International Journal of Mathematics And its Applications
Volume 5, Issue 4–B (2017), 129–131.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Absolute Summability of a Lacunary Fourier Series

Research Article

Sanjay Tripathi[1]*

1 Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda, Vadodara, Gujarat, India.

Abstract: The absolute convergence and absolute summability of Lacunary Fourier series was studied by several authors and obtained many results on it. In this direction ,N.V.Patel [2] has obtained a result on absolute summability of Lacunary Fourier series. In this paper, we have proved a results by considering more general absolute summability under same hypothesis which generalizes the results of N.V.Patel [2].

MSC: 40F05, 40G05, 42A24, 42B08, 42B05.

Keywords: Absolute CesaroSummability, Lacunary Fourier series, Absolute convergence.

1. Introduction

$$\sum_{k=1}^{\infty}(a_{n_k}\cos n_k x + b_{n_k}\sin n_k x) \tag{1}$$

be the Fourier series of a 2π-periodic functions $f \in [-\pi, \pi]$ with an infinity of gaps (n_k, n_{k+1}), where $(n_k)\,(k \in N)$ is a strictly increasing sequence of natural numbers. Several mathematicians have studied the absolute convergence and absolute summability of the Fourier series (1), as well as the order of magnitude of Fourier coefficients, by considering various properties of f.

Definition 1.1. *The series (1) is said to be summable $|c,\theta|_\tau$ $\tau \geq 1$, (see [1]) if*

$$\sum_{k=1}^{\infty} n_k^{\tau-1}\left|\sigma_{n_k}^{\theta}(x) - \sigma_{n_{k-1}}^{\theta}(x)\right|^{\tau} < \infty.$$

where $\sigma_{n_k}^{\theta}(x)$ denotes the n^{th} Cesaro mean of order θ and for a real numbers γ, which is not a negative integer and $E_n^{\gamma} = \begin{pmatrix} n+\gamma \\ n \end{pmatrix}$, where $n \in N$ and $E_0^{\gamma} = 1$.

In particular cases, if

(1). $\tau = 1$, then $|c,\theta|_\tau$ reduces to absolute summability $|c,\theta|$.

(2). $\tau = 1$ and $\theta = 1$, then $|c,\theta|_\tau$ reduces to absolute summability $|c,1|$.

* *E-mail: sanjaymsupdr@yahoo.com*

2. Main Result

Theorem 2.1 ([2, Theorem 2]). *If $f \in Lip\alpha$ at a point $x_0 \in (-\pi, \pi)$ and if (n_k) satisfies*

$$(n_{k+1} - n_k) > A n_k^{\beta} \quad (0 < \beta < 1)$$

then the Fourier series (1) of f is absolute summable $(c, 1)$ when $\alpha > \beta^{-1} - 2$.

We will prove the following results for more general absolute summability under the same hypotheses. In fact, our result is as follow:

Theorem 2.2. *If $f \in Lip\alpha$ at a point $x_0 \in (-\pi, \pi)$ and if (n_k) satisfies*

$$(n_{k+1} - n_k) > A n_k^{\beta} \quad (0 < \beta < 1),$$

then the Fourier series (1) of f is summable $|c, 1|_{\tau}$; $\tau \geq 1$ when $\alpha > \frac{2-\beta-\tau-\tau\beta}{\tau\beta}$.

Remark 2.3. *If we put $\tau = 1$ in Theorem 2.2, then $|c, 1|_{\tau}$ reduces to $|c, 1|$ summability and condition $\alpha > \frac{2-\beta-\tau-\tau\beta}{\tau\beta}$ reduces to $\alpha > \beta^{-1} - 2$ which is same as in the case of N.V.Patel [2] and hence we get Theorem 2.1.*

For the proof of the Theorem 2.2, we need the following lemma.

