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Some Transformations and Identities for I-Function of Two Variables

Research Article

A.K. Ronghe[1] and Sachin Sharma[1*]

1 Department of Mathematics, S.S.L. Jain P.G. College, Vidisha (M.P.), India.

Abstract: In this paper, some transformations, Summation formulae and identities for I-function of two variables have been evaluated. Many new as well as known relations may be derived as particular cases, which are known by identities.

Keywords: Hypergeometric function, I-function of two variables, known integrals.

1. Introduction

The double Mellin Barnes type contour integral occurring in this paper will be referred to slightly modified form for I-function of two variables [6] will be defined and represented in the following manner:

$$I\left[z_1, z_2\right] \equiv I^{0,n_1:N_1}_{p_1,q_1:N_2}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\middle| \begin{matrix} (P_1):(P_2);(P_3) \\ (Q_1):(Q_2);(Q_3) \end{matrix}\right] \quad \text{Or}$$
$$\equiv I^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\middle| \begin{matrix} (a_j;\alpha_j,A_j;\xi_j)_{1,p_1}:\ (c_j,C_j;U_j)_{1,p_2};\ (e_j,E_j;P_j)_{1,p_3} \\ (b_j;\ \beta_j,B_j;\eta_j)_{1,q_1}:\ (d_j,D_j;V_j)_{1,q_2};\ (f_j,F_j;Q_j)_{1,q_3} \end{matrix}\right]$$
$$= \frac{1}{(2\pi i)^2}\int_{L_s}\int_{L_t}\phi(s,t)\theta_1(s)\theta_2(t)z_1^s z_2^t ds dt \tag{1}$$

Where $\phi(s,t)$, $\theta_1(s)$, $\theta_2(t)$ are given by

$$\phi(s,t) = \frac{\prod\limits_{j=1}^{n_1}\Gamma^{\xi_j}\left(1-\ a_j+\ \alpha_j s\ +A_j t\right)}{\prod\limits_{j=n_1+1}^{p_1}\Gamma^{\xi_j}\left(a_j-\ \alpha_j s\ -\ A_j t\right)\prod\limits_{j=1}^{q_1}\Gamma^{\eta_j}\left(1-\ b_j+\beta_j s\ +\ B_j t\right)}, \tag{2}$$

$$\theta_1(s) = \frac{\prod\limits_{j=1}^{n_2}\Gamma^{U_j}\left(1-c_j+C_j s\right)\prod\limits_{j=1}^{m_2}\Gamma^{V_j}\left(d_j-D_j s\right)}{\prod\limits_{j=n_2+1}^{p_2}\Gamma^{U_j}\left(c_j-C_j s\right)\prod\limits_{j=m_2+1}^{q_2}\Gamma^{V_j}\left(1-d_j+D_j s\right)}, \tag{3}$$

$$\theta_2(t) = \frac{\prod\limits_{j=1}^{n_3}\Gamma^{P_j}\left(1-e_j+E_j t\right)\prod\limits_{j=1}^{m_3}\Gamma^{Q_j}\left(f_j-F_j t\right)}{\prod\limits_{j=n_3+1}^{p_3}\Gamma^{P_j}\left(e_j-E_j t\right)\prod\limits_{j=m_3+1}^{q_3}\Gamma^{Q_j}\left(1-f_j+F_j t\right)}, \tag{4}$$

Also:

* E-mail: sachin_soe@yahoo.com

- $z_1 \neq 0, z_2 \neq 0$;
- $i = \sqrt{-1}$;
- an empty product is interpreted as unity;
- $\mathcal{L}_s$ and $\mathcal{L}_t$ are suitable contours of Mellin-Barnes type. Moreover, the contour $\mathcal{L}_s$ is in the complex s-plane and runs from $\sigma_1 - i\infty$ to $\sigma_1 + i\infty$, (σ_1 real) so that all the singularities of $\Gamma^{V_j}(d_j - D_j s)(j = 1, \ldots, m_2)$ lies to the right of $\mathcal{L}_s$ and all the singularities of $\Gamma^{U_j}(1 - c_j + C_j s)(j = 1, \ldots, n_2)$, $\Gamma^{\xi_j}(1 - a_j + \alpha_j s + A_j t)(j = 1, \ldots, n_1)$lie to the left of $\mathcal{L}_s$;
- The contour $\mathcal{L}_t$ is in the complex t-plane and runs from $\sigma_2 - i\infty$ to $\sigma_2 + i\infty$, (σ_2 real) so that all the singularities of $\Gamma^{Q_j}(f_j - F_j t)(j = 1, \ldots, m_3)$lie to the right of $\mathcal{L}_t$ and all the singularities of, $\Gamma^{P_j}(1 - e_j + E_j t)(j = 1, \ldots, n_3)$, $\Gamma^{\xi_j}(1 - a_j + \alpha_j s + A_j t)(j = 1, \ldots, n_1)$lie to the left of $\mathcal{L}_t$.

