Direct Product of General Doubt Intuitionistic Fuzzy Ideals of $BCK/BCI$-algebras with Respect to Triangular Binorm

A. K. Dutta$^1$, D. K. Basnet$^2$, K. D. Choudhury$^3$ and S. R. Barbhuiya$^4$*

$^1$ Department of Mathematics, D.H.S.K. College, Dibrugarh, Assam, India
$^2$ Department of Mathematical Sciences, Tezpur University, Tezpur, Assam, India.
$^3$ Department of Mathematics, Assam University, Silchar, Assam, India.
$^4$ Department of Mathematics, Srikishan Sarda College, Hailakandi, Assam, India.

Abstract: In this paper, we introduced the concept of $(\in, \in \vee q_k)$-doubt intuitionistic fuzzy subalgebra and $(\in, \in \vee q_k)$-doubt intuitionistic fuzzy ideals in $BCK$-algebra with respect to triangular binorm by using the combined notion of not quasi coincidence ($q$) of a fuzzy point to a fuzzy set and the notion of triangular binorm. We define direct product of $(\in, \in \vee q_k)$-doubt intuitionistic fuzzy sets and direct product of $(\in, \in \vee q_k)$-doubt intuitionistic fuzzy subalgebras/ideals of $BCK/BCI$-algebras and investigate some related properties.

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1. Introduction

The name triangular norm, or simply t-norm originated from the study of generalized triangle inequalities for statistical metric spaces, hence the name triangular norm or simply t-norm. The name first appeared in a paper entitled statistical metrics [19] that was published on 27th October in 1942. The real starting point of t-norms came in 1960, when Berthold Schweizer and Abe Sklar, (two students of Menger) published their paper, statistical metric spaces [25] After a very short time, Schweizer and Sklar [27] introduced several basic notions and properties. Namely, they introduced triangular conorms (briefly t-conorms) as a dual concept of t-norms. For a given t-norm T, its dual t-conorm S is defined by $S(a, b) = 1 - T(1 - a, 1 - b)$.

They pointed out that the boundary condition is the only difference between the t-norm and t-conorm axioms. In recent years, a systematic study concerning the properties and related matters of t-norms have been made by Klement et al. [15, 16].

The concept of fuzzy sets was first proposed by Zadeh [32] in 1965. Rosenfeld [24] was the first who consider the case of a groupoid in terms of fuzzy sets. Since then these ideas have been applied to other algebraic structures such as group, semigroup, ring, field, topology, vector spaces etc. Imai and Iseki [12] introduced $BCK$-algebra as a generalization of notion

* E-mail: saidurbabhuiya@mail.com
of the concept of set theoretic difference and propositional calculus and in the same year Iseki [14] introduced the notion of BCI-algebra which is a generalization of BCK-algebra. Xi Ougen [29] applied the concept of fuzzy set to BCK-algebra and discussed some properties. Since then B-algebras was introduced in [23] by Neggers and Kim and which is related to several classes of algebras such as BCI/BCI-algebras. Huang [11] fuzzified BCI-algebras in little different ways. Jun et al. [10, 31] renamed Huang’s definition as doubt (anti) fuzzy ideals in BCK/BCI-algebras. Biswas [8] introduced the concept of anti fuzzy subgroup. The concept of doubt fuzzy BF-algebras was introduced by Saeid in [28] and the concept of doubt fuzzy ideal of BF-algebras was introduced by Barbhuiya [4].

The concept of fuzzy point introduced by Ming and Ming in [20] and also they introduced the idea of relation “belongs to” and “quasi coincident” between fuzzy point and fuzzy set. Murali [21] proposed a definition of a fuzzy point belonging to fuzzy subset under natural equivalence on fuzzy subset. Bhakat and Das [6, 7] used the relation of “belongs to” and “quasi-coincident” between fuzzy point and fuzzy set to introduced the concept of ($\in$, $\in \lor q$)-fuzzy subgroup, ($\in$, $\in \lor q$)-fuzzy subring and ($\in \lor q$)-level subset. some properties of ($\in$, $\in \lor q$)-fuzzy ideals of d-algebra was discussed by Barbhuiya and Choudhury [3]. In [5] Barbhuiya introduced ($\in$, $\in \lor q$)-intuitionistic fuzzy ideals of BCK/BCI-algebras. In fact, the ($\in$, $\in \lor q$)-fuzzy subgroup is an important generalization of Rosenfeld’s fuzzy subgroup. Further in [18] Larimi generalized ($\in$, $\in \lor q$)-fuzzy ideals to ($\in$, $\in \lor q_k$)-fuzzy ideals. Reza Ameri et al [2] introduced the notion of ($\in$, $\in \lor q_k$)-fuzzy subalgebras in BCK/BCI-algebras. In [9] Dutta et al. combined the notion of not quasi coincidence ($\overline{q}$) of a fuzzy point to a fuzzy set and the notion doubt(anti) fuzzy ideals introduced the concept of generalized doubt fuzzy subalgebra and generalized doubt fuzzy ideal in BG-algebra. In this paper, we introduced the concept of ($\in$, $\in \lor q_k$)-doubt intuitionistic fuzzy subalgebra and ($\in$, $\in \lor q_k$)-doubt intuitionistic fuzzy ideals in BCK-algebra with respect to triangular binorm by using the combined notion of not quasi coincidence ($\overline{q}$) of a fuzzy point to a fuzzy set and the notion of triangular binorm. We define direct product of ($\in$, $\in \lor q_k$)-doubt intuitionistic fuzzy sets and direct product of ($\in$, $\in \lor q_k$)-doubt intuitionistic fuzzy subalgebras/ideals of BCK/BCI-algebras and investigate some related properties.

