A New Iterative Method for Solving Nonlinear Equations Using Simpson Method

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Abstract: In this paper, we introduced the new iterative method for solving nonlinear equations. This method based on the combination of midpoint with Simpson quadrature formulas and using Newton’s method. The convergence analysis of our method is discussed. It is established that the new method has convergence order three. Numerical tests shows that the new method is comparable with the well-known existing single step iterative methods and provides better results.

Keywords: Nonlinear equations, iterative methods, Simpson method, Newton’s method, convergence.

1. Introduction

Numerical analysis is the area of mathematics and computer sciences that creates, analyzes and numerically the problems of continuous mathematics. Such problems originate generally from real-world applications of algebra, geometry and calculus and they involves variable which very continuously: these problems occur throughout the natural sciences, social sciences, engineering, medicine and business. A new iterative method for finding the approximate solutions of the nonlinear equation \( f(x) = 0 \). This numerical method has been constructed using different techniques such as Taylor series, homotopy, quadrature formula and decomposition method for more details see [1–3, 6–14]. This paper based on a combination of midpoint with Simpson quadrature formulas. There are several different methods in the literature for computation of the root of the nonlinear equation. The most famous of these methods is the classical Newton’s method (NM) [4, 5].

\[
    x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}
\]

Therefore the Newton’s method was modified by Steffensen’s method [4]

\[
    x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)}
\]

Newton’s method and Steffensen’s method are of second order converges [4, 5].

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2. Iterative Methods

Consider the nonlinear equations of the type

\[ f(x) = 0 \]  \hspace{1cm} (2)

For simplicity, we assume that \( \alpha \) is a simple root of (2) and \( \gamma \) is an initial guess sufficiently close to \( \alpha \). Now using the simpson quadrature formulas [4, 5] and fundamental theorem of calculus, one can show that the function \( f(x) \) can be approximated by the series [4].

\[ f(x) = f(\gamma) + \left( \frac{x-\gamma}{6} \right) \left[ f'(x) + 4f' \left( \frac{x+\gamma}{2} \right) + f'(\gamma) \right] \]  \hspace{1cm} (3)

Where \( f'(x) \) is the differential of. From equation (2) and (3), we have

\[ x = \gamma - \frac{6f(\gamma)}{f'(x)} \]  \hspace{1cm} (4)

Using (4), one can suggest the following iterative method for solving the nonlinear equation (2).

**Theorem 2.1.** For a given initial choice \( x_0 \), find the approximate solution \( x_{n+1} \) by the iterative schemes.

\[ x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n)} \]  \hspace{1cm} (5)

**Theorem 2.2.** For a given initial choice \( x_0 \), find the approximate solution \( x_{n+1} \) by the iterative schemes. From equation (1) and (5)

\[ x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n) - \frac{f(x_n)}{f'(x_n)}} \]  \hspace{1cm} (6)

Theorem 2.2 is called the New Iterative method (NIM) and has third order convergence.

3. Convergence Analysis

Let us now discuss the convergence analysis of the above Theorem 2.2.

**Theorem 3.1.** Let \( \alpha \in I \) be a simple zero of sufficiently differential function \( f: I \subset R \to R \) for an open interval \( I \), if \( x_0 \) is sufficiently close to \( \alpha \) then the New Iterative Method (NIM) defined by Theorem 2.2 third order convergence.

**Proof.** Let \( \alpha \) be a simple zero of \( f \). Than by expanding \( f(x_n) \) and \( f'(x_n) \) about \( \alpha \) we have

\[ f(x_n) = e_n c_1 + e_n^2 c_2 + e_n^3 c_3 + \ldots \]  \hspace{1cm} (6)

\[ f'(x_n) = c_1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + \ldots \]  \hspace{1cm} (7)

Where \( c_k = \frac{1}{k!} f^{(k)}(\alpha), \ k = 1, 2, 3, \ldots \) and \( e_n = x_n - \alpha \). From equation (6) and equation (7), we have

\[ f(x_n) = e_n - c_1 e_n^2 - e_n^3 c_3 + \ldots \]  \hspace{1cm} (8)

From equation (8), we have

\[ f'(x_n) = c_1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + \ldots \]  \hspace{1cm} (9)
From equation (6) and equation (7), we have

\[
\frac{1}{2} f'(x_n) = \frac{c_n}{2} - \frac{c_2}{2c_1} e_n^2 - \left( \frac{c_1}{c_1} + \frac{c_2}{c_1} \right) e_n^3 - \ldots \tag{10}
\]

From equation (10), we have

\[
4f'\left(x_n - \frac{1}{2} f'(x_n)\right) = 4c_1 + 4c_2 e_n + 3c_3 e_n^2 + \ldots \tag{11}
\]

From equation (7), equation (9) and equation (11), we have

\[
f' \left( x_n - f(x_n) \right) + 4f' \left( x_n - \frac{1}{2} f'(x_n) \right) + f'(x_n) = 6c_1 + 6c_2 e_n + 6 \left( c_3 + \frac{c_2}{c_1} \right) e_n^2 + \ldots \tag{12}
\]

From equation (6) and equation (12), we have

\[
\frac{6f(x_n)}{f' \left( x_n - \frac{1}{2} f'(x_n) \right) + 4f' \left( x_n - \frac{1}{2} f'(x_n) \right) + f'(x_n) } = e_n - \frac{2c_2}{c_1} e_n^3 - \left( \frac{2c_3}{c_1} + \frac{c_2}{c_1} \right) e_n^4 - \ldots \tag{13}
\]

\[
x_{n+1} = x_n + \frac{2c_2}{c_1} e_n^3 + \left( \frac{2c_3}{c_1} + \frac{c_2}{c_1} \right) e_n^4 + \ldots
\]

\[
e_{n+1} = \frac{2c_2}{c_1} e_n^3 + \left( \frac{2c_3}{c_1} + \frac{c_2}{c_1} \right) e_n^4 + \ldots
\]

This shows that Theorem 2.2 is third order convergence.

4. Numerical Results

We present some example to illustrate the root of the newly developed new iterative method, see Table 1. We compare the Newton method (NM) and Steffensen’s method (SM). All computations are performed using MATLAB. The following examples are used for numerical testing.

\[
f_1(x) = \cos(x) - x
\]

\[
f_2(x) = \sin(x) - 1 + x
\]

\[
f_3(x) = e^x - 3x
\]

\[
f_4(x) = x \tan(x) + 1
\]

\[
f_5(x) = 2 \sin(x) - x
\]

\[
f_6(x) = x + \sin(x) - x^3
\]

\[
f_7(x) = \cos(x) - xe^x
\]

\[
f_8(x) = x^2 - 9
\]

\[
f_9(x) = e^x - 1.5 - \tan^{-1}(x)
\]

\[
f_{10}(x) = x^3 - 6x + 4
\]

As for the convergence criteria, it was required that the distance of two consecutive approximations \( \delta \) and also displayed is the number of iterations to approximate the zero (IT), the approximate zero \( x_n \) and the value \( f(x_n) \).
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5. Conclusion

With the comparative study of newly developed techniques (NIM) is faster than Newton’s and Steffensen’s method. Our method can be considered as significant improvement of Newton’s and Steffensen’s method and can be considered as alternative method of solving nonlinear equations.

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Table 1. Numerical Examples and Comparison
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