For the condition of existence and condition on the various parameters of I-function of two variables $I[Z_1, Z_2]$, we refer to [2, 5] in (1) and that follows, we use the following notations for the sake of brevity. $N_1 \equiv m_2, n_2 : m_3, n_3$; $N_2 \equiv p_2, q_2 : p_3, q_3$. And sets of parameters are

$$(P_1) \equiv (a_j, \alpha_j, A_j; \xi_j)_{1,p_1}, (P_2) \equiv (c_j, C_j; U_j)_{1,p_2}, (P_3) \equiv (e_j, E_j; P_j)_{1,p_3}$$

$$(Q_1) \equiv (b_j, \beta_j, B_j; \eta_j)_{1,q_1}, (Q_2) \equiv (d_j, D_j; V_j)_{1,q_2}, (Q_3) \equiv (f_j, F_j; Q_j)_{1,q_3}$$

Following the results of Braaksma [1] and we refer to Rathie [6, 11], it can easily be shown that the function defined in (1) is analytic function of Z_1 and Z_2 if $R < 0$ and $S < 0$. Where,

$$R = \sum_{j=1}^{p_1} \xi_j \alpha_j + \sum_{j=1}^{p_2} U_j C_j - \sum_{j=1}^{q_2} \eta_j \beta_j - \sum_{j=1}^{q_2} V_j D_j, \tag{5}$$

$$S = \sum_{j=1}^{p_1} \xi_j A_j + \sum_{j=1}^{p_3} P_j E_j - \sum_{j=1}^{q_1} \eta_j B_j - \sum_{j=1}^{q_3} Q_j F_j, \tag{6}$$

And the integral (1) is convergent if,

$$\Delta_1 > 0, \ \ \Delta_2 > 0, \ \ |\arg(z_1)| < \frac{1}{2}\Delta_1 \pi, \ \ |\arg(z_2)| < \frac{1}{2}\Delta_2 \pi$$

Where,

$$\Delta_1 = \left[\sum_{j=1}^{n_1} \xi_j \alpha_j - \sum_{j=n_1+1}^{p_1} \xi_j \alpha_j - \sum_{j=1}^{q_1} \eta_j \beta_j + \sum_{j=1}^{n_2} U_j C_j - \sum_{j=n_2+1}^{p_2} U_j C_j + \sum_{j=1}^{m_2} V_j D_j - \sum_{j=m_2+1}^{q_2} V_j D_j \right], \tag{7}$$

$$\Delta_2 = \left[\sum_{j=1}^{n_1} \xi_j A_j - \sum_{j=n_1+1}^{p_1} \xi_j A_j - \sum_{j=1}^{q_1} \eta_j B_j + \sum_{j=1}^{n_3} P_j E_j - \sum_{j=n_3+1}^{p_3} P_j E_j + \sum_{j=1}^{m_3} Q_j F_j - \sum_{j=m_3+1}^{q_3} Q_j F_j \right], \tag{8}$$

Integral (1) is convergent absolutely if,

$$\Delta_1 \geq 0, \ \ \Delta_2 \geq 0, \ \ |\arg(z_1)| = \frac{1}{2}\Delta_1 \pi, \ \ |\arg(z_2)| = \frac{1}{2}\Delta_2 \pi, \tag{9}$$

The Following known result will be utilized in this paper from Luke [7] we have

$${}_3F_2 \left[\begin{matrix} a, b, c+1 \\ d, c \end{matrix} : z \right] = {}_2F_1 \left[\begin{matrix} a, b \\ d \end{matrix} : z \right] + \frac{abz}{cd} {}_2F_1 \left[\begin{matrix} a+1, b+1 \\ d+1 \end{matrix} : z \right], \tag{10}$$

And from Rainville [8] we have

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{(-1)^n}{(\alpha)_n}, \tag{11}$$

2. Transformations

The transforms involving I-function of two variables to be established are:

[T$_1$]:

$$\sum_{r=0}^{\infty}\frac{z^r}{r!}I^{0,n_1+4;N_1}_{p_1+4,q_1+3;N_2}\left[\begin{array}{c|c} z_1 & (a-r;l_1,l_2;1),(b-r;m_1,m_2;1),(c;n_1,n_2;1),(c-r-1,n_1,n_2;1),P_1;P_2;P_3 \\ z_2 & Q_1;(d-r;k_1,k_2;1),(c-r;n_1,n_2;1),(c-1;n_1,n_2;1),Q_2,Q_3 \end{array}\right]$$
$$=\sum_{r=0}^{\infty}\frac{z^r}{r!}\,I^{0,n_1+2;N_1}_{p_1+2,q_1+1;N_2}\left[\begin{array}{c|c} z_1 & (a-r;l_1,l_2;1),(b-r;m_1,m_2;1),P_1;P_2;P_3 \\ z_2 & Q_1;(d-r;k_1,k_2;1),Q_2,Q_3 \end{array}\right]$$
$$+\sum_{r=0}^{\infty}\frac{z^{r+1}}{(r+1)!}\,I^{0,n_1+3;N_1}_{p_1+3,q_1+2;N_2}\left[\begin{array}{c|c} z_1 & (a-r-1;l_1,l_2;1),(b-r-1;m_1,m_2;1),(c;n_1,n_2;1),P_1,P_2,P_3 \\ z_2 & Q_1;(d-r-1;k_1,k_2;1),(c-1;n_1,n_2;1),Q_2,Q_3 \end{array}\right], \quad (12)$$

Provided:

$$[\Delta_1+(l_1+l_2)+(m_1+m_2)-(k_1+k_2)]>0,$$
$$[\Delta_2+(l_1+l_2)+(m_1+m_2)-(k_1+k_2)]>0,$$

$$|\arg(z_1)|<\frac{1}{2}\left[\Delta_1+(l_1+l_2)+(m_1+m_2)-(k_1+k_2)\right]\pi,$$
$$|\arg(z_1)|<\frac{1}{2}\left[\Delta_2+(l_1+l_2)+(m_1+m_2)-(k_1+k_2)\right]\pi,$$

Where Δ_1 and Δ_2 are mentioned in (7) and (8) respectively.

[T$_2$]:

$$\sum_{r=0}^{\infty}\frac{z^r}{r!}I^{0,n_1+2;N_1}_{p_1+4,q_1+3;N_2}\left[\begin{array}{c|c} z_1 & (c-r-1,n_1,n_2;1),(c;n_1,n_2;1),P_1,(a-r;l_1,l_2;1),(b-r;m_1,m_2;1),P_2,P_3 \\ z_2 & Q_1;(d-r;k_1,k_2;1),(c-r;n_1,n_2;1),(c-1;n_1,n_2;1),Q_2,Q_3 \end{array}\right]$$
$$=\sum_{r=0}^{\infty}\frac{z^r}{r!}\,I^{0,n_1+2;N_1}_{p_1+2,q_1+1;N_2}\left[\begin{array}{c|c} z_1 & P_1;(a-r;l_1,l_2;1),(b-r;m_1,m_2;1),P_2;P_3 \\ z_2 & Q_1;(d-r;k_1,k_2;1),Q_2,Q_3 \end{array}\right]$$
$$+\sum_{r=0}^{\infty}\frac{z^{r+1}}{(r+1)!}\,I^{0,n_1+1;N_1}_{p_1+3,q_1+2;N_2}\left[\begin{array}{c|c} z_1 & (c;n_1,n_2;1),P_1,(a-r-1;l_1,l_2;1),(b-r-1;m_1,m_2;1),P_2,P_3 \\ z_2 & Q_1;(d-r-1;k_1,k_2;1),(c-1;n_1,n_2;1),Q_2,Q_3 \end{array}\right], \quad (13)$$

Provided:

$$[\Delta_1-(l_1+l_2+m_1+m_2+k_1+k_2)]>0,$$
$$[\Delta_2-(l_1+l_2+m_1+m_2+k_1+k_2)]>0,$$

$$|\arg(z_1)|<\frac{1}{2}\left[\Delta_1-(l_1+l_2+m_1+m_2+k_1+k_2)\right]\pi,$$
$$|\arg(z_1)|<\frac{1}{2}\left[\Delta_2-(l_1+l_2+m_1+m_2+k_1+k_2)\right]\pi,$$

Where Δ_1 and Δ_2 are mentioned in (7) and (8) respectively.