1.1. Preliminaries

Definition 1.1 ([29–31]). An algebra $(X, \ast, 0)$ of type $(2, 0)$ is called a BCK-algebra if it satisfies the following axioms:

(1). $((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0$;

(2). $(x \ast (x \ast y)) \ast y = 0$;

(3). $x \ast x = 0$;

(4). $0 \ast x = 0$;

(5). $x \ast y = 0$ and $y \ast x = 0 \Rightarrow x = y$ for all $x, y, z \in X$.

We can define a partial ordering “$\leq$” on $X$ by $x \leq y$ iff $x \ast y = 0$.

Definition 1.2 ([29–31]). A BCK-algebra $X$ is said to be commutative if it satisfies the identity $x \land y = y \land x$ where $x \land y = y \ast (y \ast x) \forall x, y \in X$. In a commutative BCK-algebra, it is known that $x \land y$ is the greatest lower bound of $x$ and $y$.

In a BCK-algebra $X$, the following hold:

(1). $x \ast 0 = x$;

(2). $(x \ast y) \ast z = (x \ast z) \ast y$;

(3). $x \ast y \leq x$;
(4). \((x \ast y) \ast z \leq (x \ast z) \ast (y \ast z)\);

(5). \(x \leq y\) implies \(x \ast z \leq y \ast z\) and \(z \ast y \leq z \ast x\).

A non-empty subset \(S\) of a \(BG\)-algebra \(X\) is called a subalgebra of \(X\) if \(x \ast y \in S\) for all \(x, y \in S\). A nonempty subset \(I\) of a \(BCK\)-algebra \(X\) is called an ideal of \(X\) if (i) \(0 \in I\) and (ii) \(x \ast y \in I\) and \(y \in I \Rightarrow x \in I\) for all \(x, y \in X\).

**Definition 1.3** ([6, 20]). A fuzzy set \(\mu\) of the form

\[
\mu(y) = \begin{cases} 
  t & \text{if } y = x, \ t \in (0, 1] \\
  0 & \text{if } y \neq x
\end{cases}
\]

is called a fuzzy point with support \(x\) and value \(t\) and it is denoted by \(x_t\) [6, 20]. Let \(\mu\) be a fuzzy set in \(X\) and \(x_t\) be a fuzzy point then

(1). If \(\mu(x) \geq t\) then we say \(x_t\) belongs to \(\mu\) and write \(x_t \in \mu\)

(2). If \(\mu(x) + t > 1\) then we say \(x_t\) quasi-coincident with \(\mu\) and write \(x_t q \mu\)

(3). If \(x_t \in \lor q \mu \Leftrightarrow x_t \in \mu\) or \(x_t q \mu\)

(4). If \(x_t \in \land q \mu \Leftrightarrow x_t \in \mu\) and \(x_t q \mu\)

The symbol \(x_t q \mu\) means \(x_t\) is not \(q\) quasi-coincident with \(\mu\) does not hold and \(\overline{x_t q \mu}\) means \(\overline{x_t} \lor q\). For a fuzzy point \(x_t\) and a fuzzy set \(\mu\) in set \(X\), Pu and Liu [20] gave meaning to the symbol \(x_t q \alpha\mu\) where \(\alpha \in \{\lor, \land, \lor q, \land q\}\).

**Definition 1.4** ([2, 18]). Let \(\mu\) be a fuzzy set in \(X\) and \(x_t\) be a fuzzy point then

(1). If \(\mu(x) < t\) then we say \(x_t\) does not belongs to \(\mu\) and write \(x_t \overline{\mu}\).

(2). If \(\mu(x) + t \leq 1\) then we say \(x_t\) not quasi-coincident with \(\mu\) and write \(x_t \overline{q} \mu\).

(3). If \(x_t \overline{\lor q} \mu \Leftrightarrow x_t \overline{\mu}\) and \(x_t \overline{q} \mu\).

