of every two members of the family. The answer to this question is not in affirmative. Consider following example in support of this.*

Let IN be the space of all natural numbers with cofinite topology. Let A be the set of all even natural numbers. Then $cl_X(A) = cl_X(IN - A) = IN$. So A and $IN - A$ are pre-open sets of IN. Let δ denotes the family of all semi-open sets of IN.

Let ς be the largest family of subsets of the space containing δ, where the closure operator is distributive over the intersection of every two members of ς. Now, using Lemma 3.4, $\{A\}\cup\delta$ is a family of subsets of X containing δ where the closure operator is distributive over the intersection of every two members of this family. So $\{A\}\cup\delta \subset \varsigma$. Similarly $\{IN-A\}\cup\delta \subset \varsigma$. Therefore $cl_X(A\cap(IN-A)) = cl_X(A)\cap cl_X(IN-A)$. This is not possible as $cl_X(A\cap(IN-A)) = \phi and cl_X(A)\cap cl_X(IN-A) = IN$. Hence there does not exist largest family of subsets of the space containing the family of all semi-open sets, where the closure operator is distributive over the intersection of any two members of the family.

As justified in Remark 3.5, in general, there does not exist the largest family of subsets of an e.d. space containing the family of all semi-open sets, where the closure operator is distributive over the intersection of every two members of the family. So one can think of finding a maximal such family. The following lemma is a step to show the existence of such a maximal family.

Lemma 3.6. *Let X be a space. Let ς be a family of subsets of a space X containing the family of all semi-open sets, where the closure operator is distributive over the intersection of any two members of ς. Then there exists a maximal family of subsets of the space containing ς where the closure operator is distributive over the intersection of every two members of ς.*

Proof. The union of a chain of families of subsets of a space where the closure operator is distributive over the intersection of every two members of the family for each family, becomes a family where the closure operator is distributive over the intersection of every two members of the family. Now the existence of a maximal such family follows using Zorn's Lemma. □

Theorem 3.7. *In an e.d. space, there exists a maximal family of subsets of the space containing the family of all semi-open sets, where the closure operator is distributive over the intersection of every two members of the family. Such a family has to be a subfamily of the family of all pre-open sets.*

Proof. The proof follows using Lemma 3.2, Remark 3.5 and Lemma 3.6. □

Theorem 3.7 assures the existence of a maximal family of subsets of an e.d. space containing the family of all semi-open sets, where the closure operator is distributive over the intersection of every two members of the family. Though in general such a largest family does not exist (see Remark 3.5). There may be some spaces where the existence of such largest family of subsets is possible. The following theorem gives necessary and sufficient conditions for the existence of such family of subsets of a space. The proof is left for the readers.

Theorem 3.8. *The following are equivalent for a space.*

(1). The space has a largest family of subsets containing the family of all semi-open sets, where the closure operator is distributive over the intersection of every two members of the family.

(2). Closure operator is distributive over the intersection of every two members of the family of all pre-open sets of the space.

4. Extremal Disconnectedness and Open Filters

First we note the following proposition.

Proposition 4.1. *Let X be a space. Then $rR = odX$.*

Theorem 4.2. *Let X be a space. If F is a prime open filter containing R, then rF is an open ultrafilter.*

Proof. Using Proposition 3.3 of [12], there exists an open ultrafilter F^* on X such that $\{int_X(cl_X(A)) : A \in F*\} \subset F$. By Proposition 2.3 (k)(2) of [15], F is contained in a unique open ultrafilter. Therefore rF is contained in a unique open ultrafilter using Proposition 3(e) of [8]. As rF is round by Proposition 3(a) of [8], so rF is an open ultrafilter by Proposition 3(f) of [8]. □

Theorem 4.3. *Let X be a space. If R is a prime open filter, then odX is the only open ultrafilter.*

Proof. By Theorem 4.2, rR is an open ultrafilter. So odX is an open ultrafilter by Proposition 4.1. Now the proposition follows by Remark 4(d) of [8]. □

Remark 4.4. *Let X be a space. If R is a prime open filter, then by Theorem 4.3, odX is the only open ultrafilter. This implies that every non-empty open subset of X is dense in X. Therefore X is an e.d. space.*

Theorem 4.5. *The following are equivalent for a space X.*

(1). X is extremally disconnected.

(2). Every prime open filter contains R.

Proof. We only prove (2)⇒(1). Suppose (2) holds. Let $x \in X$. Since ox is a prime open filter containing R, so by Theorem 4.2, rox is an open ultrafilter. Now (1) follows by Proposition 4 of [8]. □

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