[T$_3$]:

$$\sum_{r=0}^{\infty}\frac{\Gamma(a+r).z^r}{r!}\,I^{0,n_1+3;N_1}_{p_1+3,q_1+3;N_2}\left[\begin{array}{c|c} z_1 & (b-r;m_1,m_2;1),(c-r-1,n_1,n_2;1),(c;n_1,n_2;1),P_1,P_2,P_3 \\ z_2 & Q_1;(d-r;-k_1-k_2;1),(c-r;n_1,n_2;1),(c-1;n_1,n_2;1),Q_2,Q_3 \end{array}\right]$$
$$=\sum_{r=0}^{\infty}\frac{\Gamma(a+r).z^r}{r!}\;I^{0,n_1+1;N_1}_{p_1+1,q_1+1;N_2}\left[\begin{array}{c|c} z_1 & (b-r;m_1,m_2;1),P_1,P_2,P_3 \\ z_2 & Q_1,(d-r;k_1,k_2;1),Q_2,Q_3 \end{array}\right]$$
$$+\sum_{r=0}^{\infty}\frac{\Gamma(a+r+1).z^{r+1}}{r!}\;I^{0,n_1+2;N_1}_{p_1+2,q_1+2;N_2}\left[\begin{array}{c|c} z_1 & (b-r-1;m_1,m_2;1),(c;n_1,n_2;1),P_1,P_2,P_3 \\ z_2 & Q_1;(d-r-1;k_1,k_2;1),(c-1;n_1,n_2;1),Q_2,Q_3 \end{array}\right] \tag{14}$$

Provided:

$$[\Delta_1+m_1+m_2-k_1-k_2)]>0,$$
$$[\Delta_2+m_1+m_2-k_1-k_2)]>0,$$

$$|\arg(z_1)|<\frac{1}{2}\left[\Delta_1+m_1+m_2-k_1-k_2)\right]\pi,$$
$$|\arg(z_1)|<\frac{1}{2}\left[\Delta_2+m_1+m_2-k_1-k_2)\right]\pi,\quad |z|<1,$$

Where Δ_1 and Δ_2 are mentioned in (7) and (8) respectively.

[T$_4$]:

$$\sum_{r=0}^{\infty}\frac{z^r}{r!}\,I^{0,n_1;m_2,n_2+1;m_3,n_3+1}_{0,q_1+1;p_2+1,q_2;p_3+1,q_3}\left[\begin{array}{c|c} z_1 & P_1,(a-r,m_1;1),(b-r,m_2;1),P_2,P_3 \\ z_2 & Q_1,(d-r;m_1,m_2;1),Q_2,Q_3 \end{array}\right]$$
$$=\sum_{r=0}^{\infty}\frac{z^{r+1}}{r!}\;I^{0,n_1+1;m_2,n_2+1;m_3,n_3+1}_{p_1+1,q_1;p_2+1,q_2;p_3+1,q_3+2}\left[\begin{array}{c|c} z_1 & (a-r-1,m_1,m_2;1),P_1,(c;m_1;1),P_2,(c;m_2;1),P_3 \\ z_2 & Q_1,Q_2,Q_3,(d-r-1;m_1;1),(c-1,m_2;1), \end{array}\right]$$
$$+\sum_{r=0}^{\infty}\frac{z^{r+1}}{r!}\;I^{0,n_1+1;N_1}_{p_1+1,q_1;p_2+1,q_2;p_3+1,q_3+2}\left[\begin{array}{c|c} z_1 & (a-r-1,m_1,m_2;1),P_1,P_2,P_3 \\ z_2 & (c,m_2;1),Q_1,(d-r-1;m_1;1),Q_2,(c-1,m_2;1),Q_3 \end{array}\right], \tag{15}$$

$$\Delta_1>0,\quad \Delta_2>0;|\arg(z_1)|<\frac{1}{2}\Delta_1\pi,|\arg(z_2)|<\frac{1}{2}\Delta_2\pi,|z|<1$$
$$\Delta_1\geq 0,\quad \Delta_2\geq 0;|\arg(z_1)|=\frac{1}{2}\Delta_1\pi,|\arg(z_2)|=\frac{1}{2}\Delta_2\pi$$

Where Δ_1 and Δ_2 are mentioned in (7) and (8) respectively.