(4). If \(x_t \overline{\land q} \mu \Leftrightarrow x_t \overline{\mu}\) or \(x_t \overline{q} \mu\).

**Definition 1.5** ([2, 18]). Let \(\mu\) be a fuzzy set in \(X\) and \(x_t\) be a fuzzy point then

(1). If \(\mu(x) + t + k > 1\) then we say \(x_t\) is \(k\) quasi-coincident with \(\mu\) and write \(x_t q_k \mu\) where \(k \in [0, 1]\).

(2). If \(x_t \in \lor q_k \mu \Leftrightarrow x_t \in \mu\) or \(x_t q_k \mu\).

(3). If \(x_t \in \land q_k \mu \Leftrightarrow x_t \in \mu\) and \(x_t q_k \mu\).

**Definition 1.6** ([2, 18]). Let \(\mu\) be a fuzzy set in \(X\) and \(x_t\) be a fuzzy point then

(1). If \(\mu(x) + t + k \leq 1\) then we say \(x_t\) is not \(k\) quasi-coincident with \(\mu\) and write \(x_t \overline{q_k} \mu\) where \(k \in [0, 1]\).

(2). If \(x_t \overline{\lor q_k} \mu \Leftrightarrow x_t \overline{\mu}\) and \(x_t \overline{q_k} \mu\).

(3). If \(x_t \overline{\land q_k} \mu \Leftrightarrow x_t \overline{\mu}\) or \(x_t \overline{q_k} \mu\).

**Definition 1.7** ([30]). A fuzzy set \(\mu\) of a \(BG\)-algebra \(X\) is said to be \((\alpha, \beta)\)-fuzzy ideal of \(X\) if
(1). $x_1 \alpha \mu \Rightarrow 0, \beta \mu$ for all $x \in X$.

(2). $(x \ast y)_1, \gamma \alpha \mu \Rightarrow x_{m(t, s)} \beta \mu$ for all $x, y \in X$ Where $\alpha \in \land q, m \{t, s\} = \min \{t, s\}$ and $t, s \in (0, 1]$.

Definition 1.8 ([9]). A fuzzy subset $\mu$ of a $BG$-algebra $X$ is an $(\varepsilon, \in, \lor q)$-doubt fuzzy subalgebra of $X$ if

$$\mu(x \ast y) \leq \max \left\{ \mu(x), \mu(y), \frac{1 - k}{2} \right\} \text{ for all } x, y \in X.$$  

Remark 1.9. A fuzzy subset $\mu$ of a $BG$-algebra $X$ is an $(\varepsilon, \in, \lor q)$-doubt fuzzy subalgebra of $X$ iff

$$\mu(x \ast y) \leq M(\mu(x), \mu(y), 0.5)$$

Definition 1.10 ([9]). A fuzzy subset $\mu$ of a $BG$-algebra $X$ is an $(\varepsilon, \in, \lor q)$-doubt fuzzy ideal of $X$ if

(1). $\mu(0) \leq \max \{\mu(x), \frac{1 - k}{2} \}$ for all $x \in X$.

(2). $\mu(x) \leq \max \{\mu(x \ast y), \mu(y), \frac{1 - k}{2} \}$ for all $x, y \in X$.

Remark 1.11. A fuzzy subset $\mu$ of a $BG$-algebra $X$ is an $(\varepsilon, \in, \lor q)$-doubt fuzzy ideal of $X$ iff

$$\mu(0) \leq M(\mu(x), 0.5)$$

$$\mu(x) \leq M(\mu(x \ast y), \mu(y), 0.5)$$

Definition 1.12 ([1]). An intuitionistic fuzzy set (IFS) $A$ in a non-empty set $X$ is an object of the form $A = \{(x, \mu_A(x), \nu_A(x))| x \in X\}$ where $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ denote respectively the degree of membership and the degree of non membership of the element $x$ in the set $A$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \nu_A(x))| x \in X\}$.

Definition 1.13. An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \nu_A(x))| x \in X\}$ of a BCK-algebra $X$

$$x_{\alpha, \beta}(y) = \begin{cases} (\alpha, \beta) & \text{if } y = x, \\ (0, 1) & \text{if } y \neq x \end{cases}$$

is said to be an intuitionistic fuzzy point with support $x$ and value $(\alpha, \beta)$ and is denoted by $x_{(\alpha, \beta)}$. A fuzzy point $x_{(\alpha, \beta)}$ is said to intuitionistic belongs to (resp., intuitionistic quasi-coincident) with intuitionistic fuzzy set $A = \{(x, \mu_A(x), \nu_A(x))| x \in X\}$ written $x_{(\alpha, \beta)} \in A$ resp: $x_{(\alpha, \beta)}qA$ if $\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$ (resp.$\mu_A(x) + \alpha > 1$ and $\nu_A(x) + \beta < 1$). By the symbol $x_{(\alpha, \beta)}qkA$ we mean $\mu_A(x) + \alpha + k > 1$ and $\nu_A(x) + \beta + k < 1$, where $k \in (0, 1)$.

