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Basics of Development of Integration and Series in India

Research Article

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Abstract: Most of the Indian Mathematicians form the period of ryabhaa knows the arithmetic progression (AP) and sum of terms in AP. The present paper discusses the idea of Sama-ghāta-saṅkalita in Yukti-bhāṣa and Vrasaṅkalita in Gaṇita Kaumudī. These ideas create the basic development of Integration and infinite series India. This work leads to evaluate integration as the summation of series.

MSC: 03E17.

Keywords: Gaita Kaumudī, Nrayaa Pata, Sama-ghāta-saṅkalita, Yukti-bhāṣa, Jyeṣṭhadeva, Arithmetic Progression

1. Introduction

Aryabhata (476-550 A.D.) was India's great Mathematician and Astronomer. He wrote the text Aryabhatiya (498 A.D.). He mentions the result on the sum of natural numbers and sum of terms in Arithmetic Progression (AP). After Aryabhata the result on AP was found in Ganitha Sara Sangraha (850 A.D.) of Mahavira, Ganita Kaumudi (1356 A.D.) of Narayana Pandita, Brahama Sphuta Siddanta (628 A.D.) of Brahmagupta, Patiganita of Sridhara and Yukti-bhasa of Jyesthadeva (1500-1610 A.D.).

In India the significant development of series occurred after Bhaskara's time. These developments of series occur because of treatment of two ideas Varasankalita in Ganita Kaumudi and Sama-ghata-sankalita in Yukti-bhasa. This work leads to evaluate integration as the summation of series. Most of the above text contains the result of (1) Sum of first n natural numbers (2) Sum of first n terms of AP (3) Sum of squares, cubes and k^{th} power of first n natural numbers. The first and most important result stated by Narayana Pandita is the general formula for the sum of any order of triangular number (The triangular number T_n is a figurate number that can be represented in the form of a triangular grid of points where the first row contains a single element and each subsequent row contains one more element than the previous one). He also stated it as sum of sums or repeated sums or Varasankalita. He generalised the formula of Varasankalita. The first Varasankalita will give sum of first order triangular numbers, the second Varasankalita will give sum of second order triangular numbers and finally stated the result for the sum of k^{th} order triangular numbers or k^{th} Varasankalita of AP.

Yukti-bhasa of Jyesthadeva calculated the values of Sum of squares, cubes and k^{th} power of first n natural numbers and calculated the approximate value if n is very large. Nilakantha Somayaji in Aryabhatiya-Bhasya and sankara Variyar in Kriyakramakari gave geometrical demonstration to AP. The aim of this paper is to present the concept of Varasankalita of k^{th} order and Sama-ghata-sankalita (sum of k^{th} power of first n natural numbers).

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2. Sum of $1^k + 2^k + \cdots + n^k$ or sama-ghata-sankalita for Large n

The method of calculating circumference without finding squares roots is the greatest contribution of Madhava. It is based on concept of infinite series and covers the idea of Integration. This result gives us values of π (for ratio of circumference to diameter) in terms of infinite series. We use the results of sama-ghata-sankalita in Yukti-bhasa stated by Jyesthadeva for finding infinite series of π. The word used for summation in Indian mathematical text is sankalita. Aryabhata-I gave the formulae for the summation of squares and cubes of integers. These formulae are as follows

$$S_n^{(1)} = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$
$$S_n^{(2)} = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$S_n^{(3)} = 1^3 + 2^3 + \cdots + (n-2)^3 + (n-1)^3 + n^3 = [\frac{n(n+1)}{2}]^2$$

Calculation of sama-ghata-sankalita is given in Yukti-bhasa

$$S_n^{(k)} = 1^k + 2^k + \cdots + (n-2)^k + (n-1)^k + n^k$$

where n is very large. The method is as follows:

2.1. Sum of First n Natural Numbers or Mula-sankalita

$$\begin{aligned} S_n^{(1)} &= 1 + 2 + 3 + \cdots + n \\ &= n + (n-1) + (n-2) + \cdots + 2 + 1 \\ &= n + (n-1) + (n-2) + \cdots + (n-(n-2)) + (n-(n-1)) \\ &= n \times n - (1 + 2 + 3 + \cdots + (n-1)) \\ &= n \times n - S_{n-1}^{(1)} \end{aligned}$$

For large n,

$$\begin{aligned} S_n^{(1)} &\approx S_{n-1}^{(1)} \\ S_n^{(1)} &\approx n^2 - S_{n-1}^{(1)} \\ S_n^{(1)} &\approx \frac{n^2}{2} \end{aligned} \tag{1}$$

This is similar to $\int x dx = \frac{x^2}{2}$.