Lemma 2.4 ([2]). *If (n_k) satisfies $(n_{k+1} - n_k) > A n_k^{\beta} \quad (0 < \beta < 1)$ then $n_k \geq k^{\delta}$ for all $k \in N$.*

Proof of Theorem 2.2. As per the proof given in Theorem 2.1 due to [2], we have

$$\begin{aligned}\left|\sigma_{n_k}^{\theta} - \sigma_{n_{k-1}}^{\theta}\right| &= \frac{1}{n_k E_{n_k}^{\theta}} \left|\sum_{p=1}^{k} E_{n_k - n_p}^{\theta-1} n_p \left(a_{n_p} \cos n_p x + b_{n_p} \sin n_p x\right)\right| \\ &\leq \frac{1}{n_k E_{n_k}^{\theta}} \left\{\left|n_k \left(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x\right)\right|\right\} + \left|\sum_{p=1}^{k} E_{n_k - n_p}^{\theta-1} n^p \left(a_{n_p} \cos n_p x + b_{n_p} \sin n_p x\right)\right|\end{aligned}$$

Let $\theta = 1$. Then by Theorem 2.1, we have

$$\begin{aligned}\left|\sigma_{n_k}(x) - \sigma_{n_{k-1}}(x)\right| &= \frac{1}{n_k (n_k + 1)} \left|\sum_{p=1}^{k} n_p \left(a_{n_p} \cos n_p x + b_{n_p} \sin n_p x\right)\right|. \\ &= \mathrm{O}(1) \left\{\frac{k}{k^{\delta(\alpha\beta+1)}}\right\}. \\ &= \mathrm{O}(1) \left\{\frac{1}{k^{\alpha\beta\delta+\delta-1}}\right\}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}n_k^{\tau-1} \left|\sigma_{n_k}(x) - \sigma_{n_{k-1}}(x)\right|^{\tau} &= \mathrm{O}(1) \left(k^{-\delta(\tau-1)} \left\{\frac{1}{k^{\alpha\beta\delta+\delta-1}}\right\}^{\tau}\right). \\ &= \mathrm{O}(1) \frac{1}{k^{\tau(\alpha\beta\delta+\delta-1)+(\delta\tau-\delta)}}. \\ &= \mathrm{O}(1) \frac{1}{k^{\tau\alpha\beta\delta+\tau\delta-\tau+\delta\tau-\delta}}. \\ &= \mathrm{O}(1) \frac{1}{k^{\frac{\tau\alpha\beta+2\tau-\tau+\tau\beta-1}{1-\beta}}} \quad \text{for } \delta = \frac{1}{1-\beta}. \\ &= \mathrm{O}(1) \frac{1}{k^{\frac{\tau\alpha\beta+\tau+\tau\beta-1}{1-\beta}}}\end{aligned}$$

Thus we have

$$\sum_{k=1}^{\infty} n_k^{\tau-1} \left|\sigma_{n_k}(x) - \sigma_{n_{k-1}}(x)\right|^{\tau} = \mathrm{O}(1) \sum_{k=1}^{\infty} \left\{ \frac{1}{k^{\frac{\tau\alpha\beta+\tau+\tau\beta-1}{1-\beta}}} \right\}.$$

Now, $\sum\limits_{k=1}^{\infty} n_k^{\tau-1} \left|\sigma_{n_k}(x) - \sigma_{n_{k-1}}(x)\right|^{\tau} < \infty$, if

$$\frac{\tau\alpha\beta+\tau+\tau\beta-1}{1-\beta} > 1$$
$$\Rightarrow \tau\alpha\beta+\tau+\tau\beta-1 > 1-\beta$$
$$\Rightarrow \tau\beta\alpha > 1-\beta-\tau-\tau\beta+1$$
$$\Rightarrow \alpha > \frac{2-\beta-\tau-\tau\beta}{\tau\beta}$$

Hence

$$\sum_{k=1}^{\infty} n^{\tau-1} \left|\sigma_{n_k}(x) - \sigma^{\theta}_{n_{k-1}}(x)\right|^{\tau} < \infty.$$

which implies that the series (1) is $|c,1|_{\tau}$ summable and thus Theorem 2.2 is proved. □

References

[1] T.M.Flett, *Some remarks on strong summability*, Quart, J. Math. (Oxford), 10(1959), 115-139.

[2] N.V.Patel, *On the Absolute summability of a Lacunary Fourier series*, J.M.S.University of Baroda, XXIX(3)(1980), 49-56.