We use the symbol $x_t \in \mu_A$ implies $\mu_A(x) \geq t$ and $x_t \in \nu_A$ implies $\nu_A(x) \leq t$ in the whole paper.

Definition 1.14 ([1, 5]). If $A = \{(x, \mu_A(x), \nu_A(x))| x \in X\}$ and $B = \{(x, \mu_B(x), \nu_B(x))| x \in X\}$ be any two IFS of a set $X$ then: $A \subseteq B$ iff for all $x \in X, \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$; $A = B$ iff for all $x \in X, \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$; $A \cap B = \{(x, \mu_A(x), \nu_A(x))| x \in X\}$, where $(\mu_A \cap \mu_B)(x) = \min(\mu_A(x), \mu_B(x))$ and $(\nu_A \cap \nu_B)(x) = \max(\nu_A(x), \nu_B(x))$: $A \cup B = \{(x, \mu_A(x), \nu_A(x))| x \in X\}$, where $(\mu_A \cup \mu_B)(x) = \max(\mu_A(x), \mu_B(x))$ and $(\nu_A \cup \nu_B)(x) = \min(\nu_A(x), \nu_B(x))$.

An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of a BCK-algebra $X$ is said to be an intuitionistic fuzzy ideal of $X$ if
(1) $\mu_{A}(0) \geq \mu_{A}(x)$

(2) $\nu_{A}(0) \leq \nu_{A}(x)$

(3) $\mu_{A}(x) \geq \min\{\mu_{A}(x+y), \mu_{A}(y)\}$

(4) $\nu_{A}(x) \leq \max\{\nu_{A}(x+y), \nu_{A}(y)\} \ \forall \ x, y \in X.$

**Definition 1.15.** A triangular norm (t-norm) is a function $T : [0 1] \times [0 1] \rightarrow [0 1]$ satisfying the following conditions:

(T1) $T(x, 1) = x, T(0, x) = 0$ ; (boundary conditions)

(T2) $T(y, x) = T(x, y)$ ; (commutativity)

(T3) $T(x, T(y, z)) = T(T(x, y), z)$ ; (associativity)

(T4) $T(x, y) \leq T(z, w)$ ;if $x \leq z, y \leq w$ for all $x, y, z \in [0 1]$ (monotonicity)

Every t-norm $T$ satisfies $T(x, y) \leq \min(x, y)$ $\forall x, y \in [0, 1]$.

**Example 1.16.** The four basic t-norms are:

(1). The minimum is given by $T_{M}(x, y) = \min(x, y)$.

(2). The product is given by $T_{P}(x, y) = xy$.

(3). The Lukasiewicz is given by $T_{L}(x, y) = \max(x + y - 1, 0)$.

(4). The Weakest t-norm (drastic product) is given by

$$T_{D}(x, y) = \begin{cases} \min(x, y), & \text{if } \max(x, y) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 1.17.** A s-norm $S$ is a function $S : [0 1] \times [0 1] \rightarrow [0 1]$ satisfying the following conditions:

(S1) $S(x, 1) = 1, S(0, x) = x$ ; (boundary conditions)

(S2) $S(x, y) = S(y, x)$ ; (commutativity)

(S3) $S(x, S(y, z)) = S(S(x, y), z)$ ; (associativity)

(S4) $S(x, y) \leq S(z, w)$ ;if $x \leq z, y \leq w$ for all $x, y, z \in [0 1]$ (monotonicity)

Every s-norm $S$ satisfies $S(x, y) \geq \max(x, y)$ $\forall x, y \in [0, 1]$.

**Example 1.18.** The four basic t-conorm are:

(1). Maximum given by $S_{M}(x, y) = \max(x, y)$.

(2). Probabilistic sum given by $S_{P}(x, y) = x + y - xy$.

(3). The Lukasiewicz is given by $S_{L}(x, y) = \min(x + y, 1)$.

(4). Strongest t-conorm given by

$$S_{D}(x, y) = \begin{cases} \max(x, y), & \text{if } \max(x, y) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 1.19.** If for two t-norms $T_{1}$ and $T_{2}$ the inequality $T_{1}(x, y) \leq T_{2}(x, y)$ holds for all $(x, y) \in [0 1] \times [0 1]$ then $T_{1}$ is said to be weaker than $T_{2}$, and we write in this case $T_{1} \leq T_{2}$. We write $T_{1} < T_{2}$ whenever $T_{1} \leq T_{2}$ and $T_{1} \neq T_{2}$.
Remark 1.20. It is not hard to see that $T_D$ is the weakest $t$-norm and $T_M$ is the strongest $t$-norm, that is, for all $t$-norm $T$

\[ T_D \leq T \leq T_M \]

We get the following ordering of the four basic $t$-norms:

\[ T_D < T_L < T_P < T_M \]

Lemma 1.21. Let $T$ be a $t$-norm. Then $T(T(x,y) T(z,t)) = T(T(x,z) T(y,t))$ for all $x, y, z$ and $t \in [0, 1]$.