Proof. To establish (12) expressing the I-function of two variables on the left hand side as contour integral (1), we get,

$$\sum_{r=0}^{\infty}\frac{z^r}{r!}\frac{1}{(2\pi i)^2}\int_{L_s}\int_{L_t}\phi(s,t)\theta_1(s)\theta_2(t)z_1^s z_2^t\times$$

$$\frac{\Gamma(1-a+r+l_1s+l_2t).\Gamma(1-b+r+m_1s+m_2t).\Gamma(2-c+r+n_1s+n_2t).\Gamma(1-c+n_1s+n_2t)}{\Gamma(1-c+r+n_1s+n_2t).\Gamma(2-c+n_1s+n_2t).\Gamma(1-d+r+k_1s+k_2t)}dsdt$$

Now changing the order of summation and integration in view of [2] which is permissible under the conditions given in (12), we get,

$$\sum_{r=0}^{\infty} \frac{z^r}{r!} \frac{1}{(2\pi i)^2} \int_{L_s} \int_{L_t} \phi(s,t)\theta_1(s)\theta_2(t) \frac{\Gamma(1-a+r+l_1 s+l_2 t).\Gamma(1-b+r+m_1 s+m_2 t)}{\Gamma(1-d+k_1 s+k_2 t)} \times$$

$$ {}_3F_2 \left[\begin{array}{c} 1-a+l_1 s+l_2 t, 1-b+m_1 s+m_2 t, 2-c+n_1 s+n_2 t \\ 1-d+r+k_1 s+k_2 t, 1-c+n_1 s+n_2 t \end{array} : z \right] z_1^s.z_2^t ds dt$$

Now applying (10) expressing both the gauss hypergeometric function as series, changing the order of integration and summation and interchanging the result thus obtained in view of (10), we get the Right hand side of (12). Proceeding in similar way the transformations (13), (14) and (15) can also be established. □

3. Summations

In this section we derive some infinite summations form of transformation formulae discussed in Section 2.

In (12) and (13), putting $z = 1$, on both sides, and R.H.S. expressed as I-function of two variables as contour integral, changing the order of summation and integration and evaluating the summation inside the contour with the help of the Gauss theorem [3] and using (10), we get the following summations respectively:

[S$_1$]:

$$\sum_{r=0}^{\infty} \frac{1}{r!} I^{0,n_1+4;N_1}_{p_1+4,q_1+3;N_2} \left[\begin{array}{c|c} z_1 & (a-r;l_1,l_2;1),(b-r;m_1,m_2;1),(c;n_1,n_2;1),(c-r-1,n_1,n_2;1),P_1,P_2,P_3 \\ z_2 & Q_1;(d-r;k_1,k_2;1),(c-r;n_1,n_2;1),(c-1;n_1,n_2;1),Q_2,Q_3 \end{array} \right]$$
$$= I^{0,n_1+3;N_1}_{p_1+4,q_1+2;N_2} \left[\begin{array}{c|c} z_1 & (a;l_1,l_2;1),(b;m_1,m_2;1),(d-a-b+2;k_1-l_1-m_1,k_2-l_2-m_2;1),P_1,P_2,P_3 \\ z_2 & Q_1;(d-a+1;k_1-1,k_2-1;1),(d-b+1;k_1-m_1,k_2-m_2;1),(c-1;n_1,n_2;1),Q_2,Q_3 \end{array} \right]$$
$$+ I^{0,n_1+4;N_1}_{p_1+4,q_1+3;N_2} \left[\begin{array}{c|c} z_1 & (a-1;l_1,l_2;1),(b-1;m_1,m_2;1),(d-a-b+3;k_1-l_1,k_2-l_2;1),(c;n_1,n_2;1),P_1,P_2,P_3 \\ z_2 & Q_1;(d-a+1;k_1-1,k_2-1;1),(d-b+1;k_1-m_1,k_2-m_2;1),(c-1;n_1,n_2;1),Q_2,Q_3 \end{array} \right] \quad (16)$$

Provided:

$$[\Delta_1 + l_1 + m_1 - k_1 + l_2 + m_2 - k_2)] > 0,$$
$$[\Delta_2 + l_1 + m_1 - k_1 + l_2 + m_2 - k_2)] > 0,$$

$$|\arg(z_1)| < \frac{1}{2}\left[\Delta_1 + l_1 + m_1 - k_1 + l_2 + m_2 - k_2)\right]\pi,$$
$$|\arg(z_1)| < \frac{1}{2}\left[\Delta_2 + (l_1 + m_1 - k_1 + l_2 + m_2 - k_2)\right]\pi,$$

Where Δ_1 and Δ_2 are mentioned in (7) and (8) respectively.