2.2. Sum of Square of First n Natural Numbers or Varga-sankalita

Sum of squares of first n natural numbers is known as Varga-sankalita given by

$$S_n^{(2)} = 1^2 + 2^2 + 3^2 + \cdots + n^2$$
$$S_n^{(2)} = n^2 + (n-1)^2 + (n-2)^2 + \cdots + 2^2 + 1^2 \tag{2}$$

We use $nS_n^{(1)}$ and

$$(nS_n^{(1)}) = n \times (n + (n-1) + (n-2) + \cdots + 2 + 1) \tag{3}$$

Now (3)−(2)

$$\begin{aligned}(nS_n^{(1)}) - S_n^{(2)} &= n(n + (n-1) \cdots + 2 + 1) - (n^2 + (n-1)^2 + \cdots + 2^2 + 1^2) \\ &= (n(n-1) - (n-1)^2) + (n(n-2) - (n-2)^2) + \cdots + (n-1) \\ &= 1(n-1) + 2(n-2) + 3(n-3) \cdots + (n-1)1\end{aligned}$$

This can be written as

$$\begin{aligned}(nS_n^{(1)}) - S_n^{(2)} = (n-1) + (n-2) + (n-3) + \cdots + 2 + 1 \\ +(n-2) + (n-3) + \cdots + 2 + 1 \\ +(n-3) + \cdots + 2 + 1 \\ + \cdots + 2 + 1 \\ +2 + 1 \\ + 1\end{aligned}$$

$$(nS_n^{(1)}) - S_n^{(2)} = S_{n-1}^{(1)} + S_{n-2}^{(1)} + S_{n-3}^{(1)} \tag{4}$$

Right hand side of (3) is called sankalita- sankalita I and $S_n^{(1)} \approx \frac{n^2}{2}$. So equation (4) becomes

$$\begin{aligned}(nS_n^{(1)}) - S_n^{(2)} &= \frac{(n-1)^2}{2} + \frac{(n-2)^2}{2} + \frac{(n-3)^2}{2} + \dots \\ &= \frac{1}{2}((n-1)^2 + (n-2)^2 + (n-3)^2 + \dots) \\ &= \frac{1}{2} S_{n-1}^{(2)}\end{aligned} \tag{5}$$

But when n is very large $S_{n-1}^{(2)} \approx S_n^{(2)}$ and $S_n^{(1)} \approx \frac{n^2}{2}$. So equation (5) becomes

$$\begin{aligned}(nS_n^{(1)}) - S_n^{(2)} &\approx \frac{S_n^{(2)}}{2} \\ n\frac{n^2}{2} - S_n^{(2)} &\approx \frac{S_n^{(2)}}{2} \\ S_n^{(2)} &\approx \frac{n^3}{3}\end{aligned} \tag{6}$$

This is similar to $\int x^2 dx = \frac{x^3}{3}$.

2.3. Sum of Cubes of First n Natural Numbers or Ghana-sankalita

Sum of cubes of first n natural numbers is called as Ghana-sankalita given by

$$S_n^{(3)} = n^3 + (n-1)^3 + (n-2)^3 + \cdots + 2^3 + 1^3$$

Using the same process as in Varga-sankalita we have

$$\begin{aligned}(nS_n^{(2)}) - S_n^{(3)} &= S_{n-1}^{(2)} + S_{n-2}^{(2)} + S_{n-3}^{(2)} + \dots \\ (n\frac{n^3}{3}) - S_n^{(3)} &= \frac{(n-1)^3}{3} + \frac{(n-2)^3}{3} + \frac{(n-3)^3}{3} + \dots \\ \frac{n^4}{3} - S_n^{(3)} &\approx \frac{1}{3} S_{n-1}^{(3)}\end{aligned}$$

But for large n, $S_n^{(3)} \approx S_{n-1}^{(3)}$ gives $S_n^{(3)} \approx \frac{n^4}{4}$.

2.4. Sum of $1^k + 2^k + \cdots + n^k$ or sama-ghata-sankalita for Large n

Sum of k^{th} power of first n natural numbers is called as sama-ghata-sankalita given by

$$S_n^{(k)} = n^k + (n-1)^k + (n-2)^k + \cdots + 2^k + 1^k$$

So that

$$(nS_n^{(k-1)}) - S_n^{(k)} = S_{n-1}^{(k-1)} + S_{n-2}^{(k-1)} + S_{n-3}^{(k-1)} + \ldots$$

and $S_n^{(k-1)} \approx \frac{n^k}{k}$ results $S_n^{(k)} \approx \frac{n^{k+1}}{k+1}$ (In modern text we use $\int x^k dx = \frac{x^{k+1}}{k+1}$, in Indian Mathematics, repeated summations are also calculated).

3. Varasankalita

3.1. Definition Varasankalita of Natural Numbers $({}^kV_n)$

Varasankalita is defined as follows

$$\begin{aligned} {}^1V_n &= \sum n = 1 + 2 + 3 + \cdots + \text{up to n terms} \\ {}^2V_n &= \sum\sum n = 1 + 3 + 6 + 10 + \cdots + \text{up to n terms} \\ {}^3V_n &= \sum\sum\sum n = 1 + 4 + 10 + 20 + \cdots + \text{up to n terms} \end{aligned}$$

3.2. Formula for ${}^1V_n$

The first order Varasankalita is the sum of first n natural numbers. This formula is coated by most of all Indian Mathematicians. The formula coated by Narayana Pandita is

सैकपदघ्नपदार्धम् सङ्कलितम् ।

saikapadaghnapadardham sankalitam.