Definition 1.22. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two doubt intuitionistic fuzzy sets of $X_1$ and $X_2$, respectively. Then the direct product of DIFSs $A$ and $B$ with respect to triangular binorm (i.e., $(T, S)$-normed) is denoted by $A \times B = (\mu_{A \times B}, \nu_{A \times B})$ where $\mu_{A \times B} : X_1 \times X_2 \rightarrow [0, 1]$ defined by $\mu_{A \times B}(x,y) = S(\mu_A(x), \mu_B(y))$ and $\nu_{A \times B} : X_1 \times X_2 \rightarrow [0, 1]$ defined by $\nu_{A \times B}(x,y) = T(\nu_A(x), \nu_B(y))$ for all $(x, y) \in X_1 \times X_2$.

Definition 1.23 ([26]). An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of a BCK-algebra $X$ is said to be a doubt intuitionistic fuzzy subalgebra with respect to triangular binorm of $X$ if

1. $\mu_A(x \ast y) \leq S(\mu_A(x), \mu_A(y))$
2. $\nu_A(x \ast y) \geq T(\nu_A(x), \nu_A(y)) \ \forall \ x, y \in X.$

Definition 1.24 ([16, 26]). An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of a BCK-algebra $X$ is said to be a doubt intuitionistic fuzzy ideal with respect to triangular binorm of $X$ if

1. $\mu_A(0) \leq \mu_A(x)$
2. $\nu_A(0) \geq \nu_A(x)$
3. $\mu_A(x) \leq S(\mu_A(x \ast y), \mu_A(y))$
4. $\nu_A(x) \geq T(\nu_A(x \ast y), \nu_A(y)) \ \forall x, y \in X.$

2. Main Section

In this section, we define direct product of an $(\in, \in \vee \in q_k)$-doubt intuitionistic fuzzy sets with respect to triangular binorm and investigate some related properties.

Definition 2.1. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two $(\in, \in \vee \in q_k)$-intuitionistic fuzzy sets of $X_1$ and $X_2$, respectively. Then the direct product of $(\in, \in \vee \in q_k)$-intuitionistic fuzzy sets $A$ and $B$ with respect to triangular binorm (i.e., $(T, S)$-normed) is denoted by $A \times B = (\mu_{A \times B}, \nu_{A \times B})$ where $\mu_{A \times B} : X_1 \times X_2 \rightarrow [0, 1]$ defined by $\mu_{A \times B}(x,y) = S(\mu_A(x), \mu_B(y), \frac{1-\mu_A(x)}{2})$ and $\nu_{A \times B} : X_1 \times X_2 \rightarrow [0, 1]$ defined by $\nu_{A \times B}(x,y) = T(\nu_A(x), \nu_B(y), \frac{1-\nu_A(x)}{2})$ for all $(x, y) \in X_1 \times X_2$.

Definition 2.2. An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of BCK-algebra $X$ is said to be an $(\in, \in \vee \in q_k)$-doubt intuitionistic fuzzy subalgebra with respect to triangular binorm of $X$ if

1. $\mu_A(x \ast y) \leq S \left(\mu_A(x), \mu_A(y), \frac{1-\mu_A(x)}{2}\right)$ for all $x, y \in X.$
2. $\nu_A(x \ast y) \geq T \left(\nu_A(x), \nu_A(y), \frac{1-\nu_A(x)}{2}\right)$ for all $x, y \in X.$
Definition 2.3. An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of $BCK$-algebra $X$ is said to be an $(\varepsilon, \in \nu q_k)$-doubt intuitionistic fuzzy ideal with respect to triangular binorm (i.e., $(T, S)$-normed) of $X$ if

1. $\mu_A(0) \leq S \left\{\mu_A(x), \frac{1-k}{2}\right\}$ for all $x \in X$.
2. $\nu_A(0) \geq T \left\{\nu_A(x), \frac{1-k}{2}\right\}$ for all $x \in X$.
3. $\mu_A(x) \leq S \left\{\mu_A(x \ast y), \mu_A(y), \frac{1-k}{2}\right\}$ for all $x, y \in X$.
4. $\nu_A(x) \geq T \left\{\nu_A(x \ast y), \nu_A(y), \frac{1-k}{2}\right\}$ for all $x, y \in X$.