[S$_2$]:

$$\sum_{r=0}^{\infty} \frac{1}{r!} I^{0,n_1+2;N_1}_{p_1+4,q_1+3;N_2} \left[\begin{array}{c|c} z_1 & (c-r-1,n_1,n_2;1),(c;n_1,n_2;1),P_1,(a-r;l_1,l_2;1),(b-r;m_1,m_2;1)P_2,P_3 \\ z_2 & Q_1;(d-r;k_1,k_2;1),(c-r;n_1,n_2;1),(c-1;n_1,n_2;1),Q_2,Q_3 \end{array} \right]$$
$$= I^{0,n_1+1;N_1}_{p_1+3,q_1+2;N_2} \left[\begin{array}{c|c} z_1 & (d-a-b+2;(k_1+k_2)-(l_1+l_2)-(m_1+m_2);1),P_1,(a;l_1+l_2;1),(b;m_1+m_2;1),P_2,P_3 \\ z_2 & Q_1;(d-a+1;k_1-l_1,k_2-l_2;1),(d-b+1;k_1-m_1,k_2-m_2;1),Q_2,Q_3 \end{array} \right]$$
$$+ I^{0,n_1+2;N_1}_{p_1+4,q_1+4;N_2} \left[\begin{array}{c|c} z_1 & (d-a-b+3;(k_1+k_2)-(l_1+l_2)-(m_1+m_2);1),P_1,(a-1;l_1+l_2;1),(b-1;m_1+m_2;1),P_2,P_3 \\ z_2 & Q_1;(d-a+1;(k_1+k_2)-(l_1+l_2);1),(d-b+a;(k_1+k_2)-(m_1+m_2);1),(c-1,n_1+n_2;1),Q_2,Q_3 \end{array} \right]$$
$$(17)$$

Provided:

$$\{\Delta_1 - (l_1 + l_2) - (m_1 + m_2) - (k_1 + k_2)\} > 0,$$
$$\{\Delta_2 - (l_1 + l_2) - (m_1 + m_2) - (k_1 + k_2)\} > 0$$

$$|\arg(z_1)| < \frac{1}{2}\{\Delta_1 - (l_1 + l_2) - (m_1 + m_2) - (k_1 + k_2)\}\pi,$$
$$|\arg(z_2)| < \frac{1}{2}\{\Delta_2 - (l_1 + l_2) - (m_1 + m_2) - (k_1 + k_2)\}\pi,$$

Where Δ_1 and Δ_2 are mentioned in (7) and (8) respectively.

[S$_3$]: In (17) putting $l_1 + l_2 = l = n$, $m_1 + m_2 = m = n$ and $k_1 + k_2 = k = 2n$, we observe

$$\sum_{r=0}^{\infty} \frac{1}{r!} I^{0,n_1+2;N_1}_{p_1+4,q_1+3;N_2} \left[\begin{array}{c|c} z_1 & (c-r-1,n;1),(c,n;1),P_1,(a-r;n;1),(b-r;n;1)P_2,P_3 \\ z_2 & Q_1;(d-r;2n;1),(c-r;n;1),(c-1;n;1),Q_2,Q_3 \end{array} \right]$$
$$= \Gamma(a+b-d)\, I^{0,n_1;N_1}_{p_1+2,q_1+2;N_2} \left[\begin{array}{c|c} z_1 & P_1,(a;n;1),(b;n;1),P_2,P_3 \\ z_2 & Q_1;(d-a+1;n;1),(d-b+1;n;1),Q_2,Q_3 \end{array} \right]$$
$$+ \Gamma(a+b-d-2)\, I^{0,n_1+1;N_1}_{p_1+3,q_1+3;N_2} \left[\begin{array}{c|c} z_1 & (c,n;1),P_1,(a-1;n;1),(b-1;n;1)P_2,P_3 \\ z_2 & Q_1;(d-a+1;n;1),(d-b+1;n;1),(c-b;n;1),Q_2,Q_3 \end{array} \right], \tag{18}$$

Provided:

$$(\Delta_1 - 4n) > 0,\ \ (\Delta_2 - 4n) > 0,\ \ |\arg(z_1)| < \frac{1}{2}(\Delta_1 - 4n)\pi,\ \ |\arg(z_2)| < \frac{1}{2}(\Delta_2 - 4n)\pi,$$

Where Δ_1 and Δ_2 are mentioned in (7) and (8) respectively.