Meaning: multiply (n+1) [स एक–पद] by (n/2) [पद–अर्ध] for the sum (of first n natural numbers) [सङ्कलितम्]

$${}^1V_n = \frac{n \times (n+1)}{2} = \frac{n \times (n+1)}{1 \times 2}$$

3.3. Formula for ${}^2V_n$

Bhaskara II coated the formula for second order Varasankalita

सैकपदघ्नपदार्धमथैकाद्यङ्कयुतिः किल सङ्कलिताख्या ।

सा द्वियुतेन पदेन विनिघ्नी स्यात् त्रिहृता खलु सङ्कलितैक्यम् ।।

saikapadaghnapadardhamathaikadyankayutih kila sankalitakhya | *sa dviyutena padena vinighni syat trihrta khalu sankalitakhya* ||.

Meaning: The sum of n numbers starting from 1 up to n (by common difference 1) is (n+1) multiplied by $\left(\frac{n}{2}\right)$ this sum is called sankalita. That is (sum of natural numbers) is multiplied by (n+2) and then divided by 3 is the sum of sum of natural numbers (second order Varasankalita).

$${}^2V_n = \frac{n \times (n+1)}{2} \times \frac{(n+2)}{3} = \frac{n \times (n+1) \times (n+2)}{2 \times 3} = \frac{n \times (n+1) \times (n+2)}{3!}$$

This result can be proved as follows

$$\begin{aligned}
{}^2V_n &= \sum\sum n = \sum \frac{n\times(n+1)}{2}\\
&= \frac{1}{2}\sum(n^2+n)\\
&= \frac{1}{2}\left(\sum n^2+\sum n\right)\\
&= \frac{1}{2}\left(\frac{n\times(n+1)\times(2n+1)}{6}+\frac{n\times(n+1)}{2}\right)\\
&= \frac{1}{2}\times\frac{n\times(n+1)}{2}\left(\frac{(2n+1)}{3}+1\right)\\
&= \frac{n\times(n+1)\times(n+2)}{1\times2\times3}
\end{aligned}$$

Aryabhata I stated the result as

षड्भक्तः स चितिघनः सएकपदघनो विमूलो वा

Meaning: Cube of $(n+1)$ minus cube root of same root divided by 6 is the sum of sums of natural numbers thus ${}^2V_n = \frac{(n+1)^3-(n+1)}{6}$.

3.4. Formula for ${}^3V_n$

The formula for 3rd order triangular number or third order Varasankalita is

$$ {}^3V_n = \frac{n\times(n+1)\times(n+2)\times(n+3)}{4!}$$

3.5. Formula for kth Order Varasankalita in Ganita Kaumudi of Narayana Pandita

The formula stated by Narayana Pandita is

एकाधिकवारमिताः पदादिररूपोत्तराःपृथक् तेंऽशाः

एकाद्येकचयहरास्तद्घातो वारसङ्कलितम् ।

Meaning: The numbers beginning with the number of terms in the series increasing by one and equal in number to one more than the number representing the order of summation separately from the numerators. The corresponding denominators are the natural numbers beginning with one. Product of these (fractions) is the Varasankalita.

$$ {}^kV_n = \frac{n\times(n+1)\times(n+2)\times\cdots\times(n+k)}{(k+1)!}$$

4. Approach to Modern Methods

According to modern text Varasankalita is the integration of integration. For the large value of n, put $n=(n+1)$ the result is

$$\begin{aligned}
{}^1V_n &= \frac{n\times(n+1)}{1\times2}\approx\frac{n^2}{2} \quad\text{or}\quad {}^1V_n=\int x dx=\frac{x^2}{2}\\
{}^2V_n &= \frac{n\times(n+1)\times(n+2)}{1\times2\times3}\approx\frac{n^3}{6} \quad\text{or}\quad {}^2V_n=\iint x dx=\frac{x^3}{6}
\end{aligned}$$

Similarly

$$^{k}V_n = \frac{n \times (n+1) \times (n+2) \times \cdots \times (n+k)}{(k+1)!} \approx \frac{n^{k+1}}{(k+1)!} \quad \text{or}$$

$$^{k}V_n = \int\int (\text{k times}) \int x dx = \frac{x^{k+1}}{(k+1)!}$$

5. Comments

(1). The important idea of infinitesimal calculus is infinitely large and Infinitely small. So when n increases Sama-ghata-sankalita is the integration and Varasankalita is integration of integration.

(2). Madhava (Founder of Kerala School Mathematics (1340-1425) found the value of π using by the method of calculating circumference without finding square roots is one of the contributions of Madhava based on concept of infinite series. Also found the value of π in terms of infinite series. This series converges slowly.

(3). The infinite series is credited to Madhava but quoted by Sankara Variyar in his commentary Yukti-dipika and Kriya-karmakari. The method of calculating circumference without finding square roots is the greatest contribution of Madhava based on concept of infinite series. Also it covers the idea of Integration.

(4). Narayana Pandita stated the result Varasankalita of terms in AP.

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