Definition 2.4. An intuitionistic fuzzy set $A \times B$ of $BCK$-algebra $X_1 \times X_2$ is said to be an $(\varepsilon, \in \nu q_k)$-doubt intuitionistic fuzzy subalgebra of $X_1 \times X_2$ with respect to triangular binorm if

1. $\mu_{A \times B}((x_1, y_1) \ast (x_2, y_2)) \leq S \left\{\mu_{A \times B}(x_1, y_1), \mu_{A \times B}(x_2, y_2), \frac{1-k}{2}\right\}$ for all $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$.
2. $\nu_{A \times B}((x_1, y_1) \ast (x_2, y_2)) \geq T \left\{\nu_{A \times B}(x_1, y_1), \nu_{A \times B}(x_2, y_2), \frac{1-k}{2}\right\}$ for all $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$.

Definition 2.5. An intuitionistic fuzzy set $A \times B$ of $BCK$-algebra $X_1 \times X_2$ is said to be an $(\varepsilon, \in \nu q_k)$-doubt intuitionistic fuzzy ideal of $X_1 \times X_2$ with respect to triangular binorm if

1. $\mu_{A \times B}(0, 0) \leq S \left\{\mu_{A \times B}(x_1, y_1), \frac{1-k}{2}\right\}$ for all $(x_1, y_1) \in X_1 \times X_2$.
2. $\nu_{A \times B}(0, 0) \geq T \left\{\nu_{A \times B}(x_1, y_1), \frac{1-k}{2}\right\}$ for all $(x_1, y_1) \in X_1 \times X_2$.
3. $\mu_{A \times B}(x_1, y_1) \leq S \left\{\mu_{A \times B}(x_1, y_1) \ast (x_2, y_2), \mu_{A \times B}(x_2, y_2), \frac{1-k}{2}\right\}$ for all $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$.
4. $\nu_{A \times B}(x_1, y_1) \geq T \left\{\nu_{A \times B}(x_1, y_1) \ast (x_2, y_2), \nu_{A \times B}(x_2, y_2), \frac{1-k}{2}\right\}$ for all $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$.

Theorem 2.6. Let $A$ and $B$ be two $(\varepsilon, \in \nu q_k)$-doubt intuitionistic fuzzy subalgebras of $X_1$ and $X_2$, respectively. Then the Direct product $A \times B$ is an $(\varepsilon, \in \nu q_k)$-doubt intuitionistic fuzzy subalgebra of $X_1 \times X_2$.

Proof. Let $A$ and $B$ be two $(\varepsilon, \in \nu q_k)$-doubt intuitionistic fuzzy subalgebras of $X_1$ and $X_2$, respectively. For any $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$. We have

$$
\mu_{A \times B}((x_1, y_1) \ast (x_2, y_2)) = \mu_{A \times B}(x_1 \ast x_2, y_1 \ast y_2)
\leq S \left\{\mu_A(x_1 \ast x_2), \mu_B(y_1 \ast y_2), \frac{1-k}{2}\right\}
\leq S \left\{S \left\{\mu_A(x_1), \frac{1-k}{2}\right\}, S \left\{\mu_B(y_1), \frac{1-k}{2}\right\}\right\}
= S \left\{S \left\{\mu_A(x_1), \frac{1-k}{2}\right\}, S \left\{\mu_B(y_1), \frac{1-k}{2}\right\}\right\}
= S \left\{\mu_{A \times B}(x_1, y_1), \mu_{A \times B}(x_2, y_2), \frac{1-k}{2}\right\}
$$

$$
\nu_{A \times B}((x_1, y_1) \ast (x_2, y_2)) = \nu_{A \times B}(x_1 \ast x_2, y_1 \ast y_2)
\geq T \left\{\nu_A(x_1 \ast x_2), \nu_B(y_1 \ast y_2), \frac{1-k}{2}\right\}
\geq T \left\{T \left\{\nu_A(x_1), \frac{1-k}{2}\right\}, T \left\{\nu_B(y_1), \frac{1-k}{2}\right\}\right\}
= T \left\{T \left\{\nu_A(x_1), \frac{1-k}{2}\right\}, T \left\{\nu_B(y_1), \frac{1-k}{2}\right\}\right\}
= T \left\{\nu_{A \times B}(x_1, y_1), \nu_{A \times B}(x_2, y_2), \frac{1-k}{2}\right\}
$$

Hence $A \times B$ is an $(\varepsilon, \in \nu q_k)$-doubt intuitionistic fuzzy subalgebra of $X_1 \times X_2$. \hfill \Box
Theorem 2.7. Let $A$ and $B$ be two $(\varepsilon, \in \vee q_k)$-doubt intuitionistic fuzzy ideals of $X_1$ and $X_2$, respectively. Then the direct product $A \times B$ is an $(\varepsilon, \in \vee q_k)$-doubt intuitionistic fuzzy ideal of $X_1 \times X_2$.