[S$_4$]: In (14) setting $z = 1$ and applying similar method as in (16), we get,

$$\sum_{r=0}^{\infty} \frac{\Gamma(a+r)}{r!} I^{0,n_1+3;N_1}_{p_1+3,q_1+3;N_2} \left[\begin{array}{c|c} z_1 & (b-r;m_1,m_2;1),(c-r-1,n_1,n_2;1),(c;n_1,n_2;1),P_1,P_2,P_3 \\ z_2 & Q_1;(d-r;k_1,k_2;1),(c-r;n_1,n_2;1),(c-1;n_1,n_2;1),Q_2,Q_3 \end{array} \right]$$
$$= \Gamma(a)\, I^{0,n_1+2;N_1}_{p_1+2,q_1+2;N_2} \left[\begin{array}{c|c} z_1 & (b;m_1,m_2;1),(a-b+d+1;(k_1-m_1),(k_2-m_2);1),P_1,P_2,P_3 \\ z_2 & Q_1;(d+a;k_1,k_2;1),(d-b+1;(k_1-m_1),(k_2-m_2);1),Q_2,Q_3 \end{array} \right]$$
$$+ \frac{\Gamma(a+1)\Gamma(b-a-d-1)}{\Gamma(b-d)} I^{0,n_1+2;N_1}_{p_1+2,q_1+2;N_2} \left[\begin{array}{c|c} z_1 & (b-1;n_1,n_2;1),(c;n_1,n_2;1),P_1,P_2,P_3 \\ z_2 & Q_1;(d+a;n_1,n_2;1),(c-1,n_1,n_2;1),Q_2,Q_3 \end{array} \right], \tag{19}$$

Provided:

$$\Delta_1 > 0,\ \ \Delta_2 > 0,\ \ |\arg(z_1)| < \frac{1}{2}\Delta_1\pi,\ \ |\arg(z_2)| < \frac{1}{2}\Delta_2\pi,$$

Where Δ_1 and Δ_2 are mentioned in (7) and (8) respectively.

4. Particular Cases

In this section we obtain some interesting identities.

1. In (16) setting $a = c - d - 1$, $b = -1$, $m_1 = m_2 = 0$ on left hand side, expressing I-function as double contour integral, changing the order of summation and integration and using Dixio's theorem [5, p.92] we get,

$$I^{0,n_1+5;N_1}_{p_1+5,q_1+5;N_2}\left[\begin{array}{c|l} & (d;n_1-l_1,n_2-l_2;1),\left(d-\frac{c}{2}+3;\frac{n_1}{2}-l_1,\frac{n_2}{2}-l_2;1\right),(c-d-1;l_1,l_2;1),\left(\frac{c}{2}-1;\frac{n_1}{2},\frac{n_2}{2};1\right),\\ z_1 & (c;n_1,n_2;1)P_1,P_2,P_3\\ z_2 & Q_1;(d;n_1-1,n_2-1;1),(d+2;n_1-l_1,n_2-l_2;1),\left(d-\frac{c}{2}+1;\frac{n_1}{2}-l_1,\frac{n_2}{2}-l_2;1\right)(c-2;n_1,n_2;1),\\ & \left(\frac{c}{2}+1;\frac{n_1}{2},\frac{n_2}{2};1\right),Q_2,Q_3\end{array}\right]$$

$$= I^{0,n_1+3;N_1}_{p_1+2,q_1+2;N_2}\left[\begin{array}{c|c} z_1 & (2d-c+4;n_1-2l_1,n_2-2l_2;1),(c-d-1;l_1,l_2;1),P_1,P_2,P_3\\ z_2 & Q_1;(d+2;n_1-l_1;n_2-l_2;1),(2d-c+2;n_1-2l_1,n_2-2l_2;1),Q_2,Q_3\end{array}\right]$$

$$+\,2\,I^{0,n_1+3;N_1}_{p_1+3,q_1+3;N_2}\left[\begin{array}{c|c} z_1 & (2d-c+5;n_1-2l_1,n_2-2l_2;1),(c;n_1,n_2;1),(c-d-2;l_1,l_2;1),P_1,P_2,P_3\\ z_2 & Q_1;(d+2;n_1-l_1;n_2-l_2;1),(2d-c+2;n_1-2l_1,n_2-2l_2;1),(c-1;n_1,n_2;1),Q_2,Q_3\end{array}\right] \tag{20}$$

Provided:

$$[\Delta_1 - n_1 + 2l_1 - n_2 + 2l_2] > 0,$$
$$[\Delta_2 - n_1 + 2l_1 - n_2 + 2l_2] > 0,$$

$$|\arg(z_1)| < \frac{1}{2}\left[\Delta_1 - n_1 + 2l_1 - n_2 + 2l_2\right]\pi,$$
$$|\arg(z_2)| < \frac{1}{2}\left[\Delta_2 - n_1 + 2l_1 - n_2 + 2l_2\right]\pi,$$

Where Δ_1 and Δ_2 are mentioned in (7) and (8) respectively.