Theorem 2.8. If $A \times B = (\mu_{A \times B}, \nu_{A \times B})$ be an $(\varepsilon, \in \vee q_k)$-doubt intuitionistic fuzzy ideal of $X_1 \times X_2$. Then for any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X_1 \times X_2$ and $(x_1, y_1) \ast (x_2, y_2) \leq (x_3, y_3)$

(1). $\mu_{A \times B}(x_1, y_1) \leq S \{ \mu_{A \times B}(x_2, y_2), \mu_{A \times B}(x_3, y_3), \frac{1-k}{2} \}$

(2). $\nu_{A \times B}(x_1, y_1) \geq T \{ \nu_{A \times B}(x_2, y_2), \nu_{A \times B}(x_3, y_3), \frac{1-k}{2} \}$

Proof. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X_1 \times X_2$ such that $(x_1, y_1) \ast (x_2, y_2) \leq (x_3, y_3)$ then $((x_1, y_1) \ast (x_2, y_2)) \ast (x_3, y_3) = 0$. Now

(1). $\mu_{A \times B}(x_1, y_1) \leq S \{ \mu_{A \times B}((x_1, y_1) \ast (x_2, y_2), \mu_{A \times B}(x_2, y_2), \frac{1-k}{2} \}$

$\leq S \{ \mu_{A \times B}((x_1, y_1) \ast (x_2, y_2)) \ast (x_3, y_3), \mu_{A \times B}(x_3, y_3), \frac{1-k}{2} \}, \mu_{A \times B}(x_2, y_2), \frac{1-k}{2} \}$

$= S \{ S \{ \mu_{A \times B}(x_3, y_3), \frac{1-k}{2} \}, \mu_{A \times B}(x_2, y_2), \frac{1-k}{2} \}, \mu_{A \times B}(x_2, y_2), \frac{1-k}{2} \}$

$= S \{ S \{ \mu_{A \times B}(x_3, y_3), \frac{1-k}{2} \}, \mu_{A \times B}(x_2, y_2), \frac{1-k}{2} \}$

$= S \{ \mu_{A \times B}(x_3, y_3), \mu_{A \times B}(x_2, y_2), \frac{1-k}{2} \}$

(2). $\nu_{A \times B}(x_1, y_1) \geq T \{ \nu_{A \times B}((x_1, y_1) \ast (x_2, y_2)), \nu_{A \times B}(x_2, y_2), \frac{1-k}{2} \}$

$\geq T \{ \nu_{A \times B}((x_1, y_1) \ast (x_2, y_2)) \ast (x_3, y_3)), \nu_{A \times B}(x_3, y_3), \frac{1-k}{2} \}, \nu_{A \times B}(x_2, y_2), \frac{1-k}{2} \}$

$= T \{ T \{ \nu_{A \times B}(x_3, y_3), \frac{1-k}{2} \}, \nu_{A \times B}(x_2, y_2), \frac{1-k}{2} \}, \nu_{A \times B}(x_2, y_2), \frac{1-k}{2} \}$

$= T \{ T \{ \nu_{A \times B}(x_3, y_3), \frac{1-k}{2} \}, \nu_{A \times B}(x_2, y_2), \frac{1-k}{2} \}$

$= T \{ \nu_{A \times B}(x_3, y_3), \nu_{A \times B}(x_2, y_2), \frac{1-k}{2} \}$

Definition 2.9. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are intuitionistic fuzzy sets of $X_1$ and $X_2$ respectively. Define the doubt intuitionistic level set for the $A \times B$ as $(A \times B)_{\alpha, \beta} = \{(x, y) \in X_1 \times X_2 | \mu_{A \times B}(x, y) \leq \alpha, \nu_{A \times B}(x, y) \geq \beta\}$, where $\beta \in (0, \frac{1-k}{2}], \alpha \in [\frac{1-k}{2}, 1)$.

Theorem 2.10. Let $A$ and $B$ be two $(\varepsilon, \in \vee q_k)$-doubt intuitionistic fuzzy subalgebras of $X_1$ and $X_2$, respectively. Then the direct product $A \times B$ is an $(\varepsilon, \in \vee q_k)$-doubt intuitionistic fuzzy subalgebra of $X_1 \times X_2$ if and only if $(A \times B)_{\alpha, \beta} \neq \phi$ is a subalgebra of $X_1 \times X_2$.