2. In (17) setting $a = 2c - 1$, $b = 2c - d - 1$, $k_1 = l_1 = k_2 = l_2 = 2n$, $m_1 = m_2 = 0$ and proceeding as above, we get,

$$\Gamma(2c-d-2)\,I^{0,n_1+2;N_1}_{p_1+3,q_1+3;N_2}\left[\begin{array}{c|c} z_1 & (c-1;n_1,n_2;1),(c;n_1,n_2;1),P_1,(2c-1;n_1,n_2;1),P_2,P_3\\ z_2 & Q_1;(2c-2;n_1,n_2;1),(d-c+1;n;1),(d-c;n_1;n_2;1),Q_2,Q_3\end{array}\right]$$

$$= \Gamma(4c-2d-3)\,I^{0,n_1;N_1}_{p_1+1,q_1;N_2}\left[\begin{array}{c|c} z_1 & P_1,(2c-1;n_1,n_2;1),P_2,P_3\\ z_2 & Q_1;(2d-2c+2;n_1;n_2;1),Q_2,Q_3\end{array}\right]$$

$$+\,\frac{\Gamma(4c-2d-4)}{\Gamma(2c-d-2)}\,I^{0,n_1+1;N_1}_{p_1+2,q_1+2;N_2}\left[\begin{array}{c|c} z_1 & (c;n_1,n_2;1),P_1,(2c-2;n_1,n_2;1),P_2,P_3\\ z_2 & Q_1;(2d-2c+2;2n;1),(c-1;n_1,n_2;1),Q_2,Q_3\end{array}\right] \tag{21}$$

Provided:

$$(\Delta_1 - 4n) > 0,\ (\Delta_2 - 4n) > 0,\ |\arg(z_1)| < \frac{1}{2}(\Delta_1 - 4n)\pi,\ |\arg(z_2)| < \frac{1}{2}(\Delta_2 - 4n)\pi,$$

Where Δ_1 and Δ_2 are mentioned in (7) and (8) respectively.

3. In (19) setting $a = d - c + 2$, $b = -1$, $k_1 = k_2 = n_1 = n_2 = n$, $m = 0$ and proceeding as above, we get,

$$I^{0,n_1+3;N_1}_{p_1+3,q_1+4;N_2}\left[\begin{array}{c|c} z_1 & \left(d-\frac{c}{2}+3;\frac{n}{2},\frac{n}{2};1\right),\left(\frac{c}{2}-1\frac{n}{2},\frac{n}{2};1\right),(c,n,n;1),P_1,P_2,P_3\\ z_2 & Q_1;(d;2n,2n;1),\left(d-\frac{c}{2};\frac{n}{2},\frac{n}{2};1\right),(c-2;n,n;1),\left(\frac{c}{2}+1;\frac{n}{2},\frac{n}{2};1\right),Q_2,Q_3\end{array}\right]$$

$$= I^{0,n_1+1;N_1}_{p_1+1,q_1+2;N_2}\left[\begin{array}{c|c} z_1 & (2d-c;4n,4n;1),P_1,P_2,P_3\\ z_2 & Q_1;(d+2;n,n;1),(2d-c+2;n,n;1),Q_2,Q_3\end{array}\right]$$

$$+\,2(d-c+2)\,I^{0,n_1+2;N_1}_{p_1+2,q_1+3;N_2}\left[\begin{array}{c|c} z_1 & (2d-c+5;n,n;1),(c;n,n;1),P_1,P_2,P_3\\ z_2 & Q_1;(d+2;n,n;1),(2d-c;3n,3n;1),(c-1;n,n;1),Q_2,Q_3\end{array}\right] \tag{22}$$

Provided:

$$(\Delta_1 - n) > 0,\ (\Delta_2 - n) > 0,\ |\arg(z_1)| < \frac{1}{2}(\Delta_1 - n)\pi,\ |\arg(z_2)| < \frac{1}{2}(\Delta_2 - n)\pi,$$

Where Δ_1 and Δ_2 are mentioned in (7) and (8) respectively.

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