Proof. Assume $A \times B$ is an $(\varepsilon, \in \vee q_k)$-doubt intuitionistic fuzzy subalgebra of $X_1 \times X_2$. To prove $(A \times B)_{\alpha, \beta} \neq \phi$ is an subalgebra of $X_1 \times X_2$, where $\beta \in (0, \frac{1-k}{2}], \alpha \in [\frac{1-k}{2}, 1)$. Let $(x_1, y_1), (x_2, y_2) \in (A \times B)_{\alpha, \beta}$. Therefore we have
Proof. By definition, $\mu_{A \times B}(0,0) = \mu_{A \times B}((x,y) \ast (x,y)) \leq S\{\mu_{A \times B}(x,y), \mu_{A \times B}(x,y), \frac{1}{\alpha} \} = S\{\mu_{A \times B}(x,y), \frac{1}{\alpha} \}$. Therefore, $\mu_{A \times B}(0,0) \leq S\{\mu_{A \times B}(x,y), \frac{1}{\alpha} \}$ for all $(x,y) \in X_1 \times X_2$. Again, $\nu_{A \times B}(0,0) = \nu_{A \times B}((x,y) \ast (x,y)) \geq T\{\nu_{A \times B}(x,y), \nu_{A \times B}(x,y), \frac{1}{\alpha} \} = T\{\nu_{A \times B}(x,y), \frac{1}{\alpha} \}$. Therefore, $\nu_{A \times B}(0,0) \geq T\{\nu_{A \times B}(x,y), \frac{1}{\alpha} \}$ for all $(x,y) \in X_1 \times X_2$.

Lemma 2.13. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two $\epsilon \in \nu(x)$-doubt intuitionistic fuzzy subalgebras of $BCK/BCI$-algebras $X_1$ and $X_2$ respectively. Then the following are true.

(1). $\mu_A(0) \leq \mu_B(y)$ and $\mu_B(0) \leq \mu_A(x)$, for all $x \in X_1, y \in X_2$.

(2). $\nu_A(0) \geq \nu_B(y)$ and $\nu_B(0) \geq \nu_A(x)$, for all $x \in X_1, y \in X_2$.

Proof. Assume that $\mu_A(0) > \mu_B(y)$ and $\mu_B(0) > \mu_A(x)$, for some $x \in X_1, y \in X_2$. Then, $\mu_{A \times B}(x,y) = S\{\mu_A(x), \mu_B(y), \frac{1}{\alpha} \} \leq S\{\mu_A(0), \mu_A(0), \frac{1}{\alpha} \} = \mu_{A \times B}(0,0)$. That is a contradiction. Similarly, let $\nu_A(0) < \nu_B(y)$ and $\nu_B(0) < \nu_A(x)$, for some $x \in X_1, y \in X_2$. Then, $\nu_{A \times B}(x,y) = T\{\nu_A(x), \nu_B(y), \frac{1}{\alpha} \} \geq T\{\nu_A(0), \nu_A(0), \frac{1}{\alpha} \} = \nu_{A \times B}(0,0)$ That is a contradiction. Thus proving the result.

Theorem 2.14. If $A \times B$ is a $(\epsilon, \nu(x))$-doubt intuitionistic fuzzy subalgebra of $X_1 \times X_2$, then either $A$ is an $(\epsilon, \nu(x))$-doubt intuitionistic fuzzy subalgebra of $X_1$, or $B$ is an $(\epsilon, \nu(x))$-doubt intuitionistic fuzzy subalgebra of $X_2$.

Proof. Since $A \times B$ is a $(\epsilon, \nu(x))$-doubt intuitionistic fuzzy subalgebra of $X_1 \times X_2$, then for all $(x, y)$, $(x, y) \in X_1 \times X_2$, we have $\mu_{A \times B}((x, y), (x, y)) \leq S\{\mu_{A \times B}(x, y), \mu_{A \times B}(x, y), \frac{1}{\alpha} \}$
By putting \( x_1 = x_2 = 0 \), we get,

\[
\mu_{A \times B}((0, y_1) \ast (0, y_2)) \leq S \left\{ \mu_{A \times B}((0, y_1), \mu_{A \times B}(0, y_2), \frac{1-k}{2}) \right\}
\]

\[ \Rightarrow \mu_{A \times B}(((0 \ast 0), (y_1 \ast y_2)) \leq S \left\{ \mu_B(y_1), \mu_B(y_2), \frac{1-k}{2} \right\} \quad \text{using Lemma 2.13} \]

\[ \Rightarrow S\{\mu_A(0 \ast 0), \mu_B(y_1 \ast y_2)\} \leq S \left\{ \mu_B(y_1), \mu_B(y_2), \frac{1-k}{2} \right\} \]

\[ \Rightarrow \mu_B(y_1 \ast y_2) \leq S \left\{ \mu_B(y_1), \mu_B(y_2), \frac{1-k}{2} \right\} \]

Similar way we can prove, \( \nu_B(y_1 \ast y_2) \geq T\{\nu_B(y_1), \nu_B(y_2), \frac{1-k}{2}\} \). Hence \( B \) is an \((\varepsilon, \varepsilon \lor \lor_k)\)-doubt intuitionistic fuzzy subalgebra of \( X_2 \).

References


